Thrifty approximations of convex bodies by polytopes

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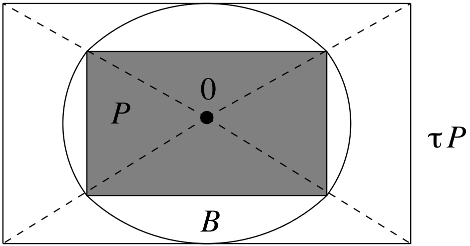
http://www.math.lsa.umich.edu/∼barvinok/papers.html

The problem

Let $B \subset \mathbb{R}^d$ be a convex body containing the origin in its interior. Given $\tau > 1$, we want to find a polytope with as few vertices as possible, such that

$$P \subset B \subset \tau P.$$



Most of the time, B is symmetric about the origin, so $B = -B$ and τ measures the Banach-Mazur distance.

The main result

Theorem

Let k and d be positive integer and let $\tau > 1$ be a real number such that

$$\left(\tau - \sqrt{\tau^2 - 1}\right)^k + \left(\tau + \sqrt{\tau^2 - 1}\right)^k \geq 6\binom{d+k}{k}^{1/2}.$$

Then for any symmetric convex body $B \subset \mathbb{R}^d$ there is a symmetric polytope $P \subset \mathbb{R}^d$ with

$$N \leq 8\binom{d+k}{k}$$

vertices such that

$$P \subset B \subset \tau P.$$

Fine and coarse approximations

Varying k, we get various asymptotic regimes. We will consider two:

- $\tau = 1 + \epsilon$, $\epsilon > 0$ is small, N is large and $k \sim \dfrac{d}{\sqrt{\epsilon}} \ln \dfrac{1}{\epsilon}$.

- N is polynomial in d, $\tau \sim \sqrt{d}$ and k is fixed.

Fine approximations

Corollary

For any

$$\gamma \; > \; \frac{e}{4\sqrt{2}} \approx 0.48$$

there exists $\epsilon = \epsilon_0(\gamma) > 0$ *such that for any* $0 < \epsilon < \epsilon_0$ *and for any symmetric convex body* $B \subset \mathbb{R}^d$ *there is a symmetric polytope* $P \subset \mathbb{R}^d$ *with*

$$N \; \leq \; \left(\frac{\gamma}{\sqrt{\epsilon}} \ln \frac{1}{\epsilon} \right)^d$$

vertices such that

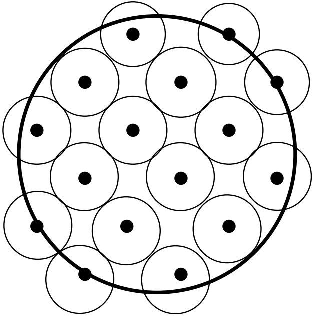
$$P \; \subset \; B \; \subset \; (1 + \epsilon) P.$$

Fine approximations

Compare with:

The "volumetric bound" (Kolmogorov and Tikhomirov 1959?)

$$N \leq \left(\frac{\gamma}{\epsilon}\right)^d$$



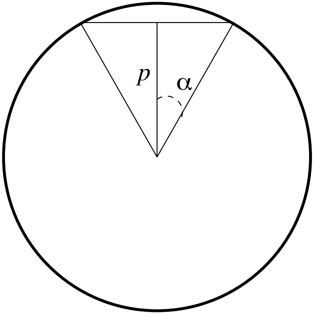
Throw as many points as possible so that the distance between any two (in the $\| \cdot \|_B$ norm) is at least ϵ.

Fine approximations

Compare with:

The C^2-smooth boundary (Gruber 1993):

$$N \leq \left(\frac{\gamma}{\epsilon}\right)^{(d-1)/2} \quad \text{for all} \quad 0 < \epsilon < \epsilon_0(B).$$



$$p = \cos\alpha = \cos\frac{\pi}{N} \approx 1 - \frac{\pi^2}{2N^2}.$$

Coarse approximations

Corollary

For any $0 < \epsilon < 1$, *for any* $d \geq d_0(\epsilon)$, *for any symmetric convex body* $B \subset \mathbb{R}^d$ *there is a symmetric polytope* $P \subset \mathbb{R}^d$ *with*

$$N \leq d^{1/\epsilon}$$

vertices such that

$$P \subset B \subset (\sqrt{\epsilon d})P.$$

Intermediate regimes

$$\tau \leq \gamma \sqrt{\frac{d}{\ln N} \ln \frac{d}{\ln N}} \quad \text{for an absolute constant} \quad \gamma > 0$$

(suggested to the author in this form by A. Litvak, M. Rudelson and N. Tomczak-Jaegermann, 2012).

Ideas of the proof: the minimum volume ellipsoid

Lemma

Let $C \subset \mathbb{R}^d$ be a compact set which spans $\mathbb{R}^d$ and let $E \subset \mathbb{R}^d$ be the (necessarily unique) ellipsoid of the smallest volume among all ellipsoids centered at the origin and containing C. Suppose that E is the unit ball. Then there exist points $x_1, \ldots, x_n \in C \cap \partial E$ and positive real $\alpha_1, \ldots, \alpha_n$ such that

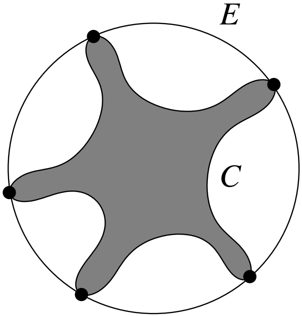
$$\sum_{i=1}^{n} \alpha_i \langle x_i, y \rangle^2 = \|y\|^2 \quad \textit{for all} \quad y \in \mathbb{R}^d.$$

Necessarily,

$$\sum_{i=1}^{n} \alpha_i = d.$$

This is F. John Theorem (1948).

Ideas of the proof: the minimum volume ellipsoid



This produces a set $X \subset C$ of

$$n \leq \frac{d(d+1)}{2} + 1$$

points such that

$$\max_{x \in X} |\ell(x)| \leq \max_{x \in C} |\ell(x)| \leq \sqrt{d} \max_{x \in X} |\ell(x)|$$

for any linear function $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$.

Ideas of the proof: sparsification

Lemma

Let $\gamma > 1$ be a real number and let $x_1, \ldots, x_n$ be vectors in $\mathbb{R}^d$ such that

$$\sum_{i=1}^{n} \langle x_i, y \rangle^2 = \|y\|^2 \quad \text{for all} \quad y \in \mathbb{R}^d.$$

Then there is a subset $J \subset \{1, \ldots, n\}$ with $|J| \leq \gamma d$ and $\beta_j > 0$ for $j \in J$ such that

$$\|y\|^2 \leq \sum_{j \in J} \beta_j \langle x_j, y \rangle^2 \leq \left(\frac{\gamma + 1 + 2\sqrt{\gamma}}{\gamma + 1 - 2\sqrt{\gamma}} \right) \|y\|^2 \quad \text{for all} \quad y \in \mathbb{R}^d.$$

This is Batson-Spielman-Srivastava Theorem (2008).

Ideas of the proof: sparsification

Given a compact $C \subset \mathbb{R}^d$, this produces a set $X \subset C$ of

$$n \ \leq \ 4d$$

points such that

$$\max_{x \in X} |\ell(x)| \ \leq \ \max_{x \in C} |\ell(x)| \ \leq \ 3\sqrt{d} \max_{x \in X} |\ell(x)|$$

for any linear function $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$.

Ideas of the proof: tensorization

Let us denote $V = \mathbb{R}^d$ and let us consider the space

$$W = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes k}.$$

Let us define a continuous map $\phi : V \longrightarrow W$ by

$$\phi(x) = 1 \oplus x \oplus x^{\otimes 2} \oplus \cdots \oplus x^{\otimes k} \quad \text{for} \quad x \in V.$$

We consider the compact set

$$C = \left\{\phi(x) : \quad x \in B\right\}, \qquad C \subset W.$$

Note that C lies in the symmetric part of W, so

$$\dim \operatorname{span}(C) \leq 1 + d + \binom{d+1}{2} + \ldots + \binom{d+k-1}{k} = \binom{d+k}{k}.$$

Ideas of the proof: tensorization

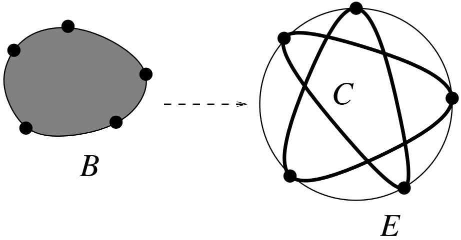
Pick a set $X \subset B$ of

$$N \leq 4\binom{d+k}{k}$$

points such that for any linear function $\mathcal{L} : W \longrightarrow \mathbb{R}$, we have

$$\max_{x \in X} |\mathcal{L}(\phi(x))| \leq \max_{x \in B} |\mathcal{L}(\phi(x))| \leq 3\binom{d+k}{k}^{1/2} \max_{x \in X} |\mathcal{L}(\phi(x))| .$$

Ideas of the proof: tensorization



Define

$$P = \mathrm{conv}\left(X \cup -X\right).$$

Ideas of the proof: Chebyshev polynomials

Recall that $V = \mathbb{R}^d$,

$$W = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus \cdots \oplus V^{\otimes k}$$

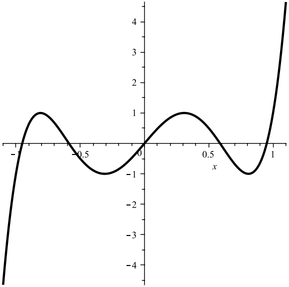
and $\phi : V \longrightarrow W$ is defined by

$$\phi(x) = 1 \oplus x \oplus x^{\otimes 2} \oplus \cdots \oplus x^{\otimes k} \quad \text{for} \quad x \in V.$$

If $\mathcal{L} : W \longrightarrow \mathbb{R}$ is a linear function then $\mathcal{L}(\phi(x))$ is a polynomial of degree k of x.
Suppose that $\ell : \mathbb{R}^d \longrightarrow \mathbb{R}$ is linear such that $|\ell(x)| \leq 1$ for all $x \in X$. To show that $|\ell(x)| \leq \tau$ for all $x \in B$ we would like to construct a polynomial p of degree k such that

$$|p(t)| \leq 1 \quad \text{if} \quad |t| \leq 1 \quad \text{and} \quad |p(\tau)| \quad \text{is the largest possible.}$$

Ideas of the proof: Chebyshev polynomials



Define

$$T_k(t) = \cos\left(k \arccos t\right) \quad \text{provided} \quad -1 \leq t \leq 1$$

$$T_k(t) = \frac{1}{2}\left(t - \sqrt{t^2 - 1}\right)^k + \frac{1}{2}\left(t + \sqrt{t^2 - 1}\right)^k \quad \text{provided} \quad |t| > 1.$$

Ideas of the proof: Chebyshev polynomials

Writing

$$T_k = \sum_{i=0}^{k} a_i t^i, \quad \text{define} \quad \mathcal{L} = \sum_{i=0}^{k} a_i \ell^{\otimes i}.$$

If $\ell(x) > \tau$ for some $x \in B$, then for that $\mathcal{L}$ we get a contradiction with

$$\max_{x \in X} |\mathcal{L}(\phi(x))| \ \leq \ \max_{x \in B} |\mathcal{L}(\phi(x))| \ \leq \ 3 \binom{d+k}{k}^{1/2} \max_{x \in X} |\mathcal{L}(\phi(x))| \, .$$