Fractional White-Noise Limit and Paraxial Approximation for Waves in Random Media

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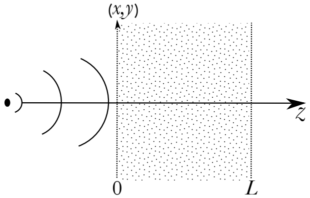
Joint work with Olivier Pinaud



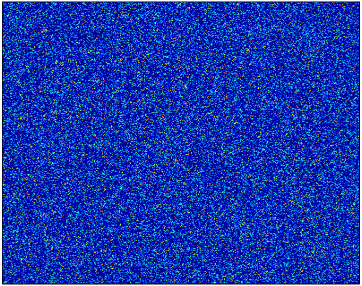
Let us consider the scalar wave equation:

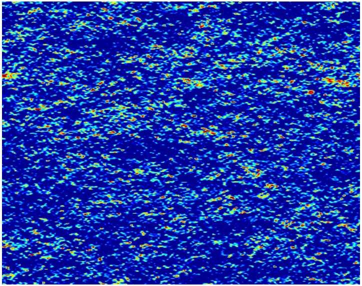
$$\Delta p - \frac{1}{c^2(z,\mathbf{x})}\partial_t^2 p = F \quad \text{with} \quad \frac{1}{c^2(z,\mathbf{x})} = \frac{1}{c_0^2}(1 + V(z,\mathbf{x})).$$

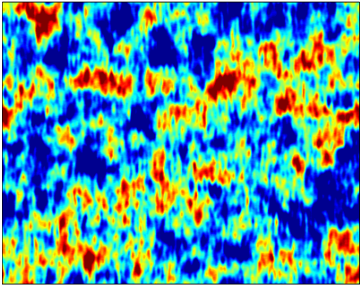
Here, V is a stationary mean-zero random field with slowly decaying correlations :

$$\mathbb{E}[V(z+s,\mathbf{x}+\mathbf{y})V(s,\mathbf{y})] \sim \frac{c}{z^{\mathfrak{H}}}R(\mathbf{x}), \qquad \mathfrak{H} \in (0,1).$$

Illustrations : Random fluctuations







Experimental results

Data collected in real environments report that propagation media with long-range correlations can be encountered in different contexts such as:

- Geophysics

  S. Dolan, C. Bean, and B. Riollet, The broad-band fractal nature of heterogeneity in the upper crust from petrophysical logs, Geophys. J. Int.,132 (1998).

- Laser beam propagation through the atmosphere

  C. Sidi and F. Dalaudier, Turbulence in the stratified atmosphere: Recent theoretical developments and experimental results, Adv. in Space Res., 10 (1990).

- Medical Imaging

  M. Feld et al., Tissue Self-Affinity and Polarized Light Scattering in the Born Approximation: A New Model for Precancer Detection, Phys. Rev. Lett., 97 (2006).

Mathematical Results

Wave propagation in random media with slowly decaying correlations

One dimensional propagation medium

- R. Marty and K. Sølna, Acoustic waves in long-range random media, SIAM J. Appl. Math, (2009).
- J. Garnier and K. Sølna, Pulse propagation in random media with long-range correlation SIAM Multiscale Model. Simul., (2009).

General propagation medium under the paraxial approximation

- G. Bal, T. Komorowski, L. Ryzhik, Asymptotics of the phase of the solutions of the random Schrödinger equation, Arch. Rat. Mech. Anal., (2011)
- C. Gomez, Wave decoherence for the random Schrödingier equation with long-range correlations, Commun. Math. Phys., (2013)
- Y. Gu, L. Ryzhik, The random Schrödinger equation: slowly decorrelating time-dependent potentials, Commun. Math. Sci., (2017)

Justification of the paraxial approximation and white-noise approximation from the random wave equation with **rapidly decaying correlations**:

- F. Bailly, J. F. Clouet, and J. P. Fouque, Parabolic and Gaussian white noise approximation for wave propagation in random media, SIAM J. Appl. Math., 56 (1996).
- J. Garnier and K. Sølna, Coupled paraxial wave equations in random media in the white-noise regime, Ann. Appl. Probab., 19 (2009).

Content

- The paraxial approximation : the homogeneous case,
- The paraxial approximation : the random case,
- Rapidly decaying correlations,
- Slowly decaying correlations,
- Ideas for the proof.

The paraxial approximation : the homogeneous case

The paraxial approximation : the homogeneous case

Let us consider first the nonrandom scalar wave equation:

$$\Delta p - \frac{1}{c_0^2}\partial_t^2 p = F \qquad \text{with} \qquad F(t,z,\mathbf{x}) = f\Big(\frac{t}{\lambda_0},\frac{\mathbf{x}}{r_0}\Big)\delta'(z),$$

where

- λ_0 is the central wavelength of the source,
- r_0 is the transverse width of the source.

We consider

- a high frequency regime, that is $\lambda_0 \ll 1$,
- a Rayleigh length of order 1, that is $r_0^2/\lambda_0 \sim 1$.

Now, introducing

$$\check{p}(\omega,\mathbf{x},z) = \frac{1}{2\pi}\int e^{i\omega t} p(\lambda_0 t, r_0\mathbf{x}, z)dt,$$

we have the following Helmholtz equation

$$\Big(\partial_z^2 + \frac{1}{r_0^2}\Delta_{\mathbf{x}} + \frac{k^2}{\lambda_0^2}\Big)\check{p} = \check{f}(\omega,\mathbf{x})\delta'(z), \qquad \text{with} \qquad k = \omega/c_0.$$

The paraxial approximation : the homogeneous case

Let us write the wave field as

$$\check{p}(\omega, \mathbf{x}, z) = \phi(\omega, z, \mathbf{x}) e^{ikz/\lambda_0},$$

so that for $z \neq 0$

$$\partial_z^2 \phi + \frac{2ik}{\lambda_0} \partial_z \phi + \frac{1}{r_0^2} \Delta_\mathbf{x} \phi = 0,$$

and

$$\underbrace{\frac{\lambda_0}{2k} \partial_z^2 \phi}_{\ll 1} + \underbrace{i \partial_z \phi + \frac{\lambda_0}{2k r_0^2} \Delta_\mathbf{x} \phi = 0}_{\text{paraxial wave equation}} .$$

Therefore, in the high frequency regime $\lambda_0 \to 0$, with $r_0^2 \sim \lambda_0$, we have

$$p(\lambda_0 t + L/c_0, r_0 x, L) \to \frac{1}{2} \int e^{-i\omega t} e^{i\Delta_\mathbf{x} L/(2k)} \check{f}(\omega, \mathbf{x}) d\omega.$$

The paraxial approximation : the random case

The paraxial approximation : the random case

We consider the following wave speed profile

$$\frac{1}{c^2(z,\mathbf{x})} = \frac{1}{c_0^2}\left(1 + \sigma V\left(\frac{z}{l_c}, \frac{\mathbf{x}}{l_c}\right)\right),$$

and our Helmholtz equation becomes

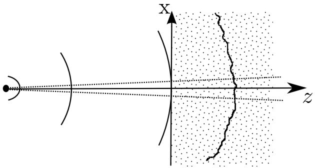
$$\left(\partial_z^2 + \frac{1}{r_0^2}\Delta_{\mathbf{x}} + \frac{k^2}{\lambda_0^2}\left(1+\sigma V\left(\frac{z}{l_c}, \frac{r_0\mathbf{x}}{l_c}\right)\right)\right)\check{p} = \check{f}(\omega,\mathbf{x})\delta'(z), \qquad \text{with} \qquad k = \omega/c_0.$$

Writing again the wave field as

$$\check{p}(\omega,\mathbf{x},z) = \phi(\omega,z,\mathbf{x})e^{ikz/\lambda_0},$$

we now have for $z \neq 0$

$$\underbrace{\frac{\lambda_0}{2k}\partial_z^2\phi}_{\ll 1} + \underbrace{i\partial_z\phi + \frac{\lambda_0}{2k r_0^2}\Delta_{\mathbf{x}}\phi + \frac{k\sigma}{2\lambda_0}V\left(\frac{z}{l_c}, \frac{r_0\mathbf{x}}{l_c}\right)\phi = 0}_{\text{random paraxial wave equation}}.$$



Considering only the paraxial wave equation

$$i\partial_z\phi + \frac{\lambda_0}{2kr_0^2}\Delta_{\mathbf{x}}\phi + \frac{k\sigma}{2\lambda_0}V\left(\frac{z}{l_c}, \frac{r_0\mathbf{x}}{l_c}\right)\phi = 0,$$

and forgetting for a moment about the Laplacian, we only have the following differential equation

$$\partial_z\phi = i\frac{k\sigma}{2\lambda_0}V\left(\frac{z}{l_c}, \frac{r_0\mathbf{x}}{l_c}\right)\phi,$$

with solution

$$\phi(z,\mathbf{x}) = e^{i\frac{k\sigma l_c}{2\lambda_0}\int_0^{z/l_c} V(u,\frac{r_0\mathbf{x}}{l_c})du}\phi(0,\mathbf{x}).$$

How to characterize the asymptotic behavior of the process

$$(z,\mathbf{x}) \mapsto \frac{\sigma l_c}{\lambda_0}\int_0^{z/l_c} V\left(u, \frac{r_0\mathbf{x}}{l_c}\right)du \quad ?$$

$$(z, \mathbf{x}) \mapsto \frac{\sigma l_c}{\lambda_0} \int_0^{z/l_c} V\left(u, \frac{r_0 \mathbf{x}}{l_c}\right) du$$

The scaling regime we consider is the following:

- $\lambda_0 = \varepsilon^2 \ll 1$ (high frequency regime),
- $r_0 = \varepsilon$ (Rayleigh length of order one),
- $l_c = \varepsilon$ (Strong lateral interaction between the wave and the medium),
- $\sigma = \varepsilon^s$.

We are therefore interested in the limiting behavior as $\varepsilon \to 0$ of the process

$$B^\varepsilon(z, x) = \varepsilon^{s-1} \int_0^{z/\varepsilon} V(u, \mathbf{x}) du.$$

Rapidly decaying correlations

Central limit theorem. Let $(X_n)_n$ be a sequence of mean zero iid random variables such that $v^2 = \mathbb{E}[|X_1|^2] < \infty$. Then,

$$N^{-1/2} \sum_{j=1}^{N} X_j \underset{N\to\infty}{\Longrightarrow} \mathcal{N}(0, v^2).$$

Here, $N^{-1/2}$ is the good scaling for the CLT since

$$var\Big(N^{-1/2} \sum_{j=1}^{N} X_j\Big) = N^{-1} \sum_{j=1}^{N} var(X_j) = v^2.$$

Rapidly decaying correlations

In our context, we have

$$\mathbb{E}[B^\varepsilon(z,\mathbf{x}+\mathbf{y})B^\varepsilon(z,\mathbf{y})] = \varepsilon^{2(s-1)} \int_0^{z/\varepsilon}\int_0^{z/\varepsilon} \mathbb{E}[V(u,\mathbf{x}+\mathbf{y})V(u',\mathbf{y})]du du'$$

$$\underset{\varepsilon\to 0}{\sim} \varepsilon^{2s-3} z \underbrace{\int_0^\infty R(u,\mathbf{x})du}_{<\infty}.$$

Therefore, the choice $s = 3/2$ leads to a nontrivial stochastic limit.

Approximation diffusion. Assuming mixing properties for the process $z \mapsto V(z,\cdot)$, we have

$$B^\varepsilon(z,x) = \varepsilon^{1/2}\int_0^{z/\varepsilon} V(u,\mathbf{x})du \underset{\varepsilon\to 0}{\Longrightarrow} B(z,\mathbf{x}),$$

where B is a Brownian field with covariance function

$$\mathbb{E}[B(z,\mathbf{x}+\mathbf{y})B(s,\mathbf{y})] = z\wedge s\int_0^\infty R(u,\mathbf{x})du.$$

See J.P. Fouque, J. Garnier, G. Papanicolaou, K. Sølna '07 for advance topics on approximation diffusion for ODEs with random coefficients.

Rapidly decaying correlations

Going back to our toy equation, under these mixing assumptions, the solution to

$$\partial_z \phi_\varepsilon = i\frac{k}{2\sqrt{\varepsilon}} V\left(\frac{z}{\varepsilon}, \mathbf{x}\right)\phi_\varepsilon$$

converges in distribution in $\mathcal{C}([0,\infty), L^2(\mathbb{R}^2))$:

$$\phi_\varepsilon(z,\mathbf{x}) = e^{i\frac{k}{2}\varepsilon^{1/2}\int_0^{z/\varepsilon} V(u,\mathbf{x})du}\phi(0,\mathbf{x}) \underset{\varepsilon\to 0}{\Longrightarrow} \phi_0(z,\mathbf{x}) := e^{i\frac{k}{2}B(z,\mathbf{x})}\phi(0,\mathbf{x}).$$

Using the **Itô's formula**, this limiting process satisfies

$$d\phi_0(z,\mathbf{x}) = i\frac{k}{2}\phi_0(z,\mathbf{x})\circ dB(z,\mathbf{x}),$$

with

$$\mathbb{E}[dB(z,\mathbf{x})dB(s,\mathbf{y})] = \delta(z-s)\int_0^\infty R(u,\mathbf{x}-\mathbf{y})du.$$

Considering a subdivision $0 = s_0 < s_1 < \cdots < s_n = z$, a Brownian motion B, and suitable assumptions on X:

The Itô integral

$$\int_0^z X(s)dB(s) \simeq \sum_{j=0}^{n-1} X(s_j)(B(s_{j+1}) - B(s_j)), \qquad \text{(Stieltjes sum)}$$

The Stratonovich integral

$$\int_0^z X(s) \circ dB(s) \simeq \sum_{j=0}^{n-1} \frac{X(s_{j+1}) + X(s_j)}{2}(B(s_{j+1}) - B(s_j)).$$

We also have

$$df(B(z)) = f'(B(z)) \circ dB(z).$$

Rapidly decaying correlations

For our toy equation

$$d\phi_0(z,\mathbf{x}) = i\frac{k}{2}\phi_0(z,\mathbf{x})\circ dB(z,\mathbf{x}),$$

we have the relation

$$\int_0^z \phi_0(s,\mathbf{x}) \circ dB(s,\mathbf{x}) = \int_0^z \phi_0(s,\mathbf{x}) dB(s,\mathbf{x}) - \frac{k^2 C(0)}{8} \int_0^z \phi_0(s,\mathbf{x}) ds,$$

with

$$C(0) = \int_0^\infty R(u,0)du.$$

Therefore, denoting $\mu(z,\mathbf{x}) = \mathbb{E}[\phi_0(z,\mathbf{x})]$, one has

$$d\mu(z,\mathbf{x}) = -\frac{k^2 C(0)}{8}\mu(z,\mathbf{x})dz,$$

so that

$$\mu(L,\mathbf{x}) = e^{-k^2 C(0) L/8}\phi_0(0,\mathbf{x}).$$

Rapidly decaying correlations

Under the paraxial scaling, the wave equation is

$$\Delta p - \frac{1}{c_0^2}\Big(1 + \varepsilon^{3/2} V\Big(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\Big)\Big)\partial_t^2 p = f\big(\frac{t}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\big)\delta'(z),$$

Theorem : J. Garnier and K. Sølna '09

Assuming the process $z \mapsto V(z, \cdot)$ is ϕ-mixing, we have

$$p(\varepsilon t + L/c_0, \varepsilon x, L) \underset{\varepsilon \to 0}{\Longrightarrow} \int e^{-i\omega t}\phi_0(\omega, L, \mathbf{x})d\omega,$$

in $L^2(\mathbb{R}^3) \cap \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^2))$, where ϕ_0 is the unique solution of the *Itô-Schrödinger equation*

$$i\, d\phi_0(z, \mathbf{x}) + \frac{1}{2k}\Delta_{\mathbf{x}}\phi_0(z, \mathbf{x}) + \frac{k}{2}\phi_0(z, \mathbf{x}) \circ dB(z, \mathbf{x}) = 0,$$

with $\phi_0(0, \mathbf{x}) = \check{f}(\omega, \mathbf{x})/2$ and

$$\mathbb{E}[dB(z, \mathbf{x})dB(s, \mathbf{y})] = \delta(z - s)\int_0^\infty R(u, \mathbf{x} - \mathbf{y})du.$$

See D.A. Dawson - G. Papanicolaou '84 for the well posedness of this SPDE.

Rapidly decaying correlations

As for the toy equation, denoting $\mu(z, \mathbf{x}) = \mathbb{E}[\phi_0(z, \mathbf{x})]$, one has

$$i\, d\mu(z, \mathbf{x}) + \frac{1}{2k}\Delta_{\mathbf{x}}\mu(z, \mathbf{x}) - \frac{k^2 C(0)}{8}\mu(z, \mathbf{x})dz = 0,$$

which can be explicitly solved :

$$\mu(L, \mathbf{x}) = e^{-k^2 C(0) L/8} e^{i\Delta_{\mathbf{x}} L/(2k)} \check{f}(\omega, \mathbf{x})/2.$$

- Study of higher order moments, spot dancing regime, scintillation.
- Application to imaging/inverse problem in random media.

(G. Bal, L. Borcea, J. Garnier, G. Papanicolaou, O. Pinaud, K. Sølna, C. Tsogka...)

Slowly decaying correlations

Slowly decaying correlations

In the previous Itô-Schrödinger equation we had

$$\mathbb{E}[dB(z,\mathbf{x}+\mathbf{y})dB(s,\mathbf{y})] = \delta(z-s)\underbrace{\int_0^\infty R(u,\mathbf{x})du}_{<\infty}.$$

- With the long-range correlations assumption we have

$$\mathbb{E}[V(z+s,\mathbf{x})V(s,\mathbf{y})] \underset{z\to\infty}{\sim} \frac{c}{z^{\mathfrak{H}}}R(\mathbf{x}-\mathbf{y}), \qquad \mathfrak{H}\in(0,1),$$

  and the above covariance function is not defined anymore.
- The "CLT scaling" $\sigma=\varepsilon^{3/2}$ is no longer the correct one.
- What is the good scaling? that is the value for s, and what would be the nature of the stochastic integral, that is the multiplicative noise?

Slowly decaying correlations

Let us start by considering again the toy equation

$$\partial_z \phi_\varepsilon = i\frac{k}{2}\varepsilon^{s-2} V\left(\frac{z}{\varepsilon}, \mathbf{x}\right)\phi_\varepsilon,$$

with solution

$$\phi_\varepsilon(z, \mathbf{x}) = e^{i\frac{k}{2}\varepsilon^{s-1}\int_0^{z/\varepsilon} V(u,\mathbf{x})du}\phi(0, \mathbf{x}).$$

We have

$$\begin{aligned}
\mathbb{E}[B^\varepsilon(z, \mathbf{x}+\mathbf{y})B^\varepsilon(z, \mathbf{y})] &= \varepsilon^{2(s-1)}\int_0^{z/\varepsilon}\int_0^{z/\varepsilon}\mathbb{E}[V(u, \mathbf{x}+\mathbf{y})V(u', \mathbf{y})]du du' \\
&\sim \varepsilon^{2(s-1)}\int_0^{z/\varepsilon}\int_0^{z/\varepsilon}|u-u'|^{-\mathfrak{H}} du' du R(\mathbf{x}) \\
&\sim \varepsilon^{2(s-2)+\mathfrak{H}} C_{\mathfrak{H}} z^{2H} R(\mathbf{x}),
\end{aligned}$$

with $H = 1 - \mathfrak{H}/2 \in (1/2, 1)$, and so that

$$s = 2 - \frac{\mathfrak{H}}{2} \in (3/2, 2).$$

$(\varepsilon^s \ll \varepsilon^{3/2})$

Slowly decaying correlations

Let Θ be a continuous odd function and $H_K(x) = (-1)^K e^{x^2/2} \frac{d^K}{dx^K} e^{-x^2/2}$ the K-th Hermite polynomial. For $X \sim \mathcal{N}(0,1)$

$$inf\,(K \geq 1 : \mathbb{E}[\Theta(X)H_K(X)] \neq 0)$$

is called the Hermite rank of Θ.

Noncentral limit theorem Let $\mathcal{V}$ a stationary Gaussian process satisfying

$$\mathbb{E}[\mathcal{V}(z+s)\mathcal{V}(s)] \underset{z\to\infty}{\sim} \frac{c}{z^\gamma}, \qquad \gamma \in (0, 1/K).$$

We have for $s = 2 - \gamma K/2$

$$\varepsilon^{s-1} \int_0^{z/\varepsilon} \Theta(\mathcal{V}(u))du \underset{\varepsilon\to 0}{\Longrightarrow} C_H\, B_H(z),$$

in $\mathcal{C}(0,\infty)$, where B_H stands for the K-th Hermite process with Hurst index $H = 1 - \gamma K/2 \in (1/2, 1)$: it has stationary increments and

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

If $K = 1$, B_H is a Gaussian process, and **if $K \geq 2$ B_H is non-Gaussian**.
See M. Taqqu '79, and R. Marty & K. Sølna '11, for application on wave propagation.

Slowly decaying correlations

We want to apply this previous theory to our toy equation

$$\partial_z \phi_\varepsilon = i\frac{k}{2}\varepsilon^{s-2} V\Big(\frac{z}{\varepsilon}, \mathbf{x}\Big)\phi_\varepsilon$$

Let Θ be a continuous odd function ($K = 1$), and $\mathcal{V}$ a stationary Gaussian field such that

$$\mathbb{E}[\mathcal{V}(z+s, \mathbf{x}+\mathbf{y})\mathcal{V}(s, \mathbf{y})] \sim \frac{c}{z^{\mathfrak{H}}} R(\mathbf{x}), \qquad \mathfrak{H} \in (0,1).$$

Setting $V(z, \mathbf{x}) = \Theta(\mathcal{V}(z, \mathbf{x}))$, we have

$$\phi_\varepsilon(z, \mathbf{x}) = e^{i\frac{k}{2}\varepsilon^{s-1}\int_0^{z/\varepsilon} V(u,\mathbf{x})du}\phi(0, \mathbf{x}) \underset{\varepsilon\to 0}{\Longrightarrow} \phi_0(z, \mathbf{x}) := e^{i\frac{k}{2}B_H(z,\mathbf{x})}\phi(0, \mathbf{x}),$$

in $\mathcal{C}([0,\infty), L^2(\mathbb{R}^2))$, where B_H stands for a fractional Brownian field with Hurst index $H \in (1/2, 1)$.
Question : Can we define a stochastic integral for which

$$d\phi_0(z, \mathbf{x}) = i\frac{k}{2}\phi_0(z, \mathbf{x})dB_H(z, \mathbf{x}) \qquad ?$$

and how to manage this integral, leading with the fractional noise, together with the paraxial approximation.

Slowly decaying correlations

The definition of the stochastic integral follows the definition of Zähle '98. Introducing the Weyl's derivative for $\alpha \in (0,1)$ and $z \in (0,L)$:

$$D_{0^+}^{\alpha} f(z) := \frac{1}{\Gamma(1-\alpha)}\Big[\frac{f(z)}{z^\alpha} + \alpha\int_0^z \frac{f(z)-f(u)}{(z-u)^{\alpha+1}}du\Big]$$
$$D_{L^-}^{\alpha} f(z) := \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\Big[\frac{f(z)}{(L-z)^\alpha} + \alpha\int_z^L \frac{f(z)-f(u)}{(u-z)^{\alpha+1}}du\Big],$$

the generalized Stieljes integral of $f \in \mathcal{C}^{\nu}(0,L)$ with respect to $g \in \mathcal{C}^{\mu}(0,L)$, with $\nu + \mu > 1$, $\nu > \alpha$, and $\mu > 1-\alpha$ is defined by

$$\int_0^L fdg := (-1)^{\alpha}\int_0^L D_{0^+}^{\alpha} f(u) D_{L^-}^{1-\alpha} g_{L^-}(u)du,$$

with $g_{L^-}(u) := g(u) - g(L^-)$, and

$$\int_0^z fdg := \int_0^L f 1_{(0,z)} dg.$$

Under proper conditions, we have

$$dF(g(t)) = F'(g(t))dg(t).$$

Slowly decaying correlations

For our purpose one can extend this integral to more general class of function

$$\Big|\int_0^L f dg\Big| \le C\|f\|_\alpha \Lambda_\alpha(g)$$

with

$$\|f\|_\alpha := \sup_{z\in(0,L)} \Big(|f(z)| + \int_0^z \frac{|f(z)-f(u)|}{(z-u)^{\alpha+1}} du\Big)$$

and

$$\Lambda_\alpha(g) := \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \sup_{0<u<t<L} |D_{t^-}^{1-\alpha} g_{t-}(u)|.$$

Using the Garsia-Rademich-Rumsey inequality, this integral is well defined for g being a fractional Brownian motion.

Slowly decaying correlations

Under the paraxial scaling, the wave equation is

$$\Delta p - \frac{1}{c_0^2}\Big(1 + \varepsilon^{2-\mathfrak{H}/2} V\Big(\frac{z}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\Big)\Big)\partial_t^2 p = f\big(\frac{t}{\varepsilon}, \frac{\mathbf{x}}{\varepsilon}\big)\delta'(z),$$

where V is a non-Gaussian random stationary field defined by

$$V(z, \mathbf{x}) = \int e^{i\mathbf{p}\cdot\mathbf{x}} \hat{V}(z, d\mathbf{p}), \qquad \text{with} \qquad \hat{V}(z, d\mathbf{p}) = m(d\mathbf{p})\Theta(\mathcal{V}(z, \mathbf{p}))$$

- m is a random measure with bounded total variation and

$$\mathbb{E}[m(d\mathbf{p})m(d\mathbf{q})] = \delta(\mathbf{p} - \mathbf{q})M(d\mathbf{p})d\mathbf{p}d\mathbf{q}$$

- $\mathcal{V}$ is a Gaussian field satisfying

$$\mathbb{E}[\mathcal{V}(z+s, \mathbf{p})\mathcal{V}(s, \mathbf{q})] \underset{z\to+\infty}{\sim} \frac{c}{z^{\mathfrak{H}}}\hat{R}(\mathbf{p}, \mathbf{q}), \qquad \mathfrak{H} \in (0, 1).$$

- Θ is a odd bounded continuous function ($K = 1$).

Slowly decaying correlations

We have the following result

Theorem : C.G and O. Pinaud '17

We have in $L^2(\mathbb{R}^3) \cap \mathcal{C}^0(\mathbb{R}, L^2(\mathbb{R}^2))$

$$p(\varepsilon t + L/c_0, \varepsilon x, L) \underset{\varepsilon \to 0}{\Longrightarrow} \int e^{-i\omega t} \phi_0(\omega, L, \mathbf{x}) d\omega,$$

where ϕ_0 is the unique pathwise solution of the *Itô-Schrödinger equation*

$$i\, d\phi_0(z, \mathbf{x}) + \frac{1}{2k} \Delta_{\mathbf{x}} \phi_0(z, \mathbf{x}) + \frac{k}{2} \phi_0(z, \mathbf{x}) dB_H(z, \mathbf{x}) = 0,$$

with $\phi_0(0, \mathbf{x}) = \check{f}(\omega, \mathbf{x})/2$. Here B_H is **non-Gaussian**:

$$B_H(z, \mathbf{x}) = \int e^{i\mathbf{p}\cdot\mathbf{x}} \hat{B}_H(z, d\mathbf{p}), \qquad \text{with} \qquad \hat{B}_H(z, d\mathbf{p}) = m(d\mathbf{p}) \underbrace{\mathcal{B}_H(z, \mathbf{p})}_{\text{Gaussian field}}$$

and

$$\mathbb{E}[B_H(t, \mathbf{x}) B_H(s, \mathbf{y})] = \frac{C_{\mathfrak{H}}}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H}) R(\mathbf{x} - \mathbf{y}),$$

or

$$\mathbb{E}[dB_H(t, \mathbf{x}) dB_H(s, \mathbf{y})] = \frac{C'_{\mathfrak{H}}}{2} \int \frac{e^{ir(t-s)}}{|r|^{2H-1}} dr\, R(\mathbf{x} - \mathbf{y}).$$

Slowly decaying correlations

- From the *Itô-Schrödinger equation* we cannot make computations as for the standard Brownian motion case!
- Removing the Laplacian term, we have

$$\phi_0(\omega, L, \mathbf{x}) = e^{i\frac{k}{2}B_H(L,\mathbf{x})}, \qquad k = \omega/c_0,$$

  and then

$$p(\varepsilon t + L/c_0, \varepsilon x, L) \underset{\varepsilon \to 0}{\Longrightarrow} \frac{1}{2} f\Big(t - \frac{B_H(L,\mathbf{x})}{2c_0}, \mathbf{x}\Big).$$

- If B_H is Gaussian, we have

$$\mathbb{E}[e^{i\frac{k}{2}B_H(L,\mathbf{x})}] = e^{-C_{\mathfrak{H}}k^2L^{2H}/8}.$$

- Nevertheless, the proof of the above theorem gives us explicit (but complicated) formulas for all the moments of ϕ_0.

Slowly decaying correlations

To study the (paraxial) wave propagation for $2-\mathfrak{H}/2 < s \leq 3/2$, we can make use of the Wigner transform:

$$W_\varepsilon(z,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int e^{i\mathbf{k}\cdot\mathbf{y}}\phi_\varepsilon\Big(z,\mathbf{x}-\varepsilon^c\frac{\mathbf{y}}{2},\xi\Big)\overline{\phi_\varepsilon\Big(z,\mathbf{x}+\varepsilon^c\frac{\mathbf{y}}{2}\Big)}d\mathbf{y}.$$

We have $\lim_\varepsilon W_\varepsilon(z,\mathbf{x},\mathbf{k}) = W(z,\mathbf{x},\mathbf{k})$ and depending on s

- $$W_1(z,\mathbf{x},\mathbf{k}) = \frac{1}{(2\pi)^d}\int d\mathbf{q}\widehat{W}_0^{\mathbf{k}}(\mathbf{x},\mathbf{q})\exp\Big(i\mathbf{k}\cdot\mathbf{q}+i\int \mathcal{B}_z(d\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}(e^{-i\mathbf{q}\cdot\mathbf{p}/2}-e^{i\mathbf{q}\cdot\mathbf{p}/2})\Big).$$
- $$\partial_z W_2 = -\sigma(\theta)(-\Delta_{\mathbf{k}})^{\theta/2}W_2,$$
  with $W_2 = \mathbb{E}[W_1]$.
- For $s = 3/2$
  $$\partial_z W_3 + \mathbf{k}\cdot\nabla_{\mathbf{x}} W_3 = \int d\mathbf{p}K(\mathbf{k},\mathbf{p})(W_3(z,\mathbf{x},\mathbf{p}) - W_3(z,\mathbf{x},\mathbf{k})).$$

C. Gomez, Wave decoherence for the random Schrödingier equation with long-range correlations, Commun. Math. Phys., (2013)

Ideas for the proof

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From the Helmholtz equation

$$\Big(\partial_z^2+\frac{1}{\varepsilon^2}\Delta_{\mathbf{x}}+\frac{k^2}{\varepsilon^4}\Big(1+\varepsilon^{2-\mathfrak{H}/2}V\Big(\frac{z}{\varepsilon},\mathbf{x}\Big)\Big)\Big)\check{p}=\check{f}(\omega,\mathbf{x})\delta'(z), \qquad \text{with} \qquad k=\omega/c_0,$$

going into the Fourier domain in $\mathbf{x}$, one has

$$\partial_z^2\hat{p}(z,\kappa)+\frac{k^2}{\varepsilon^4}\Big(1-\frac{\varepsilon^2|\kappa|^2}{k^2}\Big)\hat{p}(z,\kappa)+\frac{k^2}{\varepsilon^{2+\mathfrak{H}/2}}\int\hat{V}\Big(\frac{z}{\varepsilon},d\kappa'\Big)\hat{p}(z,\kappa-\kappa')=0$$

- If $1-\varepsilon^2|\kappa|^2/k^2>0$, κ is referred as a propagating mode.
- If $1-\varepsilon^2|\kappa|^2/k^2<0$, κ is referred as an evanescent mode.

Ideas for the proof

The main steps of the proof are :

- the contribution of the evanescent modes are negligible,
- for the propagating modes, considering the decomposition

$$\check{p}(\omega, \mathbf{x}, z) = \underbrace{a^{\varepsilon}(\omega, z, \mathbf{x})}_{\text{forward scattering}} e^{ikz/\varepsilon^2} + \underbrace{b^{\varepsilon}(\omega, z, \mathbf{x})}_{\text{backward scattering}} e^{-ikz/\varepsilon^2},$$

  we have

$$b^{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} 0$$

- and finally,

$$a^{\varepsilon}(\omega, L, x) \underset{\varepsilon \to 0}{\Longrightarrow} \phi_0(\omega, L, x) \qquad \text{in distribution.}$$

To understand the proof of the last point, let us consider one last time our toy equation

$$\partial_z \phi_\varepsilon = \frac{ik}{2\varepsilon^{2-s}} V\Big(\frac{z}{\varepsilon}\Big)\phi_\varepsilon,$$

that we rewrite as

$$\begin{aligned}\phi_\varepsilon(z) &= \phi(0) + \frac{ik}{2\varepsilon^{2-s}} \int_0^z V\Big(\frac{u}{\varepsilon}\Big)\phi_\varepsilon(u)du \\ &= \phi(0)\sum_{n\geq 0}\Big(\frac{ik}{2\varepsilon^{2-s}}\Big)^n \int_0^z du_1 \cdots \int_0^{u_{n-1}} du_n \prod_{j=1}^n V\Big(\frac{u_i}{\varepsilon}\Big).\end{aligned}$$

Considering the first order moment, one has

$$\mathbb{E}[\phi_\varepsilon(z)] = \phi(0)\sum_{n\geq 0}\Big(\frac{ik}{2\varepsilon^{2-s}}\Big)^n \int_0^z du_1 \cdots \int_0^{u_{n-1}} du_n \mathbb{E}\Big[\prod_{j=1}^n V\Big(\frac{u_i}{\varepsilon}\Big)\Big].$$

Using the strategy of M.S Taqqu '77, one can show that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^{(2-s)n}}\int_0^z du_1\cdots\int_0^{u_{n-1}} du_n\,\mathbb{E}\Big[\prod_{j=1}^n V\Big(\frac{u_i}{\varepsilon}\Big)\Big]$$

$$= c_{\mathfrak{H}}^n\int_0^z du_1\cdots\int_0^{u_{n-1}} du_n\sum_{\mathcal{F}}\prod_{(\alpha,\beta)\in\mathcal{F}}|u_\alpha-u_\beta|^{-\mathfrak{H}}$$

$$= C_{\mathfrak{H}}^n\int_0^z du_1\cdots\int_0^{u_{n-1}} du_n\lim_{A\to\infty}\mathbb{E}\Big[\prod_{j=1}^n\underbrace{\int_{-A}^{A}\frac{e^{iru_j}}{|r|^{H-1/2}}w(dr)}_{B'_{A,H}(u_j)}\Big],$$

with

$$B_{A,H}(z)=\int_{-A}^{A}\frac{e^{irz}-1}{ir|r|^{H-1/2}}w(dr),$$

and where w is a mean zero Gaussian random measure such that

$$\mathbb{E}[w(dr)w(ds)]=\delta(r+s)drds.$$

Ideas for the proof

As a result, one has

$$\mathbb{E}[\phi_\varepsilon(z)] \underset{\substack{\varepsilon\to 0\\ A\to\infty}}{\simeq} \phi(0)\sum_{n\geq 0}(ikC_{\mathfrak{H}}/2)^n \int_0^z du_1 \cdots \int_0^{u_{n-1}} du_n \mathbb{E}\Big[\prod_{j=1}^n B'_{A,H}(u_j)\Big]$$
$$\underset{\substack{\varepsilon\to 0\\ A\to\infty}}{\simeq} \mathbb{E}[\phi_A(z)].$$

where ϕ_A satisfies

$$\begin{aligned}\phi_A(z) &= \phi(0) + \frac{ikC_{\mathfrak{H}}}{2}\int_0^z \phi_A(u)B'_{A,H}(u)du\\ &= \phi(0) + \frac{ikC_{\mathfrak{H}}}{2}\int_0^z \phi_A(u)dB_{A,H}(u).\end{aligned}$$

Remembering that

$$B_{A,H}(u) = \int_{-A}^{A} dr \frac{e^{iru} - 1}{ir|r|^{H-1/2}} w(dr).$$

and that $B_{A,H} \to B_H$ (fractional Brownian motion with Hurst index H), one can show that

$$\phi_A \to \phi \qquad \text{in probability in } \mathcal{C}^{H-\theta}(0, L), \quad \forall \theta \in (0, H - 1/2),$$

where ϕ is the unique pathwise solution to

$$\phi(z) = \phi(0) + \frac{ikC_{\mathfrak{H}}}{2} \int_0^z \phi(u) dB_H(u).$$

Conclusion

- The paraxial and white-noise approximation (Itô-Schrödinger equation) is still valid for random medium with slowly decaying correlations, but the nature of the stochastic term is more complex.
- From the moment technique of the proof, one can compute moments of all order, but the formula is complex.
- The Itô-Schrödinger equation has been applied in imaging and inverse problems. Can we handle these imaging techniques in presence of random medium with slowly decaying correlations? knowing that martingale properties are not available.

Thank you!