Super-resolution and sensor calibration in imaging

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ICERM
September 25, 2017

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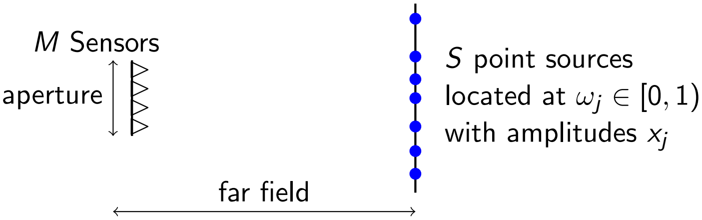
Outline

1. Super-resolution
    - Resolution in imaging
    - Super-resolution limit and min-max error
    - Super-resolution algorithms

2. Sensor calibration
    - Problem formulation
    - Uniqueness
    - An optimization approach
    - Numerical simulations

Source localization with sensor array



Point sources: $x(t) = \sum_{j=1}^{S} x_j \delta(t - \omega_j), \quad \omega_j \in [0, 1)$

Measurement at the mth sensor, $m = 0, \ldots, M-1$:

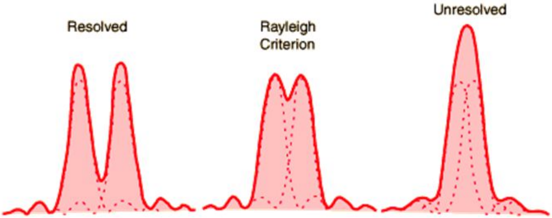
$$y_m = \sum_{j=1}^{S} x_j e^{-2\pi i m \omega_j} + e_m$$

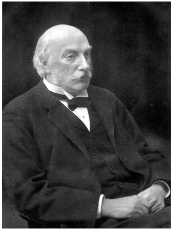
Measurements: $\{y_m : m = 0, \ldots, M-1\}$

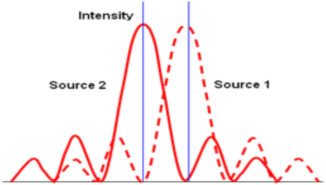
To recover: source locations $\{\omega_j\}_{j=1}^{S}$ and source amplitudes $\{x_j\}_{j=1}^{S}$.

Rayleigh criterion

$$\hat{x}(\omega) = \sum_{m=0}^{M-1} y_m \frac{e^{2\pi i m \omega}}{M}$$

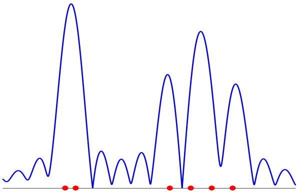




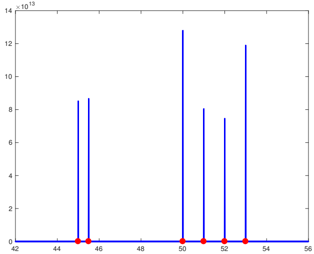


Rayleigh length $= 1/M$

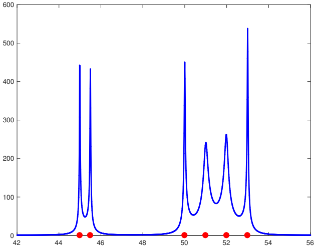
Inverse Fourier transform and the MUSIC algorithm



Multiple Signal Classification (MUSIC): [Schmidt 1981]



noise-free



noisy

Interesting questions

1. What is the super-resolution limit of the "best" algorithm?

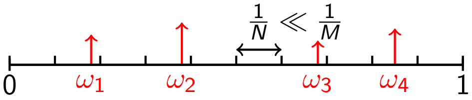
2. What is the super-resolution limit of a specific algorithm?

    - MUSIC [Schmidt 1981]
    - ESPRIT [Roy and Kailath 1989]
    - the matrix pencil method [Hua and Sarkar 1990]

Existing works

1. Super-resolution limit with continuous measurements
   - Donoho 1992, Demanet and Nguyen 2015
2. Performance guarantees for well separated point sources
   - Total variation minimization [Candès and Fernandez-Granda 2013,2014, Tang, Bhaskar, Shah and Recht 2013, Duval and Peyré 2015, Li 2017]
   - Greedy algorithms [Duarte and Baraniuk 2013, Fannjiang and L. 2012]
   - MUSIC [L. and Fannjiang 2016]
   - The matrix pencil method [Moitra 2015]
3. Performance guarantees for super-resolution
   - Total variation min for *positive* sources [Morgenshtern and Candès 2016] or sources with certain sign pattern [Benedetto and Li 2016]
   - Lasso for *positive* sources [Denoyelle, Duval and Peyré 2016]

Discretization on a fine grid



- Point sources: $\mu = \sum_{n=0}^{N-1} x_n \delta_{n/N}$ with $x \in \mathbb{C}_S^N$
- Measurement vector

$$y = \Phi x + e$$

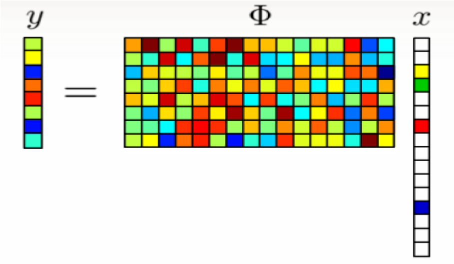
where $\Phi \in \mathbb{C}^{M \times N}$ is the first M rows of the $N \times N$ DFT matrix:

$$\Phi_{m,n} = e^{-2\pi i mn/N}$$

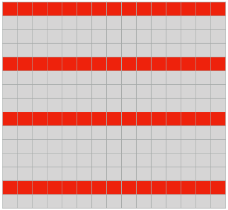
and $\|e\|_2 \leq \delta$.

Super-resolution factor (SRF) $:= \frac{N}{M}$

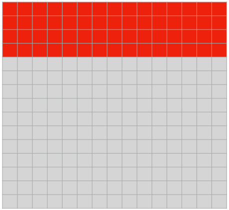
Connection to compressive sensing



Sensing matrices contain certain rows of the DFT matrix.



(a) compressive sensing



(b) super-resolution

Min-max error

Definition (S-min-max error)

Fix positive integers M, N, S such that $S \leq M \leq N$ and let $\delta > 0$. The S-min-max error is

$$\mathcal{E}(M,N,S,\delta) = \inf_{\substack{\tilde{x}=\tilde{x}(y,M,N,S,\delta)\in\mathbb{C}^N \\ y=\Phi x+e}} \sup_{x\in\mathbb{C}^N_S} \sup_{e\in\mathbb{C}^M:\|e\|_2\leq\delta} \|\tilde{x}-x\|_2.$$

Sharp bound on the min-max error

Theorem (Li and L. 2017)

There exist constants $A(S), B(S), C(S)$ such that:

1. *Lower bound. If $M \geq 2S$ and $N \geq C(2S)M^{3/2}$, then*

$$\mathcal{E}(M,N,S,\delta) \geq \frac{\delta}{2B(2S)\sqrt{M}} \mathrm{SRF}^{2S-1}.$$

2. *Upper bound. If $M \geq 4S(2S+1)$ and $N \geq M^2/(2S^2)$, then*

$$\mathcal{E}(M,N,S,\delta) \leq \frac{2\delta}{A(2S)\sqrt{M}} \mathrm{SRF}^{2S-1}.$$

The best algorithm in the upper bound:

$$\min \|z\|_0 \quad \text{subject to} \|\Phi z - y\|_2 \leq \delta$$

W. Li and W. Liao, "Stable super-resolution limit and smallest singular value of restricted Fourier matrices," preprint, arXiv:1709.03146.

Multiple Signal Classification (MUSIC)

- **Pioneering work:** Prony 1795
- **MUSIC in signal processing:** Schmidt 1981
- **MUSIC in imaging:** Devaney 2000, Devaney, Marengo and Gruber 2005, Cheney 2001, Kirsch 2002
- **Related:** the linear sampling method [Cakoni, Colton and Monk 2011], factorization method [Kirsch and Grinsberg 2008]

MUSIC

Assumption: S is known.

$$y_m = \sum_{j=1}^{S} x_j e^{-2\pi i m \omega_j}, \; m = 0, \ldots, M-1.$$

$$H = \mathrm{Hankel}(y) = \begin{bmatrix} y_0 & y_1 & \cdots & y_{M-L} \\ y_1 & y_2 & \cdots & y_{M-L+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_{L-1} & y_L & \cdots & y_{M-1} \end{bmatrix} = \underbrace{\Phi^L}_{L\times S} \underbrace{X}_{S\times S} \underbrace{(\Phi^{M-L+1})^T}_{S\times(M-L+1)}$$

where

$$X = \mathrm{diag}(x_1, \ldots, x_S)$$

$$\phi^L(\omega) = \begin{bmatrix} 1 & e^{-2\pi i \omega} & \ldots & e^{-2\pi i (L-1)\omega} \end{bmatrix}^T \in \mathbb{C}^L$$

$$\Phi^L = [\phi^L(\omega_1) \; \ldots \; \phi^L(\omega_S)] \in \mathbb{C}^{L\times S}.$$

MUSIC with noiseless measurements

$$H = \Phi^L X (\Phi^{M-L+1})^T$$

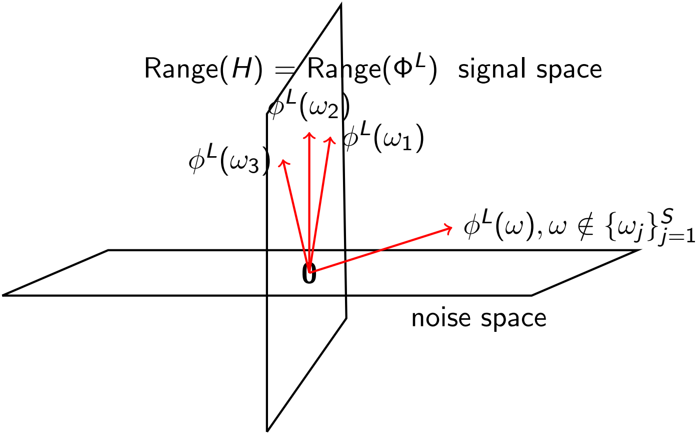
Suppose $\{\omega_j\}_{j=1}^S$ are distinct.

1. If $L \geq S$, $\mathrm{rank}(\Phi^L) = S$.
2. If $M - L + 1 \geq S$, $\mathrm{Range}(H) = \mathrm{Range}(\Phi^L)$.
3. If $L \geq S + 1$, $\mathrm{rank}([\Phi^L \;\; \phi^L(\omega)]) = S + 1$ if and only if $\omega \notin \{\omega_j\}_{j=1}^S$.

Theorem

If $L \geq S + 1$ and $M - L + 1 \geq S$, $\omega \in \{\omega_j\}_{j=1}^S$ iff $\phi^L(\omega) \in \mathrm{Range}(H)$.

Exact recovery with $M \geq 2S$ regardless of the support .

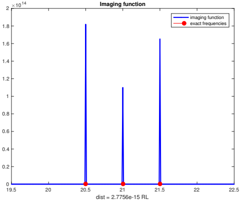


- Noise-space correlation function: $\mathcal{N}(\omega) = \frac{\|\mathcal{P}_{\text{noise}}\phi^L(\omega)\|_2}{\|\phi^L(\omega)\|_2}$
- Imaging function: $\mathcal{J}(\omega) = \frac{1}{\mathcal{N}(\omega)}$

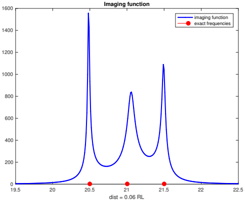
$$\mathcal{N}(\omega_j) = 0 \text{ and } \mathcal{J}(\omega_j) = \infty,\ j = 1, \ldots, S.$$

MUSIC with noisy measurements

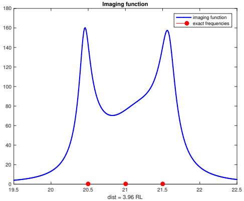
Three sources separated by 0.5 RL, $e \sim N(0, \sigma^2 I_M)$



(c) $\sigma = 0$



(d) $\sigma = 0.001$



(e) $\sigma = 0.01$

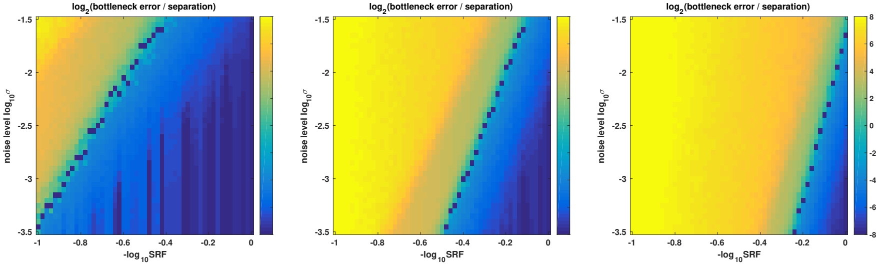
Recall upper bound of the min-max error

$$\mathcal{E}(M, N, S, \delta) \lesssim \frac{\delta}{\sqrt{M}} \mathrm{SRF}^{2S-1}$$

The noise that the “best” algorithm can handle is $\delta \sim \left(\frac{1}{\mathrm{SRF}}\right)^{2S-1}$.

Phase transition

- S consecutive point sources on the grid with spacing $1/N$
- Support error: $d(\{\omega_j\}_{j=1}^S, \{\hat{\omega}_j\}_{j=1}^S)$
- Noise $e \sim N(0, \sigma^2 I_M) + i \cdot N(0, \sigma^2 I_M)$, so $\mathbb{E}\|e\|_2 = \sqrt{2M}\sigma$.



$S = 2$ $\qquad$ $S = 3$ $\qquad$ $S = 4$

Figure: The average $\log_2[\frac{\text{Support error}}{1/N}]$ over 100 trials with respect to $\log_{10} \frac{1}{\text{SRF}}$ (x-axis) and $\log_{10} \sigma$ (y-axis).

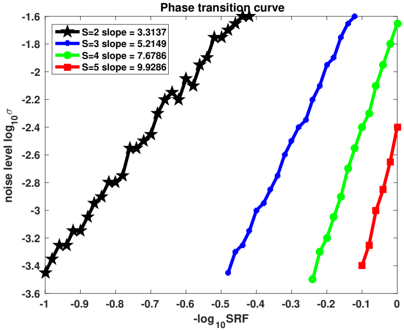
Super-resolution limit of MUSIC

The phase transition curve is

$$\sigma \sim \left(\frac{1}{\mathrm{SRF}}\right)^{p(S)}$$

where

$$2S - 1 \leq p(S) \leq 2S.$$



Future work:

$$\text{Support error by MUSC} \lesssim \mathrm{SRF}^{p(S)} \cdot \sigma.$$

Outline

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2. Sensor calibration
   - Problem formulation
   - Uniqueness
   - An optimization approach
   - Numerical simulations

Sensor calibration

Measurement at the m-th sensor, $m = 0, \ldots, M-1$:

$$y_m(t) = g_m \sum_{j=1}^{S} x_j(t) e^{-2\pi i m \omega_j} + e_m(t)$$

Multiple snapshots of measurements:

$$\{y_m(t), m = 0, \ldots, M-1, \ t \in \Gamma\}$$

To recover:

- Calibration parameters $g = \{g_m\}_{m=0}^{M-1} \in \mathbb{C}^M$
- Source locations $\{\omega_j\}_{j=1}^{S}$ and source amplitudes $x_j(t)$

Assumptions

Matrix form:

$$\underbrace{y(t)}_{\mathbb{C}^M} = \underbrace{\mathrm{diag}(g)}_{\mathbb{C}^{M\times M}} \underbrace{A}_{\mathbb{C}^{M\times S}} \underbrace{x(t)}_{\mathbb{C}^S} + \underbrace{e(t)}_{\mathbb{C}^M}$$

$$A_{n,j} = e^{-2\pi i m\omega_j}$$

$x(t) = [x_1(t) \ \ldots x_S(t)]^T, y(t) = [y_0(t) \ldots y_{M-1}(t)]^T, e(t) = [e_0(t) \ldots e_{M-1}(t)]^T$

Assumptions:

1. $|g_m| \neq 0, \ m = 0, \ldots, M-1;$
2. $\mathbb{E}x(t) = 0$ and $\mathbb{E}e(t) = 0;$
3. $R^x := \mathbb{E}x(t)x^*(t) = \mathrm{diag}(\{\gamma_j^2\}_{j=1}^S);$
4. $\mathbb{E}x(t)e^*(t) = 0;$
5. $\mathbb{E}e(t)e^*(t) = \sigma^2 I_M$ where σ represents noise level.

Uniqueness up to a trivial ambiguity

Trivial ambiguity: $\{\tilde{g}, \{\tilde{\omega}_j\}_{j=1}^S, \tilde{x}(t)\}$ is called equivalent to $\{g, \{\omega_j\}_{j=1}^S, x(t)\}$ up to a trivial ambiguity if there exist $c_0 > 0, c_1, c_2 \in \mathbb{R}$:

$$\begin{aligned} \tilde{g} &= \{\tilde{g}_m = c_0 e^{i(c_1 + mc_2)} g_m\}_{m=0}^{M-1} \\ \tilde{\mathcal{S}} &= \{\tilde{\omega}_j : \tilde{\omega}_j = \omega_j - c_2/(2\pi)\}_{j=1}^S \\ \tilde{x}(t) &= x(t) c_0^{-1} e^{-ic_1}. \end{aligned}$$

Uniqueness with infinite snapshots of noiseless measurements:

Let $f_m = \sum_{j=1}^s \gamma_j^2 e^{2\pi i m \omega_j}, \; m = 0, \ldots, M-1.$

Theorem

Suppose $|f_1| > 0$ and $M \geq S+1$. Let $\{g, \{\omega_j\}_{j=1}^S, x(t)\}$ be a solution to the calibration problem. If there is another solution $\{\tilde{g}, \{\tilde{\omega}_j\}_{j=1}^S, \tilde{x}(t)\}$, then $\{\tilde{g}, \{\tilde{\omega}_j\}_{j=1}^S, \tilde{x}(t)\}$ is equivalent to $\{g, \{\omega_j\}_{j=1}^S, x(t)\}$.

Covariance matrix

Pioneering work: Full algebraic method [Paulraj and Kailath, 1985], Partial algebraic method [Wylie, Roy and Schmitt, 1993]

$$R^y := \mathbb{E}y(t)y^*(t) = \mathrm{diag}(g)AR^xA^*\mathrm{diag}(\bar{g})$$

$$\mathcal{H} : \mathbb{C}^M \to \mathbb{C}^{M\times M} : \ \mathcal{H}(f) := \begin{bmatrix} f_0 & \bar{f}_1 & \ddots & \bar{f}_{N-1} \\ f_1 & f_0 & \ddots & \bar{f}_{N-2} \\ \ddots & \ddots & \ddots & \ddots \\ f_{N-1} & f_{N-2} & \ddots & f_0 \end{bmatrix} = AR^xA^*. \text{ Then}$$

$$R^y = \mathrm{diag}(g)\mathcal{H}(f)\mathrm{diag}(\bar{g})$$
$$R^y_{m,n} = g_m\bar{g}_n f_{m-n}$$

When $f_1 \neq 0$, the diagonal and subdiagonal entries in R^y determine the solution up to a trivial ambiguity.

Algebraic methods

Sensitivity of the partial algebraic method:

- $N \geq s+1$, $|f_1| > 0$ and sources are separated by $1/M$.
- Empirical covariance matrix is computed with L snapshots of measurements.

We proved that,

$$\mathbb{E} \min_{c_0>0, c_1, c_2 \in \mathbb{R}} \max_m |c_0 \widehat{g}_m - e^{i(c_1+mc_2)} g_m| \leq O\left(\frac{\max(\sigma, \sigma^2)}{\sqrt{L}}\right),$$

Partial algebraic method: only diagonal and subdiagonal entries in the covariance matrix are used

Full algebraic method: problem of phase wrapping

An optimization approach

$$R^y = GAR^x A^* G^* = \mathrm{diag}(g)\mathcal{H}(f)\mathrm{diag}(\bar{g})$$

Optimization problem:

$$\min_{\mathbf{g},\mathbf{f}\in\mathbb{C}^M} \mathcal{L}(\mathbf{g},\mathbf{f}) := \left\| \mathrm{diag}(\mathbf{g})\mathcal{H}(\mathbf{f})\mathrm{diag}(\bar{\mathbf{g}}) - \widehat{R}^y \right\|_F^2 .$$

- If $\widehat{R}^y = R^y$, the global minimizer of $\mathcal{L}$ is equivalent to the ground truth (g, f).

Regularized optimization

Goal: prevent $\|\mathbf{g}\| \to \infty$ and $\|\mathbf{f}\| \to 0$ (or vice versa)
$\widehat{n}_0$ is an estimator of $n_0 := \|g\|^2 \|f\|$ from the partial algebraic method.

Regularized optimization:

$$\min_{\mathbf{g},\mathbf{f} \in \mathbb{C}^N} \tilde{\mathcal{L}}(\mathbf{g},\mathbf{f}) := \mathcal{L}(\mathbf{g},\mathbf{f}) + \mathcal{G}(\mathbf{g},\mathbf{f})$$

$$\mathcal{G}(\mathbf{g},\mathbf{f}) = \rho \left[\mathcal{G}_0\left(\frac{\|\mathbf{f}\|^2}{2\widehat{n}_0} \right) + \mathcal{G}_0\left(\frac{\|\mathbf{g}\|^2}{\sqrt{2\widehat{n}_0}} \right) \right]$$

where $\mathcal{G}_0(z) = (\max(z-1,0))^2$ and $\rho \geq \frac{3\widehat{n}_0 + \|R^y - \widehat{R}^y\|_F}{(\sqrt{2}-1)^2}$

Initialization: $(\mathbf{g}^0, \mathbf{f}^0) : \|\mathbf{g}^0\|^2 \leq \sqrt{2\widehat{n}_0}, \|\mathbf{f}^0\| \leq \sqrt{2\widehat{n}_0}$

Feasible set: $\mathcal{N}_{\widehat{n}_0} = \{(\mathbf{g},\mathbf{f}) : \|\mathbf{g}\|^2 \leq 2\sqrt{\widehat{n}_0}, \|\mathbf{f}\| \leq 2\sqrt{\widehat{n}_0}\}$

Wirtinger gradient descent

for $k = 1, 2, \dots,$

- $\mathbf{g}^k = \mathbf{g}^{k-1} - \eta^k \nabla_{\mathbf{g}} \tilde{\mathcal{L}}(\mathbf{g}^{k-1}, \mathbf{f}^{k-1})$
- $\mathbf{f}^k = \mathbf{f}^{k-1} - \eta^k \nabla_{\mathbf{f}} \tilde{\mathcal{L}}(\mathbf{g}^{k-1}, \mathbf{f}^{k-1})$

end

Theorem (Eldar, L. and Tang)

If the step length is chosen such that

$$\eta^k \le \frac{2}{146\widehat{n}_0 \max(\sqrt{\widehat{n}_0}, \sqrt[4]{\widehat{n}_0}) + 8\widehat{n}_0 + 16 \max(\sqrt{\widehat{n}_0}, \sqrt[4]{\widehat{n}_0}) \|R^y - \widehat{R}^y\|_F + \frac{8\rho}{\min(\widehat{n}_0, \sqrt{\widehat{n}_0})}},$$

then Wirtinger gradient descent gives rise to $(\mathbf{g}^k, \mathbf{f}^k) \in \mathcal{N}_{\widehat{n}_0}$, *and*

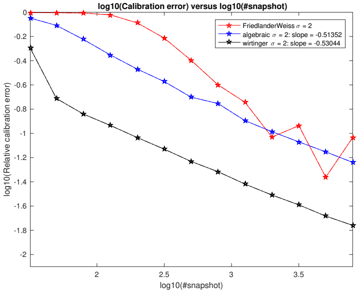
$$\|\nabla \tilde{\mathcal{L}}(\mathbf{g}^k, \mathbf{f}^k)\| \to 0, \text{ as } k \to \infty.$$

Y. C. Eldar, W. Liao and S. Tang, "Sensor calibration for off-the-grid spectral estimation," preprint, arXiv: 1707.03378.

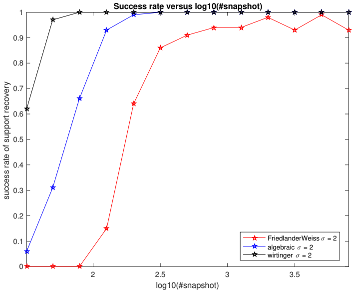
Sensitivity to the number of snapshots

1. the partial algebraic method
2. our optimization approach
3. an alternating minimization: [Friedlander and Weiss 1990]

- 20 sources separated by $2/M$ and noise level $\sigma = 2$



Relative calibration error versus L

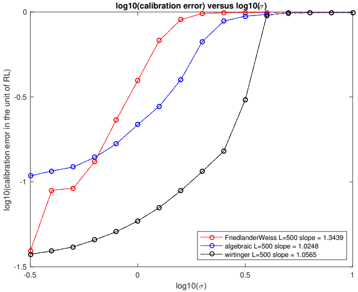


Support success rate versus L

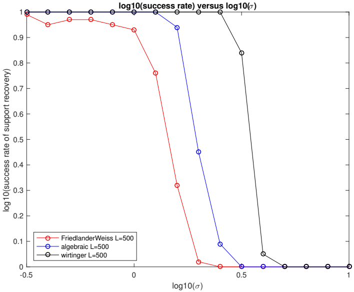
Observation: Calibration error $= O(L^{-\frac{1}{2}})$

Sensitivity to noise level σ

- 20 sources separated by $2/M$ and $L = 500$



Relative calibration error versus σ



Support success rate versus σ

Observation: Calibration error $= O(\sigma)$

Conclusion

1. Super-resolution
   - Resolution limit and a sharp bound on the min-max error
   - Resolution limit of the MUSIC algorithm
2. Sensor calibration
   - Uniqueness with infinite snapshots of noiseless data
   - The partial algebraic method and a stability analysis
   - An optimization approach and convergence to a stationary point

Thank you for your attention!

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