Mathematical questions raised by the non-uniform Doppler effect

John E. Gray

Electromagnetic and Sensor Systems Department,
Naval Surface Warfare Center Dahlgren,
18444 FRONTAGE ROAD SUITE 328,
DAHLGREN VA 22448-5161

ICERM **Mathematical and Computational Aspects of Radar Imaging**

(October 16-20, 2017)

Description: *This workshop will bring together mathematicians and radar practitioners to address a variety of issues at the forefront of mathematical and computational research in radar imaging. Some of the topics planned include shadow analysis and exploitation, interferometry, polarimetry, micro-Doppler analysis, through-the-wall imaging, noise radar, compressive sensing, inverse synthetic-aperture radar, moving target identification, quantum radar, multi-sensor radar systems, waveform design, synthetic-aperture radiometry, passive sensing, tracking, automatic target recognition, over-the-horizon radar, ground-penetrating radar, and Fourier integral operators in radar imaging.*

Abstract

The non-uniform motion Doppler effect in radar occurs when an object being tracked by a radar is undergoing any type of motion that is other than constant speed. Examples include: accelerations, jerk motion, exponential slowdown, and periodic motion such as rotation, vibration, or what is now termed micro-Doppler. I review a physics model based on a perfectly reflecting mirror which captures the essential features of the physics of non-uniform laws of motion. Then, we discuss the frequency spectrum of these types of motion as well as raise some questions for signal analysis for this type of physics. Finally, we pose some interesting questions that might be of interest to mathematicians concerning direct and inverse problems associated with observing non-uniform motion data.

NAVSEA WARFARE CENTERS
DAHLGREN

INTRODUCTION: NON-UNIFORM MOTION DOPPLER AND ONE DIMENSIONAL SIGNALS

- Since the inception of coherent waveforms, it has been realized that the effect of the motion of a non-point like object can induce structure in the return spectrum of the waveform, see for example in the electromagnetic literature.

- The explosion of interest since Chen's seminal papers has been centered on micro-Doppler, which is based on periodic motion about the central Doppler line, which have proven to have enormous applications (see Chen's recent books).

- While not as extensive in terms of potential applications, the non-uniform Doppler has applications to automobile radars, astrophysics, and moving sources, as well as other applications.

- We review the non-uniform Doppler and demonstrate some useful information that can be found in the spreading of the Doppler spectrum for the motion models: acceleration, jerk, quadric, and exponential slowdown as examples well as a characteristic of periodic motion.

INTRODUCTION: NON-UNIFORM MOTION DOPPLER AND ONE DIMENSIONAL SIGNALS

- The laws of motion for non-uniform behavior can be represented functionally by equations that are non-periodic that can be represented by Taylor series

$$r_n(t) = \pm v_R t + \sum_{i=1}^{m} (\alpha t)^{i+1} \ .$$

- The laws of motion for non-uniform behavior can be represented functionally by the equation that are

$$r_p(t) = \pm v_R t + \sum_{i=1}^{m} b_i \cos(\omega_i t)$$

for periodic motion. The periodic motion produces a modulation term in the FM portion of the signal.

- Assume that $r(t)$ represents the law of motion of a perfectly reflecting electromagnetic boundary, what is a referred to as a mirror in the optical domain; and let the return signal be $g(\tau)$ which has interacted with a moving boundary.

INTRODUCTION: NON-UNIFORM MOTION DOPPLER AND ONE DIMENSIONAL SIGNALS

- If we assume that r(t) represents the law of motion of a perfectly reflecting electromagnetic boundary, what is a referred to as a mirror in the optical domain; then the return signal, $g(\tau)$, from the boundary can be shown to be

$$g(\tau) = -\frac{dF(2h(\tau) - \tau)}{d\tau},$$

where $h(\tau)$ is the solution to the functional equation:

$$h(\tau) + \frac{r(h(\tau))}{c} = \tau.$$

- Note the broadcast waveform's functional form is $f(\tau)$ and $F(\tau) = \int^{\tau} f(x)dx$.

- Note, the functional equation for $h(\tau)$ can be solved sometimes for some types of boundaries undergoing non-uniform motion, but for most radar applications it is sufficient to use the radar approximation: $h_1(\tau) = \tau - \frac{r(\tau)}{c}$, so the return signal can be represented as $g_1(\tau) = -\frac{d}{d\tau} F_1\left(\tau - \frac{2r(\tau)}{c}\right)$.

INTRODUCTION: NON-UNIFORM MOTION DOPPLER AND ONE DIMENSIONAL SIGNALS

- Note, the functional equation for $h(\tau)$ can be solved sometimes for some types of boundaries undergoing non-uniform motion, but for most radar applications it is sufficient to use the radar approximation:

$$h_1(\tau) = \tau - \frac{r(\tau)}{c},$$

- The return signal can be represented as

$$g_1(\tau) = -\frac{d}{d\tau} F_1\left(\tau - \frac{2r(\tau)}{c}\right).$$

- For all our subsequent work, this is sufficient for analyzing the return signal's spectrum.

Observables in the Doppler Spectrum for CW Waveform

- Recall the CW spectrum for a law of motion $r(\tau)$ is

$$g_1(\tau) = A \exp\big(i\omega_0\tau - ikr(\tau)\big)\,[\Theta(t) - \Theta(t-T)]$$

where $k = \frac{2\omega_0}{c}$.

- From the definition of instantaneous frequency $\varphi_i(t)$ is for this spectral function $\varphi_i(\tau) = \omega_0 + kr'(\tau)$, so $\langle\omega\rangle$, $\langle\omega^2\rangle$, σ_ω^2 and therefore
- $\langle\omega\rangle = \int_{-\infty}^{\infty} g^*{}_1(\tau)\varphi_i(\tau)g_1(\tau) = A^2\omega_0 \int_0^{\mathrm{T}} \left(1 + \frac{2r\prime(\tau)}{c}\right) d\tau.$

- The standard deviation in the frequency is

$$\sigma_\omega^2 = \int_0^{\mathrm{T}} \left(A^2\omega_0\left(1 + \frac{2r'(\tau)}{c}\right) - \langle\omega\rangle\right)^2 d\tau\,,$$

because $A^2(\tau) = A^2$.

- The examples of different motion models in the Doppler Section can be easily computed in the τ domain.

Observables in the Doppler Spectrum for CW Waveform

- All of the subsequent integrals $G_1^{r(\tau)}(\omega)$ we consider can be written in the form

$$G_1^{r(\tau)}(\omega) = \mathrm{A}' \int_0^T e^{-i\omega'\tau} \exp\big(-ikr(\tau)\big)\, d\tau = G^{r(\tau)}(\omega) * P_T(\omega')$$

  where

$$G^{r(\tau)}(\omega) = A' \int_{-\infty}^{\infty} e^{-i\omega'\tau} \exp\big(-ikr(\tau)\big)\, d\tau,$$

  and

$$P_T(\omega') = \int_{-\infty}^{\infty} [\Theta(t) - \Theta(t-T)]\, e^{-i\omega'\tau} d\tau = \frac{\exp(-i\omega'\tau)}{i\omega'}\bigg|_0^T = \frac{T}{2} exp\left[\frac{iT\omega\prime}{2}\right] sinc\left[\frac{T\omega\prime}{2}\right].$$

- Thus, for some of the integrals we evaluate the integral explicitly, while other times we determine $G^{r(\tau)}(\omega)$ only.

- Note all of the boundaries that we present in subsequent examples do not include a velocity term.

- If one wants to include the velocity all that is needed in the subsequent examples is to reinterpret ω' as $\pm k v_o - \omega' = \omega'$ in subsequent equations.

Examples (Constant Acceleration)

- For a constant acceleration or CA-boundary, the law of motion is $r(\tau) = -\frac{1}{2}a_0\tau^2$, where a_0 is the acceleration of the boundary.

- Then, the Doppler spectrum is $k = \frac{a_0\omega_0}{c}$

$$G^{CA}{}_1(\omega) = A'\int_0^T e^{-i\omega'\tau}\exp(ik'\tau^2)\,d\tau = A'exp\left(-i\left\{\frac{\omega'}{2\sqrt{k'}}\right\}^2\right)\int_0^T \exp\left(i\left[\sqrt{k'}\tau - \frac{\omega'}{2\sqrt{k'}}\right]^2\right)d\tau,$$

- Let $\sqrt{k'}\tau - \frac{\omega'}{2\sqrt{k'}} = y,\ dy = \sqrt{k'}\tau,\ \ A'' = \frac{A'exp\left(-i\left\{\frac{\omega'}{2\sqrt{k'}}\right\}^2\right)}{k}$, then

$$G^{CA}{}_1(\omega) = A''\int_{-\frac{\omega'}{2\sqrt{k'}}}^{\sqrt{k'}T-\frac{\omega'}{2\sqrt{k'}}} \exp(iy^2)\,dy = A''\left[K\left(\frac{\omega'}{2\sqrt{k'}}\right) - K\left(\sqrt{k'}T - \frac{\omega'}{2\sqrt{k'}}\right)\right]$$

where K is the Fresnel integral.

$$\langle\omega\rangle_c = A^2\omega_0\int_0^{\mathrm{T}}\left(\pm\frac{2a_0\tau}{c}\right)d\tau = \pm A^2\omega_0 T^2\left(\frac{a_0}{c}\right)$$

and $\sigma_\omega^2 = (A^2\omega_0)^2\left(\frac{2a_0}{c}\right)^2\int_0^{\mathrm{T}}\left(\tau - \frac{T^2}{2}\right)^2\ d\tau \approx \left(\frac{2A^2\omega_0 a_0}{c}\right)^2\frac{T^3}{3}$.

Examples (Constant Jerk)

- For a law of motion $r(\tau) = \frac{1}{6} j_0 \tau^3$, where j_0 is the "constant jerk" of the boundary. Then, the spectrum given by the expression

$$G^{CJ}{}_1(\omega) = A' \int_0^T e^{-i\omega'\tau} \exp\left(-i \frac{kj_0}{6} \tau^3\right) d\tau \frac{A'}{\left(\frac{kj_0}{2}\right)^{\frac{1}{3}}} \int_0^{\left(\frac{kj_0 T^3}{2}\right)^{\frac{1}{3}}} e^{-i\omega'\left(\frac{kj_0}{2}\right)^{\frac{1}{3}} y} \exp\left(-\frac{iy^3}{3}\right) dy.$$

- Now with the definition of the Airy function: $Ai(x) = \int_{-\infty}^{\infty} \exp\left(-i\left(\frac{y^3}{3} + xy\right)\right) dy$, with, $x = \omega' \left(\frac{kj_0}{2}\right)^{\frac{1}{3}}$, we have $\left(\left(\frac{kj_0 T^3}{16}\right)^{\frac{1}{3}} \Big/ \left(\frac{kj_0}{2}\right)^{\frac{1}{3}} \right) = \frac{T}{2}$

$$G^{CJ}{}_1(\omega) = -\frac{A'T}{2} Ai\left(\omega' \left(\frac{kj_0}{2}\right)^{\frac{1}{3}}\right) * exp\left[i\left(\frac{kj_0 T^3}{16}\right)^{\frac{1}{3}} \omega'\right] sinc\left[i\left(\frac{kj_0 T^3}{16}\right)^{\frac{1}{3}} \omega'\right]$$

- Similar expressions occur in diffraction theory for circular apertures.

Examples (Constant Jerk)

- For the jerk law of motion, the central frequency is

$$\langle\omega\rangle_c = A^2\omega_0 \int_0^{\mathrm{T}} \left(\frac{j_0\tau^2}{c}\right) d\tau = \left(\frac{A^2 j_0 \omega_0 T^3}{3c}\right),$$

  and the central bandwidth is

$$\sigma_{\omega_c}^2 = \left(\frac{A^2\omega_0 j_0}{2c}\right)^2 \int_0^{\mathrm{T}} \left(\tau^2 - \left(\frac{T^3}{3}\right)\right)^2 d\tau \approx \left(\frac{2A^2\omega_0 j_0}{c}\right)^2 \frac{T^5}{5}.$$

- In order for the equivalent jerk bandwidth to be greater than or equal to an acceleration a_0 over the time period T, the jerk must obey the condition that $j_0 \geq \sqrt{\frac{20}{3}\frac{a_0}{T}}$. Unless this condition is meet, one cannot distinguish a jerk model from an acceleration model.

- Distinguishing between a jerk type motion and an acceleration requires larger sampling times, so there is going to a balancing act between noise and sampling time if this is a desired estimator.

Exponential Slowdown

- For the exponential law of motion $r(\tau) = Ae^{-\gamma\tau}$, the spectrum can be expressed in terms of Pearson incomplete gamma functions.

- Note $r'(\tau) = -\gamma A e^{-\gamma\tau}$, so the central frequency is

$$\langle\omega\rangle_c = -A^2\omega_0 \int_0^{\mathrm{T}} \left(\frac{2\gamma e^{-\gamma\tau}}{c}\right) d\tau = \left(\frac{2A^2\omega_0}{c}\right)[e^{-\gamma T} - 1],$$

  and the central bandwidth is

$$\sigma_{\omega_c}^2 = \left(\frac{2A^2\omega_0}{c}\right)^2 \int_0^{\mathrm{T}} (\gamma e^{-\gamma\tau} - [e^{-\gamma T} - 1])^2 \, d\tau \approx \left(\frac{2A^2\omega_0}{c}\right)^2 (\gamma^2 T + \cdots)$$

- This expression gives a straightforward way to estimate the ballistic slowdown coefficient γ since we can measure σ_ω directly from the frequency spectrum, and therefore we have

$$\frac{\sigma_\omega}{\left(\frac{2A^2\omega_0}{c}\right)\sqrt{T}} \approx \gamma.$$

- Thus the slowdown factor γ can be estimated in terms of parameters we know or directly measure. For example, one can use an estimate using radio based radars to predict a circular error probability (CEP) for the area a meteorite might lie within.

Non-Uniform Motion Doppler (CW Waveform) Other Polynomial Laws

- For a law of motion $r(\tau) = -\frac{f_0\tau^4}{4!}$, it is also possible to determine the spectrum. For polynomials higher than this, expressions in terms of known cataloged functions become problematic.

- However, it is possible to use *the principle of stationary phase* to evaluate the integrals approximately. Recall the expression for an arbitrary law of motion can be written as

$$G_1{}^{r(\tau)}(\omega) \approx \int_a^b g(\tau)\exp\big(ikr(\tau)\big)\,d\tau$$

- Then we can use *the method of steepest decent*, as discussed in Papoulis's books, to determine where $r'(\tau_0) = 0$ is the only zero, the integral can expressed as

$$G_1{}^{r(\tau)}(\omega) \approx \int_a^b g(\tau_0)\exp\big(ikr(\tau_0)\big) exp\left(\frac{ik}{2}(\tau-\tau_0)^2\right) d\tau.$$

- This integral can be evaluated to give

$$G_1{}^{r(\tau_0)}(\omega) \approx \sqrt{\left(\frac{2i}{kr''(\tau_0)}\right)}\, g(\tau_0)\exp\big(ikr(\tau_0)\big).$$

Periodic Motion

- A boundary that undergoes periodic motion obeys the law of motion: then $r(t+\delta) = r(t)$.

- Signals with periodic components are known to produce sidebands that have amplitudes that are proportional to Bessel functions.

- We demonstrate in the text that if the boundary is periodic, the same effect is observable in the return spectrum.

- Any source, $r(\tau)$, which is periodic, with an assumed period Ω, that is modulated by a periodic carrier, $\exp\big(-i\omega_0\tau + \omega_0 r(\tau)\big)$, can be expanded as

$$\exp\big(-i\omega_0 r(\tau)\big) = \sum_{-\infty}^{\infty} a_n\, exp(in\Omega\tau),$$

where the coefficients are determined from the expression

$$a_n = \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \exp\big(-i\omega_0 r(\tau)\big)\, exp(-in\Omega\tau)\; d\tau.$$

Periodic Motion

- If we capture this signal for a time period T, the return waveform can be expressed as

$$g_1^P(\tau) = A'\,exp(i\omega_0\tau) \sum_{n=-\infty}^{\infty} a_n\, exp(in\Omega\tau)\;\; P_T(\tau).$$

- After evaluating the integral, so we have spectrum

$$G_1^P(\omega) = \frac{A'T}{2} \sum_{n=-\infty}^{\infty} a_n \exp\left(i\{n\Omega - \omega'\}\tfrac{T}{2}\right) sinc\left(\{n\Omega - \omega'\}\tfrac{T}{2}\right).$$

- Note,

$$\lim_{T\to\infty} G_1^P(\omega) \to A' \sum_{n=-\infty}^{\infty} a_n \delta(n\Omega - \omega + \omega_0),$$

- A periodic boundary produces an infinite spectrum of equally spaced amplitudes a_n located at the points $\pm n\Omega + \omega_0$.

- For the boundary $r(\tau) = Xsin(\Omega\tau)$ (X is the amplitude of the motion), we have

$$G_1^P(\omega) = \sum_{n=-\infty}^{\infty} J_n\left(\frac{4\pi X}{\lambda}\right) \exp\left(i\{n\Omega - \omega'\}\tfrac{T}{2}\right) sinc\left(\{n\Omega - \omega'\}\tfrac{T}{2}\right).$$

Periodic Motion

- If there are multiple periodic modes, the spectrum is a product of the individual components: the finite spectrum is

$$G_1^{PL}(\omega) = \frac{AT}{2} \prod_{l=1}^{L} \sum_{n_l=-\infty}^{\infty} \frac{J_{n_l}\left(\frac{4\pi X_l}{\lambda}\right) \exp\left(i\{n_l\Omega_l - \omega + \omega_0\}\frac{T}{2}\right) sinc\left(\{n_l\Omega_l - \omega + \omega_0\}\frac{T}{2}\right)}{\Omega_l}.$$

- Note, in the limit that the sampling time becomes infinite, this expression becomes:

$$G_1^{PL}(\omega) = \frac{A'T}{2} \prod_{l=1}^{L} \sum_{n_l=-\infty}^{\infty} \frac{J_{n_l}\left(\frac{4\pi X_l}{\lambda}\right) \delta(n_l\Omega_l - \omega + \omega_0)}{\Omega_l}.$$

- Note these models for periodic motion spectra appear in many application areas including: communication systems, micro-Doppler (rotations of solids can be shown to be equivalent to this as well), systems' identification problems, and electrical interference problems.

- The general expression for the spectra is the product of a one dimensional spectra.

Observables

- If the spacing between the periods of the contributions, Ω_l, are sufficiently separated, the logarithm of the spectrum further separates the spectrum into distinctive parts.

- The return waveform from that is modulated by a periodic component does not provide a useful formula similar to those for the spread in the signal. Bandwidth does not appear to be a useful discriminate.

- However the amplitude of the sidebands can by using the spacing between them to estimate the size of the periodic component. Recall, the spectrum is
$$\sum_{n=-\infty}^{\infty} J_n\left(\frac{4\pi X}{\lambda}\right) \exp\left(i\{n\Omega - \omega'\}\tfrac{T}{2}\right) sinc\left(\{n\Omega - \omega'\}\tfrac{T}{2}\right),$$
so from knowledge of the location of the zero's of the Bessel function one can determine the amplitude X, because the sensor wavelength λ is known.

- The period, Ω, are equally spaced, so we can estimate the period of vibration or rotation. If there are not equally spaced spikes, then there is more than one periodic mode. If this is the case, one just needs to find how many distinctive spacing there are, which is indicative of how many periodic modes there are.

Observables

- A table of zero's that have specific spacing's of the heights of the Bessel function or the spacing of the zeroes of the Bessel function can be used to find the amplitude of vibration X, since we know the wavelength $\left(\frac{4\pi X_l}{\lambda}\right) = x$.

k	$J_0(x)$	$J_1(x)$	$J_2(x)$	$J_3(x)$	$J_4(x)$	$J_5(x)$
1	2.4048	3.8317	5.1356	6.3802	7.5883	8.7715
2	5.5201	7.0156	8.4172	9.7610	11.0647	12.3386
3	8.6537	10.1735	11.6198	13.0152	14.3725	15.7002
4	11.7915	13.3237	14.7960	16.2235	17.6160	18.9801
5	14.9309	16.4706	17.9598	19.4094	20.8269	22.2178

- Recognizing the analytic form allows one to distinguish the characteristics of single versus multiple sources of the periodic feature.

Observables

- If there are multiple peaks that have different spacing, then one can determine both the periodic components Ω_l as well as determine the amplitudes X_l from the locations of the Bessel functions.

- Both of these parameters provide structural characteristics as well as motion characterization about a center of mass.

- In a certain sense this provides an ideal way of thinking about the nature of non-uniform motion types: non-uniform motion of the center of mass, while those termed micro-Doppler are about the center of mass.

- Note, Chen's books discuss many applications of periodic Doppler, what he terms micro-Doppler related to identification problems associated with human factors as well as rotations about the center of mass.

- Our approach based upon spread or bandwidth associated with a specific law of motion offers an alternative to this. We believe that these methods are best used in combination as complementary methods to maximize the extraction of information.

Summary

- For these non-uniform motion models, one is able to obtain useful phenomenological models for an unknown associated with an object one could be tracking with a radar.

- The sampled spectrum of the return from a radar typically would have enough data points collected with it so that the bandwidth can be estimated.

- A waterfall chart of the various expected mean frequency can be used to estimate the parameters that characterize the various types of non-uniform motion.

- Phenomenological unknown such as acceleration or jerk or the slowdown constant can be estimated using these formulas to derive an empirical estimate from the return data of the radar.

- When higher order estimates are needed to estimate the phenomenological constants, this is also straight forward to calculate from the radar data.

- The non-uniform Doppler Effect offers enormous richness in its' applications to radar and other forms of remote sensing; we have only scratched the surface.

Summary (Continued)

- We have talked observables in this paper, we have not discussed signal processing.

- Time-frequency methods should always be considered for estimating information from the non-uniform spectrum.

- Non-uniform motion models are also useful as models for testing time-frequency methods of which wavelets are a special case.

- The wideband form of the return signal for the uniform Doppler effect is

$$\mathrm{s(t)} = \sqrt{\mu}\mathrm{s}\left(\sqrt{\mu}\mathrm{t} + \tau\right)$$

  where the delay is $\tau = 2\frac{R}{c}$ and the Doppler shift is $\sqrt{\mu}$ is the rescaling of the time axis or time warping.

TIME WARPING (AUDIO)

- Time warping is a way to think about how the channel the signal is passing through distorts the functional form of the signal.

- There are a number distinctive ways the signal's time dependence can be distorted or warped.

  - *Time warping* (**TW**) in audio technology refers to the stretching or contracting of the time axis of a signal which is a function of time.

  - **TW** can be viewed as a non-linear transformation of the signal time axis, so a signal $s(t)$ is transformed into the function $s(h(t))$.

  - In audio technology, the following are examples of time warping:

1. A disk jockey varies the speed of a record on a turn table to create sound effects in the music.

2. Vibrato, regular, pulsating change of pitch, of a string instrument is an example warping. (It is can be used to add expression to vocal and instrumental music.) Vibrato can be realized electronically by delay circuits.

3. The Doppler shift of a siren of an ambulance siren with the approaching or departing of the vehicle is a common example of linear TW.

TIME WARPING (AUDIO)

- Mathematically, a *linear time warp* is one that can be written in the form $\varphi_\alpha(t) = \left(1 + \frac{1}{2}\alpha t\right)t;\ \ so\ \varphi'_\alpha(t) = (1 + \alpha t)$.

- The instantaneous frequency is defined as
$$f(t) = (1 + \alpha t)f_c,$$
so it can be expressed in terms of a linear **TW** as $f(t) = \varphi'_\alpha(t)f_c$where $\varphi'_\alpha(t)$ is the chirp rate of the waveform.

- Now for a linear chirp waveform with a constant amplitude A, the waveform is $g(t) = Acos(2\pi f_c \int(1 + \alpha t)dt) = Acos\left(2\pi f_c \varphi_\alpha(t)\right)$.

- Note, there is an inflection point for the instantaneous frequency at $t = -\frac{1}{\alpha}$, where the frequencies become negative (non-physical), so t is limited to the interval $\frac{1}{\alpha}[-1,1]$.

- One can define a chirp rate as the frequency increase rate relative to the mean or center frequency f_0: $\alpha = \frac{f\prime(t)}{f_c}$, which is only valid for constant chirp rates.

TIME WARPING (AUDIO)

- For a chirp waveform

$$\alpha = \frac{f'(t)}{f_c} = \frac{\left(f_c \varphi'_\alpha(t)\right)'}{f_c} = \varphi''_\alpha(t)$$

thus we have $\alpha = \varphi''_\alpha(t) = \alpha + \mathcal{O}(t)$.

- When the time warping is nonlinear, non-constant chirp rate, then the waveform is the integral of the chirp rate.

- The "inverse warp function can be found by the time integration of the inverse of the warp rate:

$$\Psi(t) \approx \int (\varphi'(t))^{-1} dt = \int \frac{f(0)}{f(t)} dt.$$

- This expression has an intuitive explanation. The inverse warp rate $\Psi'(t)$ for some point z should stretch out frequency modulation in f(t), which is roughly equal to $(\varphi'(t))^{-1}$ does. Note one may have to introduce a bias to insure $\Psi(t) \leq 0$.

- As an example, if one applies the definition to the linear chirp, then the inverse warp rate is $\Psi_\alpha(t) \approx \int (1 + \alpha t)^{-1} dt = \frac{1}{\alpha} \log(1 + \alpha t) \approx (\alpha < 1) \frac{1}{\alpha}\ \alpha t = t,$ which is a good approximation for a small chirp rate. For linear chirp signals, $f(t) =\ \varphi'_\alpha(t) f_c$ can be transformed back to stationary signals by the inverse warp function.

OPERATOR APPROACH TO DOPPLER INFORMATION

- The return signal can be represented approximately as:

$$g_1(\tau) = -\frac{d}{d\tau} F_1\left(\tau - \frac{2r(\tau)}{c}\right).$$

- For all our subsequent work, this equation is sufficient for analyzing the return signal's spectrum.

- Note, that in Chen's discussion of micro-Doppler, he develops operators which in many situations can be viewed as another approach to ferreting out details as to the structure of a spectrum which can be related to what I have termed *quantum inspired mathematics*.

- "*Given a law of motion that is known, namely x(t), is there an operator that gives rise to it (preferably unique)*?

- If that operator is equivalent to the one that induces the law of motion on the signal, then we can consider the possibility of post-selection for the return signal to determine if we can detect the presence or absence of the operator acting on the signal.

- In the language of mathematics, the question we are asking is: *for a given x(t), is there an operator $\hat{A}$ such that:*

$$|x(t)\rangle = e^{t\hat{A}}|x(0)\rangle?$$

OPERATOR APPROACH TO DOPPLER INFORMATION

- The physics of the interaction of the waveform with the object can be expressed mathematically as an operator acting on a waveform.

- This provides a new approach to understanding the return signal at the receiver as a measurement problem; *the goal of the receiver designer is to obtain the expected value of an operator acting on a signal.*

- Thus the design of a matched filter is generalized to operators which represent measurements.

- This language reveals a more detailed understanding of the underlying interactions within the return signal that are not usually brought out by standard signal processing design techniques.

- The bottom line is a common framework for solving the measurement problem for radar, sonar, and quantum mechanics by casting them in the language of quantum mechanics as a Rigged Hilbert Space based on the *Aharonov Ansatz.*

OPERATOR APPROACH TO DOPPLER INFORMATION: AHARONOV ANSATZ

The *Aharonov Ansatz* for sensing can be summarized as:

1. Any (sensor) measurement process, whether active or passive can be thought of as determining the mathematical operator's characteristics based on a signal's interaction with an object.

2. Certain types of interaction operators can be "post-selected" for in the return signal when the broadcast signal is known for either a single or multiple operators. Thus a receiver or measurement apparatus design can be optimized for these operators.

3. In principle, detector design can be "matched" to signal interaction or optimized so that mathematical solutions to receiver (in the classical sense) design or the design of apparatus in the quantum mechanical sense for difficult to measure quantum interactions can be improved, (Examples of such improvements have been reported in the literature).

4. Matching to or post-selection of a given operator, when possible, maximizes ability to detect a "signal" or the characteristics of an interaction. This changes our ability to find the hard to find, and possibly to detect the new.

OPERATOR APPROACH TO DOPPLER INFORMATION

- The question that we address in this paper is "what operators generate a given law of motion?"

- Once this is done, we then consider the matched filters for these specific operators.

- It also provides a means to "post-select"/correlate the return signal so the receiver design for radars/quantum sensors can be optimized and detector algorithms can be implemented in banks designed for characteristics specific to distinctive types of interaction operators.

- In addition, one can also do the same for three dimensional periodic motion using the rotation matrices as well.

- From this example, we can develop the matched filter response for a rotating object in terms of the rotation matrix.

OPERATOR APPROACH TO DOPPLER INFORMATION
NON-UNIFORM DOPPLER EFFECT

- Then, if the answer is yes, can we say that the non-uniform Doppler return signal:

$$g_1(\tau) = -\frac{d}{d\tau} F_1\left(\tau - \frac{2r(\tau)}{c}\right)$$

  is **generated** by $e^{t\hat{A}}$?

- Given our new approach to understanding the return signal at the receiver as a measurement problem; then *the goal of the receiver designer is to obtain the expected value of an operator.*

- For linear Doppler, we know what these operators are, the design of a matched filter is in use is for the scale or displacement operators which represent measurements of range or Doppler shifts.

- Detector algorithms can be implemented in banks designed for characteristics specific to distinctive types of interaction operators.

OPERATOR APPROACH TO DOPPLER INFORMATION

Signal Operators

- The operator $\mathbb{W} = e^{i\tau\widehat{\mathcal{W}}}$ can be interpreted as a *time translation operator* on a function s(t):

$$\mathbb{W}\mathrm{s}(\mathrm{t}) = e^{i\tau\widehat{\mathcal{W}}}\mathrm{s}(\mathrm{t}) = \mathrm{s}(\mathrm{t}+\tau).$$

- The *frequency translation operator* $\mathbb{T} = e^{i\theta\hat{\mathcal{T}}}$ has exactly the same effect:

$$\mathbb{T}\mathrm{S}(\omega) = S(\omega+\theta).$$

  The *compression operator* $\hat{\mathbb{C}}$ has the property that it transforms a signal s(t) according to the rule:

$$\hat{\mathbb{C}}s(t) = \sqrt{\mu}s\left(\sqrt{\mu}t\right)$$

- Both the translation operator $\widehat{\mathbb{W}}$ and the compression operator $\hat{\mathbb{C}}$ are Hermitian operators.

- So the wideband Doppler return can be written in terms of two operators

$$\mathbb{W}\,\hat{\mathbb{C}}\mathrm{s}(\mathrm{t}) = e^{i\tau\widehat{\mathcal{W}}}e^{i\mu\hat{\mathcal{C}}}\mathrm{s}(\mathrm{t}) = \sqrt{\mu}\mathrm{s}\left(\sqrt{\mu}\mathrm{t}+\tau\right).$$

- Then the Doppler signal kernel is

$$\rho_s = \sqrt{\mu}\mathrm{s}^*\left(\sqrt{\mu}\mathrm{t}+\tau\right)s(t) = \mathbb{W}\,\hat{\mathbb{C}}\mathrm{s}(\mathrm{t})$$

  which is the building block for the analysis of signals.

OPERATOR APPROACH TO DOPPLER INFORMATION

Signal Operators

- The wideband ambiguity function with $\omega = 0$, can be written as:

$$\chi_s(0,\tau) = \left\langle s(t) \middle| \hat{\mathbb{C}}\widehat{\mathbb{W}} s(t) \right\rangle = \langle \hat{\mathbb{C}}\widehat{\mathbb{W}} \rangle,$$

  which is the expected value of the product of the operators $\langle \hat{\mathbb{C}}\widehat{\mathbb{W}} \rangle_s$ for a signal s.

- This is exactly the type of expression we would expect from quantum mechanics with $\Psi \triangleq s$.

- We would expect more complicated interactions of the signal with the target to be the expected value of the product of additional operators, so the interaction (I) ambiguity function, ${\chi_s}^I(\omega,\tau)$, can be defined as:

$${\chi_s}^I(\omega,\tau) = \left\langle s(t)e^{i\omega t} \middle| \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}} s(t) \right\rangle,$$

  then a measurement of these operators with a radar is the expected value of:

$${\chi_s}^I(0,\tau) = \left\langle s(t) \middle| \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}} \middle| s(t) \right\rangle = \langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}} \rangle_s$$

- This observation is the basis for rethinking how to design a receiver based on the notion that measurement is the expectation of operators associated with what can be observed about an object.

OPERATOR APPROACH TO DOPPLER INFORMATION

Signal Operators

- We can also consider a cross ambiguity function, ${}_{s}^{r}\chi_{s}{}^{I}(0,\tau)$, which is defined as:

$$ {}_{s}^{r}\chi_{s}{}^{I}(0,\tau) = \left\langle r(t) \middle| \hat{\mathcal{O}}_1 \ldots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}} \middle| s(t) \right\rangle = \chi_{s}^{r} \langle \hat{\mathcal{O}}_1 \ldots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}} \rangle ? $$

- The underlying goal for this project is: For a given operator set of operators, $\mathcal{O}_n$, determine whether there is a post-selection waveform $r(t)$ *or* $r_n(t)$ that does the equivalent to what the matched filter does for $\hat{\mathbb{C}}\widehat{\mathbb{W}}$.

- Cross-correlation measures the similarity of two different signals while autocorrelation is a means to find repeated periodic patterns within the signal.

- A non-zero auto-correlation tells us that there is an underlying pattern in the components of a signal despite the apparent randomness of s(t) that might be all that is apparent from visual inspection.

OPERATOR APPROACH TO DOPPLER INFORMATION

Generating Operators

- If the "dynamics" of evolution of a system can be characterized in term of a single parameter α, then the dynamics of a variable is expressed as $u = u(\alpha)$ which describes a curve parameterized by α.

- Now let $F = F(q, p)$, where the variables $\mathrm{p}(\alpha)$ and $\mathrm{q}(\alpha)$ are termed phase space variables. Then

$$\frac{dF(q,p)}{d\alpha} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial q_i}{\partial \alpha} + \frac{\partial F}{\partial p_i}\frac{\partial p_i}{\partial \alpha}\right) + \frac{\partial F}{\partial \alpha} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial p_i}\right) + \frac{\partial F}{\partial \alpha},$$

  so we can write: $\frac{dF(p,q)}{d\alpha} = [F, G]_{q,p} + \frac{\partial F}{\partial \alpha}$,

  - If F does not explicitly depend on α, then we can write this equation instead as:

$$\frac{dF(p,q)}{d\alpha} = [F, G]_{q,p},$$

  and we would say that G generates $F(\alpha)$.

  - Since we can assume that $u(\alpha)$ is analytical, we can expand it in the series:

$$u(\alpha) = u_0 + \alpha \left.\frac{du}{d\alpha}\right|_{\alpha=0} + \frac{\alpha^2}{2!}\left.\frac{d^2u}{d\alpha^2}\right|_{\alpha=0} + \cdots.$$

OPERATOR APPROACH TO DOPPLER INFORMATION

Generating Operators

- Now, if we rewrite this equation from the generator perspective:

$$u(\alpha) = u_0 + \alpha[u, G]|_{\alpha=0} + \frac{\alpha^2}{2!}\left[[u, G], G\right]\Big|_{\alpha=0} + \cdots$$

  which can also be written as the operator equation:

$$u(\alpha) = exp\left(\alpha\hat{\mathcal{G}}\right)u(\alpha)\,|_{\alpha=0}$$

  where $\hat{\mathcal{G}}$ is the operator that generates $u(\alpha)$.

- For a mechanical system, the Poisson bracket variables might be taken to be position and momentum and the evolution parameter is time.

- For a time-frequency space (time and frequency taken as the canonical variables), position could be taken as time while frequency is taken to be momentum; then the evolution parameter might be with respect to the energy density, E, as the evolution parameter.

- For a normal dynamic system, we illustrate with specifics which create a given law of motion expressed in terms of the time series:

$$u(t) = u_0 + t[u, H]|_{t=0} + \frac{t^2}{2!}\left[[u, H], H\right]\Big|_{t=0} + \quad \ldots.. \quad (*)$$

OPERATOR APPROACH TO DOPPLER INFORMATION

Generating Operators

Example 2 *(constant acceleration): Let* $\mathcal{G}$ *be*

$$\mathcal{G} = \frac{p^2}{2\beta} + a_0 q.$$

Then brackets for this generator are:

$$[q,\mathcal{G}] = a_0\frac{p}{\beta}, [[q,\mathcal{G}],\mathcal{G}] = \left[a_0\frac{p}{\beta},\mathcal{G}\right] = -\frac{a_0}{\beta},$$

so (*) becomes $\left[[[p,\mathcal{G}],\mathcal{G}],\mathcal{G}\right] = 0$*, and the dynamics equation is:*

$$q(t) = q_0 + \frac{p_0}{\beta}t - \frac{a_0}{2}t^2.$$

Example 3 (periodic motion): *Let* $\mathcal{G}$ *be* $\mathcal{G} = \frac{p^2}{2\beta} + \frac{1}{2}\beta\omega^2 q^2$*. Then the brackets for this generator are:*

$$[q,\mathcal{G}] = \frac{p}{\beta}, [[q,\mathcal{G}],\mathcal{G}] = -\omega^2 q, and \ \left[[[q,\mathcal{G}],\mathcal{G}],\mathcal{G}\right] = -\omega^2\frac{p}{\beta}$$

and the dynamical equation is: $q(t) = q_0 sin(\omega t) + \frac{p_0}{\beta\omega}cos(\omega t)$*. This method might be termed the phase space method for generating the dynamics.*

OPERATOR APPROACH TO DOPPLER INFORMATION

Generating Operators for Doppler

- We work explicitly with the operator equation: $u(t) = exp\left(t\hat{\mathcal{G}}\right) u(0)$ with a known $u(t)$ and try to determine either $exp\left(t\hat{\mathcal{G}}\right)$ or $\hat{\mathcal{G}}$ if we are dealing with a scalar dynamics equation or if we are dealing with a vector dynamics equation:

$$|u(t)\rangle = exp\left(t\hat{\mathcal{G}}\right) |u(0)\rangle.$$

- There are two forms the return signal can take depending on whether we are talking about the narrowband form of the return signal:

$$g_1(\tau) = f_1\left(\tau - \frac{2r(\tau)}{c}\right)$$

or

$$g_1(\tau) = -\frac{d}{d\tau}F_1\left(\tau - \frac{2r(\tau)}{c}\right)$$

if we are dealing with a wideband form for the broadcast signal. Therefore the operator problem we are trying to solve is this: for a known $r(\tau)$, find an operator such that either it satisfies the narrow band equation: $f_1\left(\tau - \frac{2r(\tau)}{c}\right) \approx exp\left(it\hat{\mathcal{G}}/c\right)\hat{\mathbb{C}}\widehat{\mathbb{W}}f_1(\tau)$

OPERATOR APPROACH TO DOPPLER INFORMATION

Generating Operators for Doppler

- The operator is assumed to satisfy the equation: $\hat{\mathbb{C}}\,\widehat{\mathbb{W}}\,f_1(\tau) = f_2(\tau)$

$$exp\left(\frac{i\tau\hat{\mathcal{G}}}{c}\right) f_2(\tau) = f_1\left(\frac{2r'(\tau)}{c}\right) = f_1\left(\frac{e^{i\tau\hat{A}}|r(0)\rangle}{c}\right).$$

- This amounts to assuming the wideband equation can be written as:

$$-\frac{d}{d\tau}F_1\left(\tau - \frac{2r(\tau)}{c}\right) = \widehat{\mathbb{W}}\; exp\left(\frac{it\hat{\mathcal{G}}}{c}\right) F_1(\tau).$$

- If such an operator existed for some boundaries law of motion, then the cross-correlation or weak value in quantum mechanics of $exp\big(t\hat{\mathcal{G}}\big)$ could be written as

$${}_{f_p}\langle exp\big(t\hat{\mathcal{G}}\big)\rangle_{F_1} = \frac{\left\langle f_p(\tau)\middle| exp\big(t\hat{\mathcal{G}}\big)\middle| -\frac{d}{d\tau}F_1(\tau)\right\rangle}{\left\langle f_p(\tau)\middle| -\frac{d}{d\tau}F_1(\tau)\right\rangle} \text{ (wideband form)}$$

- The cross-correlation or weak value in quantum mechanics of $exp\big(t\hat{\mathcal{G}}\big)$ could be written as

$${}_{f_p}\langle exp\big(t\hat{\mathcal{G}}\big)\rangle_{f_1} = \frac{\left\langle f_p(\tau)\middle| exp\big(t\hat{\mathcal{G}}\big)\middle| f_1(\tau)\right\rangle}{\left\langle f_p(\tau)\middle| f_1(\tau)\right\rangle} \quad \text{(narrowband form)}$$

OPERATOR APPROACH TO DOPPLER INFORMATION

Generating Operators for Doppler

- It is natural to ask what operator plays the role of a Hamiltonian in this formulation.
- The joint-time frequency operator $\hat{H}_S = \hat{\mathcal{T}}\hat{\mathcal{W}}$ which obeys the quantization condition that the quantum mechanical equivalent $\hat{H}_Q = \hat{Q}\hat{P}$ obeys (this might be termed the viral operator) might serve as the time-frequency equivalent of the Hamiltonian.
- Since $\hat{H}_S = \hat{\mathcal{C}} - \frac{i}{2}$, and the properties of the compression operator, 1-6, we have the desired behavior for the signal Hamiltonian that $[\hat{\mathcal{T}}, \hat{\mathcal{C}}] = i\hat{\mathcal{T}}$, and $[\hat{\mathcal{W}}, \hat{\mathcal{C}}] = -\hat{\mathcal{W}}$ which can be interpreted as saying that these operators are the generators of complex behavior in the time-frequency space since the higher order commutators are 0.
- Furthermore, more complex time behavior can always be generated by a *signal Hamiltonian* of the form:

$$\hat{H}_S = \hat{\mathcal{T}}\hat{\mathcal{W}} + F\left(\hat{\mathcal{T}}, \widehat{\mathcal{W}}\right).$$

- Thus, much of the group's theoretic treatment of complex operators (parity, space-time, angular momentum) has analogs in signal processing.

OPERATOR APPROACH TO DOPPLER INFORMATION
Three Dimensional Rotation Operators

- Chen introduced what he termed micro-Doppler to account for periodic spectral lines that occur abut the main Doppler line.

- They can occur when scattering off either a rotating (due to rotational motion about the center of mass of a three dimensional object or rotating propeller blades) or vibrating object (such as a car surface due to engine noise).

- As we have noted, the effect of non-uniform Doppler on a waveform can always be thought of as $exp\left(\frac{it\hat{\mathcal{G}}}{c}\right)\hat{\mathbb{C}}\widehat{\mathbb{W}}f_1(\tau)$.

- This is exactly the mathematical form one wants when considering periodic motion with $exp\left(\frac{it\hat{\mathcal{G}}}{c}\right)$ being the generator of the periodic motion.

OPERATOR APPROACH TO DOPPLER INFORMATION

Three Dimensional Rotation Operators

- For a position in fixed space, where **R** is the positive vector of the radar, **V** is the translation velocity and **Ω** is the angular motion about the center of mass **r**, the velocity observed at the sensor v is:

$$\boldsymbol{v} = \frac{d}{dt}(\boldsymbol{R} + \boldsymbol{r}) = \boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{r}.$$

- Note, **Ω** can always be expressed in terms of the three Euler angles ψ, θ, φ as:

$$\boldsymbol{\Omega} \times \boldsymbol{r} = \alpha_X \boldsymbol{R}_X + \alpha_Y \boldsymbol{R}_Y + \alpha_Z \boldsymbol{R}_Z$$

where $\boldsymbol{R}_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi & sin\psi \\ 0 & -sin\psi & cos\psi \end{bmatrix}$, $\boldsymbol{R}_Y = \begin{bmatrix} cos\theta & 0 & -sin\theta \\ 0 & 1 & 0 \\ sin\theta & 0 & cos\theta \end{bmatrix}$, $\boldsymbol{R}_Z = \begin{bmatrix} cos\varphi & sin\varphi & 0 \\ -sin\varphi & cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

- Note that there are three possibilities for the time dependence of $\boldsymbol{\Omega} \times \boldsymbol{r}$. either:
  1. the $\alpha_i{}'s$ and the $\boldsymbol{R_i}'s$ are not functions of time,
  2. $\alpha_i = \alpha_i(t)$ and the $\boldsymbol{R_i}'s$ are not functions of time,
  3. $\alpha_i = \alpha_i(t)$ and the $\boldsymbol{R_i} = \boldsymbol{R_i}(t)$.

Three Dimensional Rotation Operators

- Then for Case 1,

$$exp\left(\frac{it\hat{\mathcal{G}}}{c}\right) = exp\left(\frac{it(\boldsymbol{\Omega} \times \boldsymbol{r})}{c}\right) = exp\left(\frac{it(\alpha_X \boldsymbol{R}_X + \alpha_Y \boldsymbol{R}_Y + \alpha_Z \boldsymbol{R}_Z)}{c}\right),$$

so the functional form of the return waveform is:

$$g_1(\tau) = exp\left(\frac{it(\alpha_X \boldsymbol{R}_X + \alpha_Y \boldsymbol{R}_Y + \alpha_Z \boldsymbol{R}_Z)}{c}\right) \hat{\mathbb{C}}\widehat{\mathbb{W}} f_1(\tau).$$

- The second case can be expressed as:

$$g_1(\tau) = exp\left(\frac{it(\alpha'_X(t)\, \boldsymbol{R}_X + \alpha_{\prime Y}(t)\, \boldsymbol{R}_Y + \alpha'_Z(t) \boldsymbol{R}_Z)}{c}\right) \hat{\mathbb{C}}\widehat{\mathbb{W}} f_1(\tau).$$

where the ' indicates derivative with respect to time.

- The third case with $\boldsymbol{R_i}(t)$ really requires a more precise formulation of the physics based moment of inertia varying at non-constant rate, which would cause the rotation angles to wobble, and causing the rotation angle rates to vary.

OPERATOR APPROACH TO DOPPLER INFORMATION
NOISE OPERATORS

- A unitary operator $\hat{U}$ can always be written in the form $\hat{U} = e^{i\hat{H}}$ where $\hat{H}$ is a Hemitian operator.

- The expression:

$$e^{i\hat{H}}\hat{A}e^{-i\hat{H}} = \hat{A}'$$

  is the unitary transformation of the operator $\hat{A}$ into the operator $\hat{A}'$.

- Likewise, the unitary transformation of a state is $\chi' = e^{i\hat{H}}\chi$.

- The expression for $e^{\xi\hat{H}}\hat{A}e^{-\xi\hat{H}}$ allows us to make rather interesting observation about the connection between operators and signals.

OPERATOR APPROACH TO DOPPLER INFORMATION
NOISE OPERATORS

- If we let $[\hat{H}, \hat{A}] = [\hat{D}, \hat{X}]$, and note $[\hat{D}, \hat{X}] = -i$, by their definition, and let $\xi = \tilde{\beta}$, a random variable with a given probability distribution.

- Then using the for these operators gives:

$$\exp(i\tilde{\beta}\hat{D})\,\hat{X}\,\exp(-i\tilde{\beta}\hat{D}) = \hat{X} - i(-i\tilde{\beta}) = \hat{X} + \tilde{\beta}.$$

- Thus, when the signal can be thought of as an operator, the notion of signal plus noise can be thought of as a unitary transformation of an operator by a *random operator*.

- By *random operator*, we mean an unitary operator of the form $\exp(i\tilde{\beta}\hat{D})$, where $\tilde{\beta}$ is a random variable with a given probability distribution.

- This is not the usual understanding of random operators in quantum mechanics, but it is "natural" in the signal processing sense.

OPERATOR APPROACH TO DOPPLER INFORMATION
NOISE OPERATORS

- For example, if we were to think about signal processing problems as the result of random unitary operators into the measurement operator $\hat{A}'$, does this provide a different way to address the signal-to-noise improvement problem?

- Also, are the common expressions for non-linear combinations of signal and noise expressible in this fashion?

- It is our hypothesis that the answer is yes.

- We plan to address this in a future paper that also considers the first question and some mathematical issues associated with these types of operators.

OPERATOR CURRENT

Definition: The post-selection operator (or cross-correlation for classical applications) current density (PSOCD) is defined as

$$\hat{\rho}^{A}_{\varphi^*,\psi} = \left(\varphi^* \hat{A}\psi\right);$$

when $\varphi = \psi$, this becomes:

Definition: The operator current density (OCD) is defined as

$$\hat{\rho}^{A}_{\psi^*,\psi} = \left(\psi^* \hat{A}\psi\right);$$

Definition: The expected value of the PSOCD is

$$\langle \hat{\rho}^{A}_{\varphi^*,\psi} \rangle = \int \varphi^* \hat{A}\psi dV = \int \hat{\rho}^{A}_{\varphi^*,\psi}\, dV\,;$$

which reduces to the expected value $\langle A \rangle_{,\psi}$ of an operator

$$\langle A \rangle_{,\psi} = \int \psi^* \hat{A}\psi\, dV = \int \hat{\rho}^{A}_{\psi^*,\psi}\ dV = \langle \hat{\rho}^{A}_{\psi^*,\psi} \rangle$$

where $\langle \cdot \rangle \triangleq \int \quad \cdot\ dV$.

OPERATOR CURRENT

Theorem: "*If a field* $g(x,z)$ *satisfies the diffusion equation (*the Schrodinger equation is an example of the diffusion equation*)*

$$\frac{\partial^2 g}{\partial x^2} + 2ik\frac{\partial g}{\partial z} = 0,$$

the it's ambiguity function satisfies $\chi(x,v,z)$ *the wave equation*

$$v^2\frac{\partial^2 \chi}{\partial x^2} - k^2\frac{\partial^2 \chi}{\partial z^2} = 0."$$

In radar and optics wave, propagation can usually be interpreted as solution the diffusion equation [11] with z interchanged for time in the Schrodinger equation.)

- From translation & compression operators, once transform the functional form of a signal to produce matched filter kernel (the optimal response of a radar receiver to noise), which is the wideband ambiguity function of radar with $\omega = 0$, the ambiguity function can be interpreted as:

$$\chi_s(0,\tau) = \left\langle s(t) \middle| \hat{\mathbb{C}}\widehat{\mathbb{W}} s(t) \right\rangle = \langle \hat{\mathbb{C}}\widehat{\mathbb{W}} \rangle = \langle \hat{\rho}^{\hat{\mathbb{C}}\widehat{\mathbb{W}}}_{s(t)^*,s(t)} \rangle.$$

OPERATOR CURRENT

- In the language of physics, it is the expected value of the product of the operators, $\langle \hat{\mathbb{C}}\widehat{\mathbb{W}} \rangle_s$ for a signal s.
- This is exactly the type of expression we would expect from quantum mechanics with $\psi \triangleq s$.
- More complicated interactions of the signal with the target lead to the expected value of the product of additional operators.
- So the interaction ambiguity function, ${\chi_s}^I(\omega, \tau)$, can be defined as

$${\chi_s}^I(\omega, \tau) = \left\langle s(t)e^{i\omega t} \middle| \hat{\mathcal{O}}_1 \ldots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}} s(t) \right\rangle = \langle \hat{\rho}^{\hat{\mathcal{O}}_1 \ldots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}}}_{e^{-i\omega t} s(t)^*, s(t)} \rangle.$$

or the Wigner function ${W_s}^I(p, \tau) = \left\langle \psi(\mathrm{x})e^{ipx} \middle| \hat{\mathcal{O}}_1 \ldots \hat{\mathcal{O}}_n \hat{\mathbb{C}}\widehat{\mathbb{W}} \psi(\mathrm{x}) \right\rangle$.

OPERATOR CURRENT

- Then, a measurement of these operators is the expected value of

$$W_s{}^I(0,\tau) = \left\langle \psi(x) \middle| \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_n \hat{\mathbb{C}} \widehat{\mathbb{W}} \middle| \psi(x) \right\rangle = \langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_n \hat{\mathbb{C}} \widehat{\mathbb{W}} \rangle_\psi$$

- This observation is the basis for rethinking how we design a measurement device (or a receiver for a classical device) based on the notion that measurement is the expectation of operators associated an object.

- If we consider a cross ambiguity function, ${}^r_s\chi_s{}^I(0,\tau)$, which is defined as:

$$ {}^r_s\chi_s{}^I(0,\tau) = \left\langle r(t) \middle| \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_n \hat{\mathbb{C}} \widehat{\mathbb{W}} \middle| s(t) \right\rangle = \chi^r_s \langle \hat{\mathcal{O}}_1 \dots \hat{\mathcal{O}}_n \hat{\mathbb{C}} \widehat{\mathbb{W}} \rangle,$$

  then we are dealing with a post-selection or cross correlation current.

- The matched filter for operators can be used to explore how to post-select a waveform to "optimize" the wavefunction for a given operator.

- The concept may play a role in some aspects of experimental physics in the future.

VARIATIONAL FUNCTIONAL

- If we just play with an equation that was defined long before weak measurement was proposed, the Variational Functional one encounters quantum mechanical perturbation theory:

$$\Lambda = \frac{\langle \varphi | \hat{A} | \psi \rangle}{\langle \vartheta | \psi \rangle}$$

  what does it tell us?

- Normally, we just use it to get the lowest energy eigenvalue when we can't find an exact solution (see Ballintine for numerous examples).

- By varying Λ with respect to $|\vartheta\rangle$ and $|\psi\rangle$, then one obtains two Schrodinger like equations for $|\psi\rangle$ and the complex conjugate of $|\varphi\rangle$. When $\hat{A} = \hat{A}^{\#}$, the operator is Hermitian operator and we are back to normal quantum mechanics. But what if we don't?

- Use Λ as the defintion of what the measurement of the operator $\hat{A}$ is subject to constraints such as $\langle \psi | \psi \rangle = 1$ and $\langle \vartheta | \vartheta \rangle = 1$, $1 > |\langle \vartheta | \psi \rangle| = \varepsilon > 0$, noise, and the interpretation of the inner product ("New Interpretation of the Scalar Product in Hilbert Space", A[3], PRL, Vol 47 # 15.). What does that tell us about state evolution?

- There are a variety questions that a functional starting point for weak measurement leaves open that are worth considering.

- Landau and Lifshitz *Quantum Mechanics* (pp 56-8): "*Schrodinger's equation can be obtained from the variational principle* $\delta \int \psi^* (\hat{H} - E) \psi dq = 0$. *Since* ψ *is complex, we can vary* ψ *and* ψ^* *independently. ..., we obtain the required solution* $(\hat{H} - E)\psi = 0$. *The variation of* ψ *gives nothing different.*"

REMOTE SENSING

- Remote Sensing can be cast into the language operator currents, particularly when one is getting a return signal from a multi-static sensor network.

- LIGO and radio astronomy are examples where one assumes one knows the return signal, but not the broadcast signal, so one needs to pre-select the signal in a manner that determines $\odot$ in $\langle E^R(t)|\widehat{M}_S|\odot\rangle$ we can assume to have measured the remainder.

- Neutrino physics is naturally formulated as a two state system, so operator current remains a viable option if a Mach-Zhender interferometer or a polarizer technique for neutrinos could be found.

- Polarization radar exists and is starting to be used in a variety platforms including satellites. Operator currents for specific scattering attributes of a landscape can be formulated in terms of the polarization operators.

- In general, any sensing problem that can be cast in terms of the polarization matrices, or their higher dimensional analogs, is subject to operator currents.

FINAL THOUGHTS

1. The operator approach to the physics of the interaction of the scatter with the is a natural approach to signal analysis.

2. The operator current is a natural way to think about signals interaction.

3. It is possible to generalize the matched filter concept to operators.

4. Signal processing & analysis techniques can be extended using these ideas when combined with digital technology to create new engineering analysis of signals.

5. References, see Research Gate.