When is the single-scattering approximation valid?

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Topics

1. SAR models and data acquisition geometries
2. The single-scattering/Born approximation
3. Fréchet differentiability and bilinear operators
4. What remains to be done

Joint work with Margaret Cheney, Raluca Felea, Romina Gaburro and Cliff Nolan

Synthetic Aperture Radar

- Sources (S) and receivers (R) pass over landscape
- Pulses of EM waves emitted by S, reflect off obstacles, possibly multiple times, are detected by R
- Many data acquisition geometries:
  - Monostatic ($R = S$) or not
  - One flight path (2D data) or multiples passes (3D)
  - Straight vs. curved, etc.
- Edge/singularity detection:
  - Characterization of artifacts
  - Removal if possible
  - Guidance for filter design if not

Microlocal approach

- Good for finding the locations and orientations of edges (and other singularities)
- Many geometries have been studied: work of Cheney, Nolan; Felea; Cheney, Yarman, Yazici; Ambartsoumian,Felea,Krishnan,Nolan,Quinto; Gaburro
- Based on a single-scatter (Born) approximation, ignoring multiple reflections
- $\leftrightarrow$ A formal linearization DF of the nonlinear map F sending the propagation speed to the data

Q. Under what conditions is this linearization justified?

Problem: Show that F is Fréchet differentiable.

Previous work on Fréchet diff. of forward maps:

- Very general results of Blazek, Stolk and Symes (2013)
- More specific work of Kirsch and Rieder (2014)

Our eventual goal is to establish Fréchet diff. between Banach function spaces (for wave speed and data) that reflect known operator degeneracies of DF, which are known to be sensitive to the data acquisition geometry.

Mathematical model

Time dependent wave eqn without source term:

$$\left(\nabla^2 - c(x)^{-2}\partial_t^2\right)U(x,t) = 0$$

$c(x)$ = propagation speed.

Source at location $x = s$ emits pulse: spatial-temporal waveform $W(x-s,t)$, e.g., $\delta(x-s)\delta(t)$.

E-field component/wave $U(s,x,t)$ satisfies

$$\left(\nabla_x^2 - c(x)^{-2}\partial_t^2\right)U(s,x,t) = W(x-s,t),\ U \equiv 0,\ t << 0$$

Write $U = U^{in} + U^{sc}$, with incident field

$$U^{in} = G_0 * W(x - s, t), \quad G_0 = -\frac{\delta(t - |x|/c_0)}{4\pi|x|}$$

satisfying free-space WE,

$$\left(\nabla_x^2 - c_0^{-2}\partial_t^2\right) U^{in}(s, x, t) = W(x - s, t),\ U \equiv 0,\ t << 0$$

$\Longrightarrow$ U^{sc} satisfies

$$\left(\nabla_x^2 - c_0^{-2}\partial_t^2\right) U^{sc}(s, x, t) = -V(x) \cdot \partial_t^2 U,\ U^{sc} \equiv 0,\ t << 0,$$

$V(x) = c_0(x)^{-2} - c(x)^{-2} =$ reflectivity function.

SAR Problem: Recover $V(x)$, hence $c(x)$, from

$$u_{\mathbb{D}}(s,r,t) = U^{sc}(s, x = r, t)|_{\mathbb{D}}$$

for various data acquisition geometries $\mathbb{D}$.

- **Monostatic:** $R = S \in \Gamma$, flight path, straight or curved
- **Bistatic:** $S \in \Gamma_1$, $R \in \Gamma_2$, possibly at different altitudes and speeds
- **Single** or **multiple** passes: dim($\mathbb{D}$)=2 or 3.

Microlocal SAR Problem:

Detect edges or other singularities of $c(x)$
(at least their locations and orientations) from $u_{\mathbb{D}}$.

Many $\mathbb{D}$ studied, based on a
single scattering/Born approx./formal linearization.

Two common features:

- Ambiguity artifacts: multiple locations/orientations of edges can give rise to same data.
- Degeneracy artifacts: operator theory and estimates worse than might expect.

Map $F : c(x) \to u_{\mathbb{D}}(s, r, t)$ is a nonlinear mapping.
Want to understand validity of the linearization.

Formal linearization

Convenient to use $\gamma_0 := c_0(x)^2$, $\gamma = \gamma(x) := c(x)^2$.

Let $\Box_\gamma = \gamma(x)\nabla_x^2 - \partial_t^2$, and write $\gamma = \gamma_0 + \delta\gamma$, $u_{\mathbb{D}} = u = u_0 + \delta u$. Then,

$$\begin{aligned}\Box_\gamma u &= \big((\gamma_0 + \delta\gamma)\nabla_x^2 - \partial_t^2\big)(u_0 + \delta u) \\ &= \Box_{\gamma_0} u_0 + (\delta\gamma)\nabla_x^2 u_0 + \Box_{\gamma_0}(\delta u) \quad \mathrm{mod}\ \delta^2.\end{aligned}$$

$\Longrightarrow$

$$DF\big(\gamma_0\big)(\delta\gamma) := \delta u = -\Box_{\gamma_0}^{-1}\big((\nabla^2 u_0)\cdot \delta\gamma\big),$$

where $\Box_{\gamma_0}^{-1}$ is the forward solution operator for $\Box_{\gamma_0}$.

Differentiability of F

Def. Let X and Y be Banach spaces, $F : X \to Y$ a map, and $x_0 \in X$ and $y_0 = f(x_0) \in Y$. Then F is **Fréchet differentiable at** x_0 if there exists a a bounded linear operator $DF(x_0) : X \to Y$ such that

$$F(x) = y_0 + DF(x_0)(x - x_0) + o\big(||x - x_0||_X\big)$$

as $||x - x_0||_X \to 0$.

In our setting, reasonable to aim for a quadratic bound:

$$||u - u_0 - DF\big(\gamma_0\big)(\gamma - \gamma_0)||_Y \leq C||\gamma - \gamma_0||_X^2.$$

Problem: Find pairs of function spaces, X for $\gamma(x)$ and Y for $u_{\mathbb{D}}(s, r, t)$, for which this holds.

Set

$$v := u - u_0 - DF\big(\gamma_0\big)(\gamma - \gamma_0) = u - u_0 - \Box_{\gamma_0}^{-1}\big((\nabla^2 u_0) \cdot \delta\gamma\big).$$

Apply $\Box_\gamma$ to v. Find:

$$v = \Box_\gamma^{-1}\Big(\delta\gamma \cdot \nabla^2 \Box_{\gamma_0}^{-1}\big((\nabla^2 u_0) \cdot \delta\gamma\big)\Big).$$

Recalling $u_0 = \Box_{\gamma_0}^{-1}(W^s), \quad W^s(x,t) := W(x-s,t)$,

$\rightarrow$ form bilinear operator,

$$B(f,g)(s,r,t) := \Box_\gamma^{-1}\left(g \cdot \nabla^2 \Box_{\gamma_0}^{-1}\left(\left(\nabla^2 \Box_{\gamma_0}^{-1} W^s\right) \cdot f\right)\right)$$

Problem: Find pairs of function spaces X for γ and Y for u such that

(i) For $\gamma(x) \in \Gamma_+$, the strictly positive cone of X, the forward source problem $\Box_\gamma U = W$ has a solution with $u = U^{sc}|_{\mathbb{D}} \in Y$.

(ii) For $\gamma \in \Gamma_+$, the formal $DF(\gamma) : X \to Y$ is a bounded operator.

(iii) For some $M < \infty$,

$$||B(f,g)||_Y \leq M||f||_X \cdot ||g||_X.$$

We search for such X, Y among standard L^2-based Sobolev spaces, $H^p = W^{2,p} = L^2_p$, $p \in \mathbb{R}$.

Three assumptions

1. No caustics. The background propagation speed $c_0(x)$ has simple ray geometry (no multi-pathing/caustics).
$\implies$ Well-defined time-of-travel metric, $d_0(x, y)$.

2. No short-range scattering: If incident wave from s scatters at x' to x'' and back up to r, then $|x' - x''| \geq \epsilon > 0$.

Note: (1) and (2) are stable conditions and hold for any speed $c(x)$ close to c_0 in C^3-norm. In particular, such $c(x)$ also has a metric, $d_c(x, y)$.

3. Conormal wave-form. The wave-form W is conormal for the origin in space-time, of some order $m \in \mathbb{R}$:

$$W(x,t) = \int_{\mathbb{R}^{3+1}} e^{i[x\cdot\xi + t\tau]} a_m(\xi,\tau)\, d\xi\, d\tau,$$

with $a_m \in S^m_{1,0}$, a symbol of order $m \in \mathbb{R}$. Such W are smooth away from $x=0$, $t=0$, e.g., $\delta(x)\cdot\delta(t)$ is of order $m=0$.

N.B.The spaces for which we currently have results are too regular to include one model reflectivity function:

$$V(x) = c_0(x)^{-2} - c(x)^{-2} = g(x_1,x_2)\cdot\delta(x_3 - h(x_1,x_2))$$

where g = ground reflectivity and h = altitude.

Prop. If $c_0 \in C^\infty$, the formal $DF(\gamma_0)$ has Schwartz kernel

$$K_{DF}(s,r,t,x') = \int e^{i[t - d_{c_0}(s,x') - d_{c_0}(x',r)]\tau} a_{m+2}(\tau)\, d\tau.$$

Thus, $DF(\gamma_0)$ is a linear generalized Radon transform $\Longrightarrow$ a Fourier integral operator (FIO) of order

$$m + 1 - \frac{dim(\mathbb{D}) - 3}{4}$$

and has canonical relation

$$C_{DF} = N^*\{d_{c_0}(s,x') + d_{c_0}(x',r) = t\}' \subset T^*\mathbb{D} \times T^*\mathbb{R}^3$$

which is nondegenerate. Thus, for all $p \in \mathbb{R}$,

$$DF(\gamma_0) : H^p(\mathbb{R}^3) \to H^{p-m-\frac{5-dim(\mathbb{D})}{2}}(\mathbb{D}).$$

$$B(f,g)(s,r,t) = \int e^{i\phi(s,r,t,x',x'';\tau)} a(\tau) f(x') g(x'')\, d\tau\, dx'\, dx'',$$

where a is a symbol of order $m+4$ and

$$\phi(s,r,t,x',x'';\tau) := \big[t - d_{c_0}(s,x') - d_{c_0}(x',x'') - d_c(x'',r)\big]\tau.$$

which encodes double-scattering events.

Note: first two metrics are for $c_0(x)$, but last is for $c(x)$.

B is a bilinear generalized Radon transform / FIO.

No general theory, so use *ad hoc* methods.

Can think of B as a linear gen. Radon transf., $\widetilde{B}$, applied to $f \otimes g = f(x') \cdot g(x'')$ on $\mathbb{R}^{3+3}$.

Prop. If assumptions (1)-(3) hold and c_0, $c \in C^\infty$, $\widetilde{B}$ is a linear FIO of order

$$m + \frac{9 - dim(\mathbb{D})}{2} - \frac{6 - \dim(\mathbb{D})}{4}$$

and has canonical relation $C_{\widetilde{B}} \subset T^*\mathbb{D} \times T^*\mathbb{R}^6$ which is nondegenerate. Thus, for all $p \in \mathbb{R}$,

$$\widetilde{B} : H^p(\mathbb{R}^6) \to H^{p-m-\frac{9-dim(\mathbb{D})}{2}}(\mathbb{D}).$$

Using additional information about $C_{\widetilde{B}}$ and tensor products $f \otimes g$, for $p \geq 0$ can be improved to

$$B : H^p(\mathbb{R}^3) \times H^p(\mathbb{R}^3) \to H^{2p-m-\frac{9-dim(\mathbb{D})}{2}}(\mathbb{D}).$$

Comparing the estimates for $DF(\gamma_0)$ and B, see that we can take $X = H^p(\mathbb{R}^3)$ and $Y = H^{p-m-\frac{5-dim(\mathbb{D})}{2}}(\mathbb{D})$ if $p \geq 2$. Also need $H^p \hookrightarrow C^3(\mathbb{R}^3)$ for stability of Assumptions 1 and 2.

By Sobolev embedding, any $p > 9/2$ suffices.

What remains to be done

1. We believe can extend this to general $\gamma \in \Gamma_+ \subset H^p(\mathbb{R}^3)$ close to $\gamma_0 \in C^\infty$.This would give Fréchet differentiability at smooth backgrounds c_0.

2. Extending this to get Fréchet diff. at general γ_0 (not necessarily C^∞) will be more challenging.

3. Lowering the regularity assumptions to include reasonable models of surface reflectors.

Thank you!

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