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# Reduction methods for the Riccati equation, and the solution of the linear-quadratic regulator problem

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My contribution to Vladimir's scientific excellence  
or  
How it all started...

*Extended Krylov subspaces: approx. of the matrix square root and related functions*

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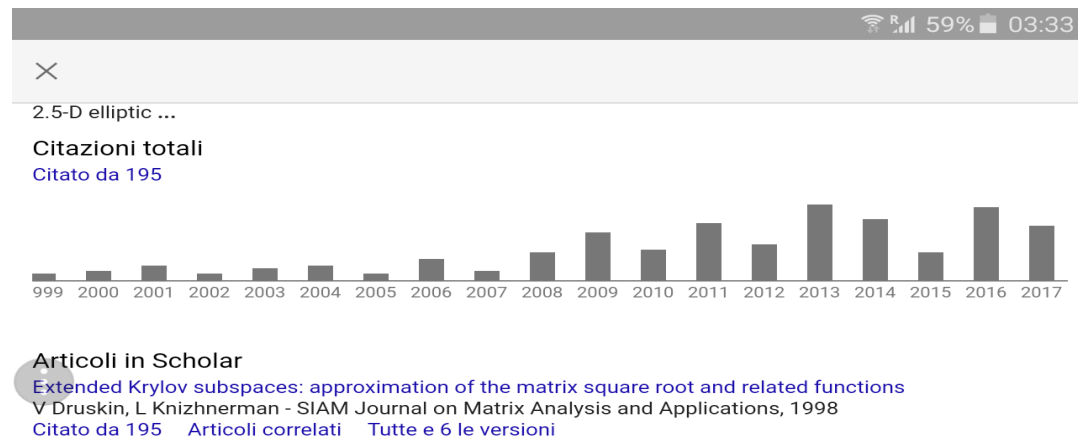
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(average of 4 cit until 2007, then 9 (2008), 16 (2009), 10 (2010), 19 (2011), 25 (2013)....)

## The problem

Find  $\mathbf{X} \in \mathbb{R}^{n \times n}$  such that

$$A\mathbf{X} + \mathbf{X}A^\top - \mathbf{X}BB^\top\mathbf{X} + C^\top C = 0$$

with  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times p}$ ,  $C \in \mathbb{R}^{s \times n}$ ,  $p, s = \mathcal{O}(1)$

**Rich literature on analysis, applications and numerics:**

Lancaster-Rodman 1995, Bini-Iannazzo-Meini 2012, Mehrmann et al 2003 ...

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**We focus on the large scale case:**  $n \gg 1000$

Different strategies

- (Inexact) **Kleinman iteration** (Newton-type method)
- **Projection methods**
- Invariant subspace iteration
- (Sparse) multilevel methods
- ....

## Newton-Kleinman iteration

Assume  $A$  stable. Compute sequence  $\{\mathbf{X}_k\}$  with  $\mathbf{X}_k \rightarrow_{k \rightarrow \infty} \mathbf{X}$

$$(A - \mathbf{X}_k B B^\top) \mathbf{X}_{k+1} + \mathbf{X}_{k+1} (A^\top - B B^\top \mathbf{X}_k) + C^\top C + \mathbf{X}_k B B^\top \mathbf{X}_k = 0$$

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- 1: Given  $\mathbf{X}_0 \in \mathbb{R}^{n \times n}$  such that  $\mathbf{X}_0 = \mathbf{X}_0^\top$ ,  $A^\top - B B^\top \mathbf{X}_0$  is stable
- 2: **For**  $k = 0, 1, \dots$ , until convergence
- 3:     **Set**  $\mathcal{A}_k^\top = A^\top - B B^\top \mathbf{X}_k$
- 4:     **Set**  $\mathcal{C}_k^\top = [\mathbf{X}_k B, C^\top]$
- 5:     **Solve**  $\mathcal{A}_k \mathbf{X}_{k+1} + \mathbf{X}_{k+1} \mathcal{A}_k^\top + \mathcal{C}_k^\top \mathcal{C}_k = 0$



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### Critical issues:

- The full matrix  $\mathbf{X}_k$  cannot be stored (sparse or low-rank approx)
- Need a computable stopping criterion
- Each iteration  $k$  requires the solution of the Lyapunov equation

(Benner, Feitzinger, Hylla, Saak, Sachs, ...)

## Galerkin projection method for the Riccati equation

Given the orth basis  $V_k$  for an approximation space, determine

$$\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$$

to the **Riccati solution matrix** by orthogonal projection:

|                     |  |
|---------------------|--|
| Galerkin condition: | Residual orthogonal to approximation space |
|---------------------|--|

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giving the **reduced** Riccati equation

$$(V_k^\top A V_k) \mathbf{Y} + \mathbf{Y} (V_k^\top A^\top V_k) - \mathbf{Y} (V_k^\top B B^\top V_k) \mathbf{Y} + (V_k^\top C^\top) (C V_k) = 0$$

$\mathbf{Y}_k$  is the stabilizing solution (Jaimoukha-Kasenally 1994, Heyouni-Jbilou 2009)

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**Key questions:**

- **Which** approximation space?
- Is this meaningful from a Control Theory perspective?

## Dynamical systems and the Riccati equation

$$A\mathbf{X} + \mathbf{X}A^\top - \mathbf{X}BB^\top\mathbf{X} + C^\top C = 0$$

Time-invariant linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \\ y(t) = Cx(t), \end{cases}$$

$u(t)$  : control (input) vector;       $y(t)$  : output vector

$x(t)$  : state vector;       $x_0$  : initial state

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Minimization problem for a Cost functional:      (simplified form)

$$\inf_u \mathcal{J}(u, x_0) \quad \mathcal{J}(u, x_0) := \int_0^\infty (x(t)^\top C^\top C x(t) + u(t)^\top u(t)) dt$$

## Dynamical systems and the Riccati equation

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THEOREM Let the pair  $(A, B)$  be stabilizable and  $(C, A)$  observable. Then there is a unique solution  $\mathbf{X} \geq 0$  of the Riccati equation. Moreover,

i) For each  $x_0$  there is a unique optimal control, and it is given by

$$u_*(t) = -B^\top \mathbf{X} \exp((A - BB^\top \mathbf{X})t)x_0 \quad \text{for } t \geq 0$$

ii)  $\mathcal{J}(u_*, x_0) = x_0^\top \mathbf{X} x_0$  for all  $x_0 \in \mathbb{R}^n$

see, e.g., Lancaster & Rodman, 1995



## Order reduction of dynamical systems by projection

Let  $V_k \in \mathbb{R}^{n \times d_k}$  have orthonormal columns,  $d_k \ll n$

Let  $T_k = V_k^\top A V_k$ ,  $B_k = V_k^\top B$ ,  $C_k^\top = V_k^\top C^\top$

Reduced order dynamical system:

$$\begin{cases} \dot{\hat{x}}(t) = T_k \hat{x}(t) + B_k \hat{u}(t), & \hat{x}(0) = \hat{x}_0 := V_k^\top x_0 \\ \hat{y}(t) = C_k \hat{x}(t) \end{cases}$$

$$x_k(t) = V_k \hat{x}(t) \approx x(t)$$

Typical frameworks:

- Transfer function approximation
- Model reduction

## The role of the projected Riccati equation

Consider again the reduced Riccati equation:

$$(V_k^\top A V_k) \mathbf{Y} + \mathbf{Y} (V_k^\top A^\top V_k) - \mathbf{Y} (V_k^\top B B^\top V_k) \mathbf{Y} + (V_k^\top C^\top)(C V_k) = 0$$

that is

$$T_k \mathbf{Y} + \mathbf{Y} T_k^\top - \mathbf{Y} B_k B_k^\top \mathbf{Y} + C_k^\top C_k = 0 \quad (*)$$

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**THEOREM.** Let the pair  $(T_k, B_k)$  be stabilizable and  $(C_k, T_k)$  observable. Then there is a unique solution  $\mathbf{Y}_k \geq 0$  of  $(*)$  that for each  $\hat{x}_0$  gives the feedback **optimal control**

$$\hat{u}_*(t) = -B_k^* \mathbf{Y}_k \exp((T_k - B_k B_k^* \mathbf{Y}_k)t) \hat{x}_0, \quad t \geq 0$$

for the reduced system.

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♣ If there exists a matrix  $K$  such that  $A - BK$  is passive, then the pair  $(T_k, B_k)$  is stabilizable.

## Projected optimal control vs approximate control

★ Our projected optimal control function:

$$\hat{u}_*(t) = -B_k^\top \mathbf{Y}_k \exp((T_k - B_k B_k^\top \mathbf{Y}_k)t) \hat{x}_0, \quad t \geq 0$$

with  $\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$

★ Commonly used approximate control function:

If  $\tilde{\mathbf{X}}$  is some approximation to  $\mathbf{X}$ , then

$$\tilde{u}(t) := -B^\top \tilde{\mathbf{X}} \tilde{x}(t)$$

where  $\tilde{x}(t) := \exp((A - BB^\top \tilde{\mathbf{X}})t)x_0$

$$\hat{u}_* \neq \tilde{u}$$

They induce different actions on the functional  $\mathcal{J}$ , even for  $\tilde{\mathbf{X}} = \mathbf{X}_k$

## Projected optimal control vs approximate control

$$\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$$

$$\text{Residual matrix: } R_k := A\mathbf{X}_k + \mathbf{X}_k A - \mathbf{X}_k B B^\top \mathbf{X}_k + C^\top C$$

★ Projected optimal control function:

$$\hat{u}_*(t) = -B_k^\top \mathbf{Y}_k \exp((T_k - B_k B_k^\top \mathbf{Y}_k)t)$$

THEOREM. Assume that  $A - B B^\top \mathbf{X}_k$  is stable and that  $\tilde{u}(t) := -B^\top \mathbf{X}_k x(t)$  approx control. Then

$$|\mathcal{J}(\tilde{u}, x_0) - \hat{\mathcal{J}}_k(\hat{u}_*, \hat{x}_0)| = \mathcal{E}_k, \quad \text{with } \mathcal{E}_k \leq \frac{\|R_k\|}{2\alpha} x_0^\top x_0,$$

where  $\alpha > 0$  is such that  $\|e^{(A - B B^\top \mathbf{X}_k)t}\| \leq e^{-\alpha t}$  for all  $t \geq 0$ .

Note:  $|\mathcal{J}(\tilde{u}, x_0) - \hat{\mathcal{J}}_k(\hat{u}_*, \hat{x}_0)|$  is nonzero for  $R_k \neq 0$

## On the choice of approximation space

Approximate solution  $\mathbf{X}_k = V_k \mathbf{Y}_k V_k^\top$ , with

$$(V_k^\top A V_k) \mathbf{Y}_k + \mathbf{Y}_k (V_k^\top A^\top V_k) - \mathbf{Y}_k (V_k^\top B B^\top V_k) \mathbf{Y}_k + (V_k^\top C^\top)(C V_k) = 0$$

**Krylov-type subspaces:** (extensively used in the linear case)

- $\mathcal{K}_k(A, C^\top) := \text{Range}([C^\top, AC^\top, \dots, A^{k-1}C^\top])$  (Polynomial)
- $\mathcal{EK}_k(A, C^\top) := \mathcal{K}_k(A, C^\top) + \mathcal{K}_k(A^{-1}, A^{-1}C^\top)$  (EKS, Rational)
- $\mathcal{RK}_k(A, C^\top, \mathbf{s}) :=$

$$\text{Range}([C^\top, (A - s_2 I)^{-1}C^\top, \dots, \prod_{j=1}^{k-1} (A - s_{j+1} I)^{-1}C^\top])$$

(RKS, Rational)

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★ Matrix  $B B^\top$  **not involved** (nonlinear term!)



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★ Parameters  $s_j$  (adaptively) chosen in field of values of  $-A$

Druskin-Simoncini, 2011

## Performance of solvers

Problem:  $A$ : 3D Laplace operator,  $B, C$  randn matrices,  $\text{tol}=10^{-8}$

$(n, p, s) = (125000, 5, 5)$

|                  | its | inner its | time | space dim | rank( $X_f$ ) |
|------------------|-----|-----------|------|-----------|---------------|
| Newton $X_0 = 0$ | 15  | 5, ..., 5 | 808  | 100       | 95            |
| GP-EKS           | 20  |           | 531  | 200       | 105           |
| GP-RKS           | 25  |           | 524  | 125       | 105           |

$(n, p, s) = (125000, 20, 20)$

|                  | its | inner its | time | space dim | rank( $X_f$ ) |
|------------------|-----|-----------|------|-----------|---------------|
| Newton $X_0 = 0$ | 19  | 5, ..., 5 | 2332 | 400       | 346           |
| GP-EKS           | 15  |           | 622  | 600       | 364           |
| GP-RKS           | 20  |           | 720  | 400       | 358           |

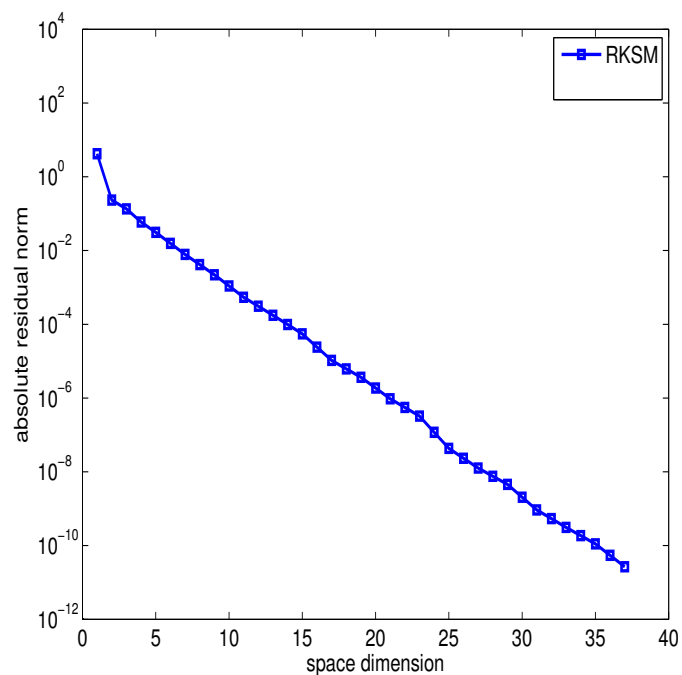
GP=Galerkin projection

(V.Simoncini & D.Szyld & M.Monsalve, 2014)

## A numerical example on the role of $BB^\top$

Consider the  $500 \times 500$  Toeplitz matrix

$$A = \text{toeplitz}(-1, \underline{2.5}, 1, 1, 1), \quad C = [1, -2, 1, -2, \dots], \quad B = \mathbf{1}$$



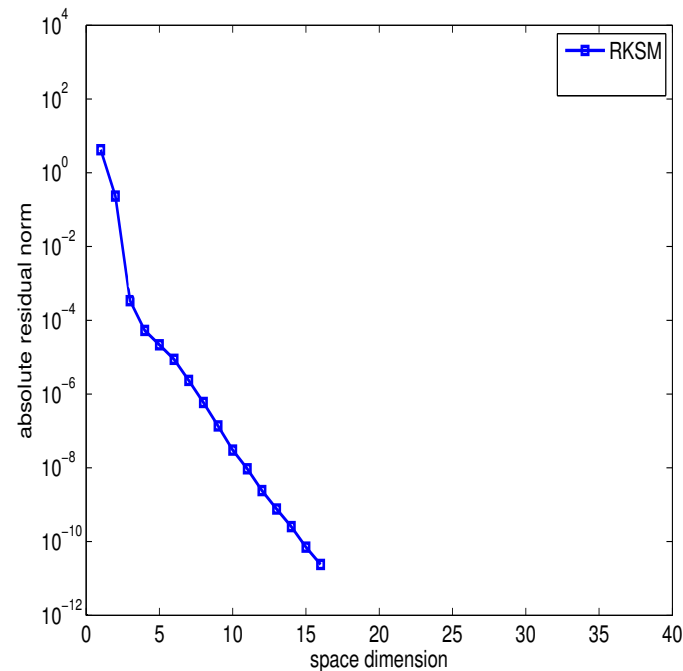
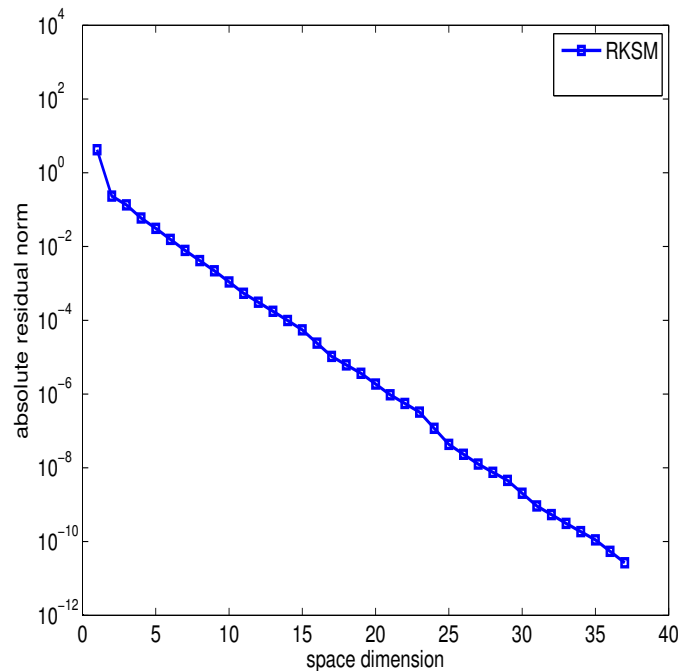
Parameter computation:

**Left:** adaptive RKS on  $A$

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Parameter computation:

**Left:** adaptive RKS on  $A$       **Right:** adaptive RKS on  $A - BB^\top \mathbf{X}_k$

(Lin & Simoncini 2015)

## On the residual matrix and adaptive RKS

$$R_k := A\mathbf{X}_k + \mathbf{X}_k A - \mathbf{X}_k B B^\top \mathbf{X}_k + C^\top C$$

THEOREM. Let  $\mathcal{T}_k = T_k - B_k B_k^\top \mathbf{Y}_k$ . Then

$$R_k = \hat{R}_k V_k^\top + V_k \hat{R}_k^\top, \quad \text{with} \quad \hat{R}_k = A V_k \mathbf{Y}_k + V_k \mathbf{Y}_k \mathcal{T}_k^\top + C^\top (C V_k)$$

so that  $\|R_k\|_F = \sqrt{2} \|\hat{R}_k\|_F$

At least formally:

$\Rightarrow V_k \mathbf{Y}_k V_k^\top$  is a solution to the Riccati equation ( $R_k = 0$ ) if and only if  $Z_k = V_k \mathbf{Y}_k$  is the solution to the Sylvester equation ( $\hat{R}_k = 0$ )

## On the residual matrix and adaptive RKS

$$R_k = \hat{R}_k V_k^\top + V_k \hat{R}_k^\top$$

Expression for the semi-residual  $\hat{R}_k$ :

THEOREM. Assume  $C^\top \in \mathbb{R}^n$ ,  $\text{Range}(V_k) = \mathcal{RK}_k(A, C^\top, \mathbf{s})$ . Assume that  $\mathcal{T}_k = T_k - B_k B_k^\top \mathbf{Y}_k$  is diagonalizable. Then

$$\hat{R}_k = \psi_{k, T_k}(A) C^\top C V_k (\psi_{k, T_k}(-\mathcal{T}_k^\top))^{-1}.$$

where

$$\psi_{k, T_k}(z) = \frac{\det(zI - T_k)}{\prod_{j=1}^k (z - s_j)}$$

(see also Beckermann 2011 for the linear case)

On the choice of the next parameters  $s_{k+1}$

$$\widehat{R}_k = \psi_{k,T_k}(A)C^\top CV_k(\psi_{k,T_k}(-\mathcal{T}_k^\top))^{-1}.$$

with  $\psi_{k,T_k}(z) = \frac{\det(zI - T_k)}{\prod_{j=1}^k (z - s_j)}$

★ **Greedy strategy:** Next shift should make  $(\psi_{k,T_k}(-\mathcal{T}_k^\top))^{-1}$  smaller

⇓

Determine for which  $s$  in the spectral region of  $\mathcal{T}_k$  the quantity  $(\psi_{k,T_k}(-s))^{-1}$  is large, and add a root there

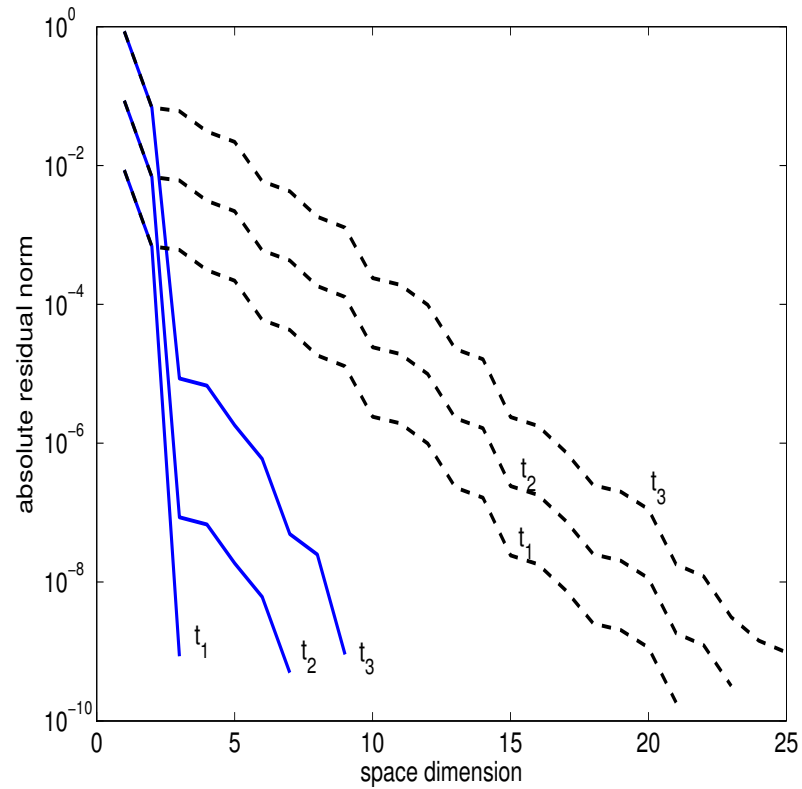
$$s_{k+1} = \arg \max_{s \in \partial \mathbb{S}_k} \left| \frac{1}{\psi_{k,T_k}(s)} \right|$$

$\mathbb{S}_k$  region enclosing the eigenvalues of  $-\mathcal{T}_k = -(T_k - B_k B_k^\top \mathbf{Y}_k)$

(This argument is new also for linear equations)

## Selection of $s_{k+1}$ in RKS. An example

$A$ :  $900 \times 900$  2D Laplacian,  $B = t \mathbf{1}$  with  $t_j = 5 \cdot 10^{-j}$ ,  
 $C = [1, -2, 1, -2, 1, -2, \dots]$



RKS convergence with and without modified shift selection as  $t$  varies

**Solid curves:** use of  $\mathcal{T}_k$

**Dashed curves:** use of  $T_k$



## Further results not presented but relevant

- Stabilization properties of the approx solution  $\mathbf{X}_k$
- Accuracy tracking as the approximation space grows
- Interpretation via invariant subspace approximation

(V.Simoncini, 2016)

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## Wrap-up

- ♡ Projection-type methods fill the gap between MOR and Riccati equation
- ♡ Clearer role of the non-linear term during the projection

## Outlook

- ♠ **Petrov-Galerkin projection (à la Balanced truncation)**  
(work in progress, with A. Alla, Florida State Univ.)
- ♠ **Projected Differential Riccati equations**  
(see, e.g., Koskela & Mena, tr 2017)
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