Polynomial Optimzation in Quantum Information Theory

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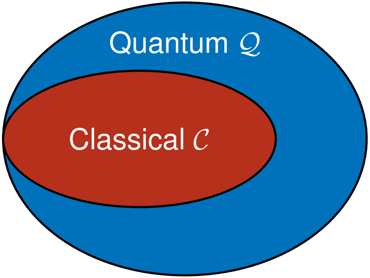
Real Algebraic Geometry and Optimization

Warm Up

- Entanglement is one of the key features in Quantum Information
- Bell '64:



- How to distinguish $\mathcal{C}$ and $\mathcal{Q}$?
- What is the correct definition for $\mathcal{Q}$? Does it matter?
- Can Polynomial Optimization help to understand these sets?

RAG and POP basics

Polynomial Optimization

- $f \in \mathbb{R}[\underline{X}]$ polynomial in commuting variables
- $g_0 = 1, g_1, \dots, g_r \in \mathbb{R}[\underline{X}]$ defining a semi-algebraic set:

$$K = \{\underline{a} \in \mathbb{R}^n \mid g_0(\underline{a}) \geq 0, \dots, g_r(\underline{a}) \geq 0\}$$

- Want to minimize f over K

$$\boxed{\begin{aligned} f_* &= \inf f(\underline{a}) && \text{s.t. } \underline{a} \in K \\ &= \sup a \in \mathbb{R} && \text{s.t. } f - a \geq 0 \text{ on } K \end{aligned}}$$

- NP-hard

RAG and POP basics

RAG helps

$$f_* = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \geq 0 \text{ on } K$$

NP-hard ☹

- $M(g) := \{p = \sum_j h_j^2 g_{i_j} \text{ for some } h_i \in \mathbb{R}[\underline{X}]\}$
- sos relaxation

$$f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M(g)$$

"SDP" ☺

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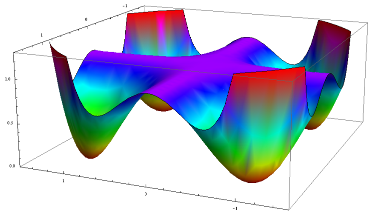
"SDP" ☺

- f_{sos} is always a lower bound but might be strict
- If $M(g)$ is archimedean: $f_* = f_{sos}$



$x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1$

RAG and POP basics

SOS hierarchy

- $M(g)_t := \{p = \sum_j h_j^2 g_{i_j} \text{ for some } h_i \in \mathbb{R}[\underline{X}]_t\}$
- sos hierarchy

$$\boxed{f_t = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M(g)_t}$$

SDP ☺

- We have
  - $f_t \leq f_{t+1} \leq f_*$
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- Certificate of exactness:
  - Flatness of dual solution
  - Allows extraction of optimizers

NC-RAG and NC-POP

NC Polynomials

- Want to replace scalar variables by matrices/operators
- Free algebra $\mathbb{R}\langle \underline{X} \rangle$ with noncommuting variables $X_1, \ldots, X_n$
- Polynomial

$$f = \sum_w f_w w$$

- Let $\underline{A} \in (\mathcal{S}^d)^n$: $f(\underline{A}) = f_1 I_d + f_{X_1} A_1 + f_{X_2 X_1} A_2 A_1 \ldots$

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- Add involution $*$ on $\mathbb{R}\langle \underline{X} \rangle$
  - fixes $\mathbb{R}$ and $\{X_1, \ldots, X_n\}$ pointwise
  - $X_i^* = X_i$
- Consequence

$$f^* f(\underline{A}) = f(\underline{A})^T f(\underline{A}) \succeq 0$$

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NC-RAG and NC-POP

Eigenvalue optimization

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$

$$f_{nc} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \succeq 0 \text{ on } K$$

NP-hard ☹

- Observation: Checking if $f = \sum_i h_i^* h_i$ is an SDP
  so as well checking $f = \sum_j h_j^* g_{i_j} h_j$ (with degree bounds)

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- sos relaxation
  $M_{nc}(g) := \{p = \sum_j h_j^* g_{i_j} h_j \text{ for some } h_i \in \mathbb{R}\langle \underline{X} \rangle\}$

$$\boxed{f_{sos} = \sup a \in \mathbb{R} \quad \text{s.t. } f - a \in M_{nc}(g)}$$

- Fact: $f_{sos} \leq f_{nc}$
- Theorem (Helton et al.): If $M_{nc}(g)$ is archimedean, then $f_{sos} = f_{nc}$.

NC-RAG and NC-POP

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- f_t converges to f_{sos} as $t \to \infty$
- If $M_{nc}(g)$ is archimedean: $f_{sos} = f_{nc}$ and hence $f_t \to f_{nc}$ as $t \to \infty$

NC-RAG and NC-POP

Trace optimization

- Let $f \in \mathbb{R}\langle \underline{X} \rangle$

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NP-hard ☹

- K contains only operators, for which a trace is defined
- If $f = \sum_j h_j^* g_{i_j} h_j + \sum_k [p_k, q_k]$ then $\operatorname{Tr}(f(\underline{A})) \geq 0$ for all $\underline{A} \in K$
- sos relaxation

  $M_{tr}(g) := \{\sum_j h_j^* g_{i_j} h_j \text{ for some } h_i \in \mathbb{R}\langle \underline{X} \rangle\} + [\mathbb{R}\langle \underline{X} \rangle, \mathbb{R}\langle \underline{X} \rangle]$

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Back to Quantum Information

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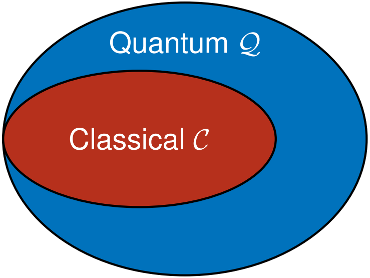

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Basics of quantum theory

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- ψ is entangled if it is not a product state

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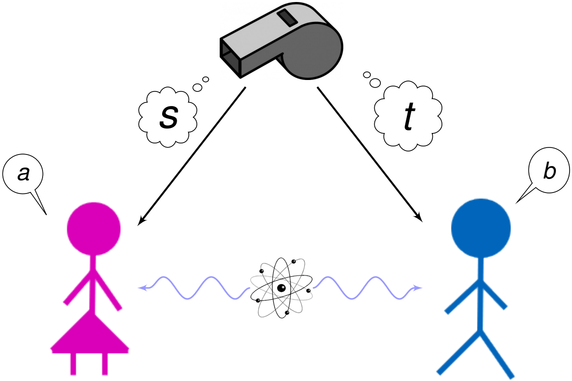
$$\psi_A \otimes \psi_B \text{ with } \psi_A \in \mathcal{H}_A, \psi_B \in \mathcal{H}_B$$

- A state $\psi \in \mathcal{H}$ can be measured
  - outcomes $a \in A$
  - POVM: a family $\{E_a\}_{a \in A} \subseteq B(\mathcal{H})$ with $E_a \succeq 0$ and $\sum_{a \in A} E_a = 1$
  - probablity of getting outcome a is $p(a) = \psi^T E_a \psi$.

Nonlocal bipartite correlations

- Question sets S, T, Answer sets A, B
- No (classical) communication



- Which correlations $p(a, b \mid s, t)$ are possible?

Correlations

Classical strategy $\mathcal{C}$

Independent probability distributions $\{p_s^a\}_a$ and $\{p_t^b\}_b$:

$$p(a, b \mid s, t) = p_s^a \cdot p_t^b$$

shared randomness: allow convex combinations

Correlations

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shared randomness: allow convex combinations

Quantum strategy $\mathcal{Q}$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \otimes F_t^b) \psi$$

- Nonlocality: $(E_s^a \otimes 1)(1 \otimes F_t^b) = (1 \otimes F_t^b)(E_s^a \otimes 1)$
- If $\psi = \psi_A \otimes \psi_B$ then we have classical correlation

More correlations

Quantum strategy $\mathcal{Q}$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on Hilbert spaces $\mathcal{H}_A, \mathcal{H}_B$, $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$:

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Quantum strategy $\mathcal{Q}_c$

POVMs $\{E_s^a\}_a$ and $\{F_t^b\}_b$ on a joint Hilbert space, but $[E_x^a, F_y^b] = 0$:

$$p(a, b \mid s, t) = \psi^T (E_s^a \cdot F_t^b) \psi$$

Fact

$$\mathcal{C} \subseteq \mathcal{Q} \subseteq \overline{\mathcal{Q}} \subseteq \mathcal{Q}_c$$

Tsirelson's problem

Fact

$$\mathcal{C} \subseteq \mathcal{Q} \subseteq \overline{\mathcal{Q}} \subseteq \mathcal{Q}_c$$

- Bell: $\mathcal{C} \neq \mathcal{Q}$
- closure conjecture [Slofstra '16]: $\mathcal{Q} \neq \overline{\mathcal{Q}}$
- weak Tsirelson [Slofstra '16]: $\mathcal{Q} \neq \mathcal{Q}_c$
- Dykema et al. '17: Concrete example in a decent subset of $\mathcal{Q}$
- strong Tsirelson (open): Is $\overline{\mathcal{Q}} = \mathcal{Q}_c$?
- strong Tsirelson is equivalent to Connes embedding problem

Nonlocal games

- Characterized by
  - 2 sets of questions S, T, asked with probability distribution π
  - 2 sets of answers A, B
  - A winning predicate $V : A \times B \times S \times T \to \{0, 1\}$

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- Characterized by
  - 2 sets of questions S, T, asked with probability distribution π
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  - A winning predicate $V : A \times B \times S \times T \to \{0, 1\}$
- Winning probability (value of the game)

$$\omega = \sup_p \sum_{s \in S, t \in T} \pi(s,t) \sum_{a \in A, b \in B} V(a,b;s,t) p(a,b|s,t)$$
$$= \sup_p \sum_{a,b,s,t} f_{abst} p(a,b \mid s,t)$$

- optimize over correlations $p \in \{\mathcal{C}, \mathcal{Q}, \mathcal{Q}_c\}$

SOS relaxation over $\mathcal{C}$

$$\omega_{\mathcal{C}} = \sup_{p} \sum_{a,b,s,t} f_{abst} p_s^a \cdot p_t^b$$

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  - $f((\underline{p}, \underline{q})) := \sum_{a,b,s,t} f_{abst} p_s^a \cdot q_t^b \in \mathbb{R}[\underline{p}, \underline{q}]$
  - $K = \{(\underline{p}, \underline{q}) \mid p_s^a, q_t^b \geq 0, \sum_a p_s^a = \sum_b q_t^b = 1\}$
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- Hence

$$\begin{aligned} \omega_{\mathcal{C}} &= \sup f(\underline{p}, \underline{q}); && \text{s.t. } (\underline{p}, \underline{q}) \in K \\ &= \inf a \in \mathbb{R} && \text{s.t. } a - f \geq 0 \text{ on } K \\ &= \inf a \in \mathbb{R} && \text{s.t. } a - f \in M(g) \quad (f_{sos}) \\ &\leq \inf a \in \mathbb{R} && \text{s.t. } a - f \in M(g)_t \quad (f_t) \end{aligned}$$

- Converging hierarchy of SDP upper bounds

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SOS relaxation over $\mathcal{Q}$

$$\omega_{\mathcal{Q}} = \sup \sum_{a,b,s,t} f_{abst} \operatorname{Tr}(E_s^a \otimes F_t^b)$$

- Cameron et al.: For most games we have $p(a, b \mid s, t) = \operatorname{Tr}(\tilde{E}_s^a \tilde{F}_t^b)$ with $\tilde{E}_s^a, \tilde{F}_t^b \succeq 0, \sum_a \tilde{E}_s^a = \sum_b \tilde{F}_t^b = D$ with $\operatorname{Tr}(D^2) = 1$

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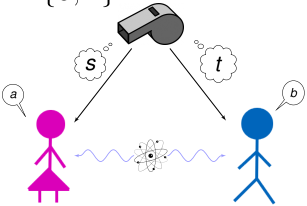
- Hence

$$\begin{aligned} \omega_{\mathcal{C}} = &\sup \operatorname{Tr} f(\underline{E}, \underline{F}); && \text{s.t. } (\underline{E}, \underline{F}, D) \in K \\ \leq &\inf a \in \mathbb{R} && \text{s.t. } a - f \in M_{tr}(g) \\ \leq &\inf a \in \mathbb{R} && \text{s.t. } a - f \in M_{tr}(g)_t \end{aligned}$$

- Converging sequence of upper SDP bounds

CHSH Game

- Questions $S = T = \{0, 1\}$, Answers $A = B = \{0, 1\}$
- Alice & Bob win, if $a + b \equiv st \mod 2$



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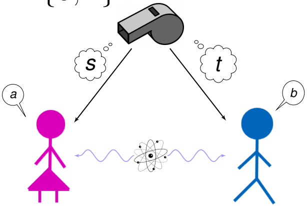
- $\omega_{\mathcal{C}} = \frac{3}{4}$
- $\omega_{\mathcal{Q}} = \omega_{\mathcal{Q}_c} = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.854$
- 1^{st} level of SOS hierarchies are exact



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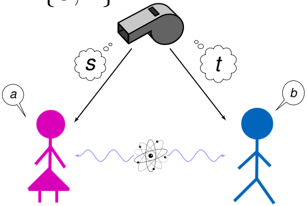

- Alternative formulation:
- 2 measurements with 2 outcomes each: $E_s^0, E_s^1, F_t^0, F_t^1$
- Setting $E_s := E_s^0 - E_s^1$, $F_t := F_t^0 - F_t^1$ one obtains the CHSH inequality

$$f_{CHSH} := E_0 F_0 + E_0 F_1 + E_1 F_0 - E_1 F_1$$

- Optimizing f_{CHSH} over variants of $\mathcal{C}, \mathcal{Q}$ give $\omega_{\mathcal{C}}, \omega_{\mathcal{Q}}$

I_{3322} inequality

- Questions $S = T = \{0, 1, 2\}$, Answers $A = B = \{0, 1\}$

$$f := E_0 F_0 + E_0 F_1 + E_0 F_2 + E_1 F_0 + E_1 F_1 - E_1 F_3 + E_2 F_0 - E_2 F_1 - E_0 - 2F_0 - F_1$$

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$$

- Maximizing over $\mathcal{C}$: $f_* \leq 0$
- Best lower bound: 0.250875384

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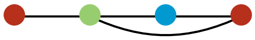
- Maximizing over $\mathcal{C}$: $f_* \leq 0$
- Best lower bound: 0.250875384
- NC-SOS upper bounds:

level	psd	trace
1	0.375	0.375
2	0.25094006	0.2509397
3	0.25087556	0.2508754

- Pal & Vertesi computed (eigenvalue) SOS-bounds for 240 Bell inequalities of which 20 are not matching ($\geq 10^{-4}$) the lower bound. 4 of them get exact ($\leq 10^{-8}$) using trace SOS-bounds, about 1/2 of them improve

Quantum coloring as feasibility problem



Quantum coloring as feasibility problem

$$\chi(G) = \min t \in \mathbb{N} \text{ s.t. } x_u^i \in \{0,1\}, u \in V(G), i \in [t],$$

$$\sum_{i \in [t]} x_u^i = 1 \quad \forall u \in V(G),$$

$$x_u^i x_u^j = 0 \quad \forall\, i \neq j, \forall u \in V(G),$$

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Quantum coloring as feasibility problem

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& x_u^i x_u^j = 0 \quad \forall\, i \neq j, \forall u \in V(G), \qquad (*) \\
& x_u^i x_v^i = 0 \quad \forall uv \in E(G) \\
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- We can write this as

$$\min t \in \mathbb{N} \text{ s.t. } \exists \text{ operator solution of } (*)$$

Nullstellensätze

Let $g_1, \ldots, g_r \in \mathbb{C}[\underline{X}]$

Theorem (weak Nullstellensatz)

Let $I = (g_1, \ldots, g_r)$, $V(I) := \{\underline{a} \in \mathbb{C}^n \mid g_1(\underline{a}) = \cdots = g_r(\underline{a}) = 0\}$. Then

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Let $g_1, \ldots, g_r \in \mathbb{C}\langle \underline{X} \rangle$

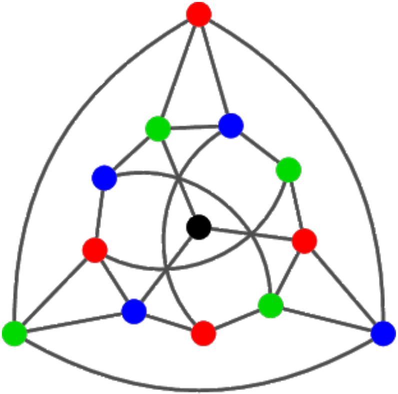
Theorem (Amitsur Nullstellensatz)

Let $Z(I) := \{\underline{A} \in R^n \mid R \text{ primitive ring }, g_1(\underline{A}) = \cdots = g_r(\underline{A}) = 0\}$. Then

$$Z(I) = \varnothing \Leftrightarrow 1 \in (g_1, \ldots, g_r).$$

- We have an algorithm to compute NC Gröbner bases, but it might not terminate...

Against all odds...[1]



- Gröbner basis: $4 \leq \chi_q(G_{13})$

[1] with Piovesan, Mancinska, Roberson

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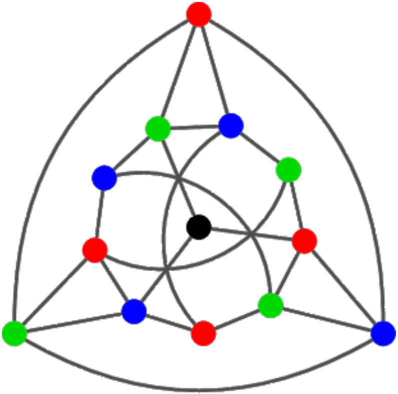

- Gröbner basis: $4 \leq \chi_q(G_{13}) \leq \chi(G_{13}) = 4$
- Consequence $\chi_q(G_{14}) = 4 < 5 = \chi(G_{14})$

[1] with Piovesan, Mancinska, Roberson

Final Remarks

- Quantum theory gives archimedean property for NC-SOS relaxations
- dual side (linear forms & moments) offers even more bounds (Laurent et al.)
- We can transfer the flatness machinery & might obtain concrete optimizer/strategies

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Open problems

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- Is there always a finite dimensional solution/strategy for a finite game?
- How can we detect optimality if there is no finite dimensional solution?

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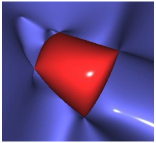
Thank you for your attention.

POEMA

Polynomial Optimization, Efficiency through Moments and Algebra

Marie Skłodowska-Curie Innovative Training Network

2019-2022



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3. Sorbonne Université, Paris, France (Mohab Safey el Din)
4. NWO-I/CWI, Amsterdam, the Netherlands (Monique Laurent)
5. Univ. Tilburg, the Netherlands (Etienne de Klerk)
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