The Pythagoras numbers of projective varieties.

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Outline.

1. The Pythagoras numbers of homogeneous polynomials.
2. The Pythagoras numbers of projective varieties.
3. Applications.

Definition.

A polynomial $f \in \mathbb{R}[X_0, \ldots, X_n]_{2d}$ *is a sum-of-squares if there exist an integer* $t > 0$ *and polynomials* $g_1, \ldots, g_t \in \mathbb{R}[X_0, \ldots, X_n]_d$ *such that*

$$f = g_1^2 + \cdots + g_t^2.$$

Obs. A given sum-of-squares f has many different representations.

Example:

$$2x^2 + 2y^2 = \left(\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 + \left(-\frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}y\right)^2 + x^2 + y^2$$

Definition.

The **sum-of-squares length** *of f, denoted $\ell(f)$ is the smallest integer r such that there exist forms $g_1, \dots, g_r \in \mathbb{R}[X_0, \dots, X_n]_d$ with*

$$f = g_1^2 + \cdots + g_r^2.$$

Denote by $\Sigma_{n,d}$ the set of sums-of-squares of forms of degree d in $n+1$-variables.

Definition. (Choi-Lam-Reznick, Scheiderer)

The **Pythagoras number** $\Pi_{n,d}$ *is the smallest number of squares that suffices to express EVERY sum of squares in $\Sigma_{n,d}$. Equivalently*

$$\Pi_{n,d} := \max_{f \in \Sigma_{n,d}} \ell(f)$$

Motivation: Finding SOS certificates

Finding SOS certificates is a good method for verifying the nonnegativity of f. If f is indeed SOS, how do we find such an expression?

Observation. *Let $\vec{m}$ be the vector of all monomials of degree at most d. Then f is SOS iff there exists a positive semidefinite matrix $A \succeq 0$ such that*

$$f = \vec{m}^t A \vec{m}.$$

Remark.

f can be written as a sum of r-squares of linear combinations of the components of $\vec{m}$ iff some A with rank r satisfies $f = \vec{m}^t A \vec{m}$.

Observation. *Let $\vec{m}$ be the vector of all monomials of degree at most d. Then f is SOS iff there exists a positive semidefinite matrix $A \succeq 0$ such that*

$$f = \vec{m}^t A \vec{m}.$$

- $(+)$ The computational complexity of this problem is unknown but such identites can often be found numerically in polynomial time and often rounded to prove the existence of exact certificates.
- (-) Even numerical solutions to the above problem can only be computed when the matrix is relatively small (max 10000×10000 using state of the art augmented Lagrangian solvers). The dimensions of A are $\binom{n+d}{d} \times \binom{n+d}{d}$.

The Burer-Monteiro approach

An alternative approach:

1. *Replace our PSD matrix A with a factorization $A = Y^t Y$ where Y has size $r \times \binom{n+d}{d}$ where r is such that the identity $f = \vec{m}^t A \vec{m}$ has a solution of rank r and*
2. *Use non-convex (i.e. local) optimization algorithms to* $\min_Y \|f - \vec{m}^t Y^t Y m\|$

- (Advantage) The above problem has a much smaller search space for Y. Although non-convex it often finds global minima [Bumal, Voroninski, Bandeira, 2018].
- (Key point) In order to guarantee correct behavior **we would like an a-priori bound for** r, depending on information which is easy to obtain from the problem at hand (for instance number of variables and degree).

The numbers $\Pi_{n,d}$ give Burer-Monteiro bounds.

How to find a bound for r?

Note that any $r \geq \ell(f)$ would work and $r = \Pi_{n,d}$ would work for EVERY $f \in \Sigma_{n,d}$.

Theorem. (Scheiderer)

If the Iarrobino-Kanev conjectures about Hilbert functions of $I(Z)^2$ for generic points Z hold then the number $\Pi_{n,d} \sim d^{\frac{n}{2}}$

Outline

In this talk we will generalize the Pythagoras numbers $\Pi_{n,d}$ *to all projective varieties*.

The natural operations of projective geometry will allow us to obtain *computable* lower and upper bounds for this quantity (via **Quadratic Persistence** and **Algebraic Treewidth**).

We will study situations in which the upper and lower bounds coincide. As applications we obtain a classification of extremal varieties and the exact computation of Pythagoras numbers for some varieties.

Real algebraic varieties

Let $S := \mathbb{R}[X_0, \ldots, X_n]$ *and let* $I \subseteq S$ *be a homogeneous, real-radical ideal which does not contain any linear form.*

To the ideal I *we associate:*

1. *A (totally) real projective variety*

$$X := V(I) \subseteq \mathbb{P}^n$$

   not contained in any hyperplane.

2. *A graded* $\mathbb{R}$*-algebra*

$$\mathbb{R}[X] := S/I$$

   the homogeneous coordinate ring of X*.*

The Pythagoras number of a variety X

Definition.

Let Σ_X be the set of sums of squares of linear forms in R_2,

$$\Sigma_X := \left\{q \in \mathbb{R}[X]_2 : \ \exists k \in \mathbb{N} \text{ and } s_i \in \mathbb{R}[X]_1 (q = s_1^2 + \cdots + s_k^2)\right\}$$

Definition.

If $f \in \Sigma_X$ the sum-of-squares length of f is defined as

$$\ell(f) = \min \left\{ k \in \mathbb{N} : \exists k, g_1, \ldots, g_k \in \mathbb{R}[X]_1 \left(f = \sum_{i=1}^{k} g_i^2 \right) \right\}.$$

Definition.

The Pythagoras number of X is defined as $\Pi(X) := \max_{f \in \Sigma_X} \ell(f)$.

Sanity Check

Example:

What is the Pythagoras number of $\mathbb{P}^n$?

Example:

Let $X = V(X^2 + Y^2 - Z^2) \subseteq \mathbb{P}^2$. What is the Pythagoras number of X?

Sanity Check

Example:

What is the Pythagoras number of $\mathbb{P}^n$*?*
If $q = x^t Ax$ *then by changing coordinates we can write*

$$q = \sum_{i=0}^{n} \lambda_i X_i^2 = \sum_{i=0}^{n} \left(\sqrt{\lambda_i} X_i \right)^2$$

so $\Pi(X) = n + 1$*.*

Example:

If $X \subseteq \mathbb{P}^2$ *is defined by the ideal* $(X^2 + Y^2 - Z^2) \subseteq \mathbb{R}[X, Y, Z]$ *then*

$$g := X^2 + Y^2 + Z^2 = 2Z^2.$$

so $\ell(g) = 1$*. In fact* $\Pi(X) = 2$*.*

Veronese embeddings

Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ be the map sending $[x_0 : \cdots : x_n]$ to the vector of all monomials of degree d in variables $x_0, \ldots, x_n$.
$X := \nu_d(\mathbb{P}^n)$

Example:

$$\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$$

$$\nu_2([x : y : z]) = [x^2 : y^2 : z^2 : xy : xz : yz]$$

$X := \nu_2(\mathbb{P}^2)$ *is the Veronese surface in* $\mathbb{P}^5$

The linear forms in $X := \nu_d(\mathbb{P}^n)$ *correspond to forms of degree* d *in* $\mathbb{P}^n$ *and the quadratic forms in* X *correspond to forms of degree* $2d$ *in* $\mathbb{P}^n$*.*

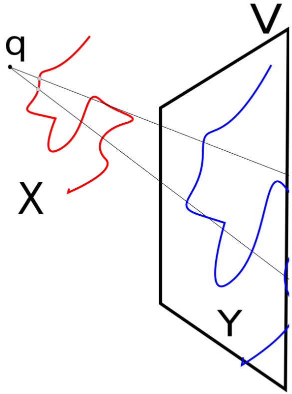
Example:

If $X := \nu_d(\mathbb{P}^n)$ *then* $\Pi(X) := \Pi_{n,d}$.

Lower bounds from quadratic persistence.

Definition.

Let $q \in \mathbb{P}^n$ be a point in projective space and let V be a hyperplane not containing q. we define the **projection away from** *q, $\pi_q : \mathbb{P}^n \setminus \{q\} \to V \cong \mathbb{P}^{n-1}$ by sending a point x to the unique point of intersection between the line $\langle q, x \rangle$ and V.*



Properties of projections.

If $q \in X \subseteq \mathbb{P}^n$ is a generic real point then

1. *$Y := \overline{\pi_q(X)} \subseteq \mathbb{P}^{n-1}$ is a non-degenerate variety and*
2. *(**Key property**) The cone Σ_Y is isomorphic to the face F of Σ_X consisting of sums of squares vanishing at q.*

Properties of projections.

Lemma.

If $Y = \pi_q(X)$ *then* $\Pi(X) \geq \Pi(Y)$

Proof.

$$\Pi(X) = \max_{f \in \Sigma_X} \ell(f) \geq \max_{f \in \Sigma_Y} \ell(f) = \Pi(Y)$$

$\square$

Lower bounds from projections away from real points.

Lemma.

If $Y = \pi_q(X)$ then $\Pi(X) \geq \Pi(Y)$

So we would like to keep projecting away from points until we can actually compute the right hand side. This would be very easy if Y was projective space itself...

Definition.

The **quadratic persistence** $\mathrm{qp}(X)$ *of a variety X is the cardinality s of the smallest set of points $q_1, \ldots, q_s$ for which the ideal of the projection $\pi_{\{q_1,\ldots,q_s\}}(X) \subseteq \mathbb{P}^{n-s}$ contains no quadrics.*

A lower bound Theorem

Theorem. (Blekherman, Smith, Sinn, -)

If $X \subseteq \mathbb{P}^n$ is irreducible and nondegenerate then the following inequalities hold:

$$\mathrm{qp}(X) \leq \mathrm{codim}(X)$$

$$\Pi(X) \geq n + 1 - \mathrm{qp}(X)$$

It follows that $\Pi(X) \geq \dim(X) + 1$.

Question.

1. *What are the varieties with small Pythagoras number?*
2. *What are the varieties with large quadratic persistence?*

Classification of varieties with small Pythagoras number

Theorem. (Blekherman, Plaumann, Sinn, Vinzant)

Assume X is irreducible. The variety X is of minimal degree if and only if $\Pi(X) = \dim(X) + 1$.

Theorem. (Blekherman, Sinn, Smith, -)

Assume X is irreducible. The variety X is of minimal degree if and only if $\mathrm{qp}(\mathrm{X}) = \mathrm{codim}(\mathrm{X})$.

Remark.

For these varieties the lower bound is an equality
$\Pi(X) = n + 1 - \mathrm{qp}(\mathrm{X})$

Varieties of minimal degree

The degree of any non-degenerate projective variety $X \subseteq \mathbb{P}^n$ satisfies the inequality

$$\deg(X) \geq \operatorname{codim}(X) + 1.$$

Definition.

$X \subseteq \mathbb{P}^n$ *is* **of minimal degree** *if the equality holds*

The classification of varieties of minimal degree in projective spaces is known since the 1880s [Castelnuovo, Del Pezzo]. They are cones over

1. A quadric hypersurface or
2. The Veronese surface $\nu_2(\mathbb{P}^2) \subseteq \mathbb{P}^5$ or
3. A rational normal scroll, the projective toric variety corresponding to a Lawrence prism with heights $(a_0, \ldots, a_n)$.

Classification of varieties with small Pythagoras number

Theorem. (Blekherman, Smith, Sinn, -)

Let X be irreducible and Arithmetically Cohen-Macaulay. The following statements are equivalent:

1. $\Pi(X) = \dim(X) + 2$
2. $\mathrm{qp}(\mathrm{X}) = \mathrm{codim}(\mathrm{X}) - 1.$

In particular $\Pi(X) = n + 1 - \mathrm{qp}(\mathrm{X})$

Theorem. (Blekherman, Smith, Sinn, -)

Such varieties can be classified as either:

1. *X is a variety of almost minimal degree (i.e. $\deg(X) = \mathrm{codim}(\mathrm{X}) + 2$) or*
2. *X is a subvariety of codimension one in a variety of minimal degree.*

Back to quadratic persistence...

We know that quadratic persistence is an algebraic invariant which:

1. Takes values in $[0, 1, \ldots, \mathrm{codim}(X)]$
2. It assumes its maximum value only on varieties of minimal degree.

Definition.

The length of the linear strand of $\mathbb{C}[X]$ is given by

$$b(X) := \min\{i \in \mathbb{N} : Tor_j(\mathbb{C}[X], \mathbb{C})_{j+1} = 0 \text{ for all } j \geq i\}$$

Theorem. (Green's $K_{p,1}$-Theorem)

The length of the linear strand of the free resolution of X is at most $1 + \mathrm{codim}(\mathrm{X})$ and equality holds iff X is a variety of minimal degree.

Using the BGG correspondence we prove,

Theorem. (Blekherman, Sinn, Smith, -)

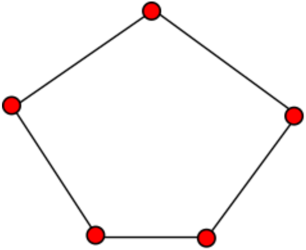
The inequality $b(X) \leq 1 + \mathrm{qp}(X)$

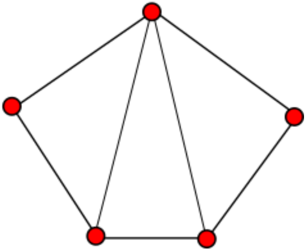
Upper bounds: Chordal graphs and combinatorial treewidth

Let G be an undirected, loopless graph.

Definition.

A graph G is chordal if it does not contain induced cycles of length $\ell \geq 4$.

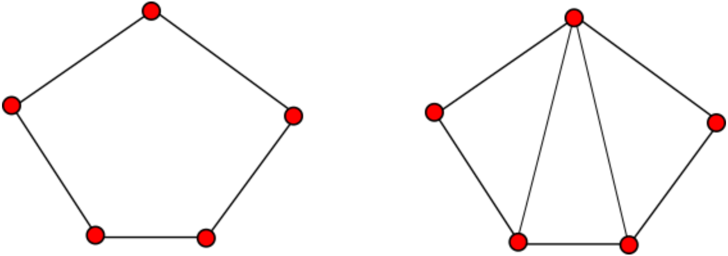




Chordal graphs and combinatorial treewidth

Definition.

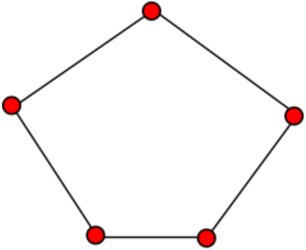
A graph C is a chordal cover of a graph G if $V(G) = V(C)$, $E(G) \subseteq E(C)$ and C is chordal.

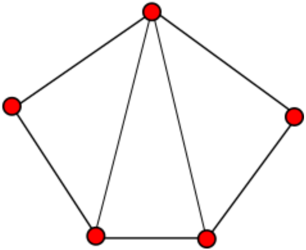


Chordal graphs and combinatorial treewidth

Definition.

The treewidth of a graph G is the smallest clique number of its chordal covers (minus one).





Informally, tree-width measures how tree-like is a graph. It is an important concept because NP-complete problems are "easy" on graphs with small treewidth. Computing TreeWidth is NP hard.

An upper bound from combinatorial treewidth

To a graph G in $[n]$ we can associate an ideal $I_G = (x_i x_j : (i,j) \notin E(G)) \subseteq k[x_1, \ldots, x_n]$ and a variety $X_G := V(I_G) \subseteq \mathbb{P}^{|V|-1}$.

Theorem. (Laurent, Varvitsiotsis)

The inequality $\Pi(X_G) \leq 1 + \mathrm{tw}(G)$ *holds.*

Commutative algebra and chordal graphs

To a graph G in $[n]$ we can associate an ideal $I_G = (x_i x_j : (i,j) \notin E(G)) \subseteq k[x_1, \ldots, x_n]$.

Theorem. (Fröberg)

The graph G is chordal if and only if the ideal I_G has Castelnuovo-Mumford regularity 2.

Varieties of regularity two are the "chordal graphs" of algebraic geometry.

Chordal varieties

Theorem. (Fröberg)

The graph G is chordal if and only if the ideal I_G has Castelnuovo-Mumford regularity 2.

What are all "chordal-like" varieties? (i.e. those of regularity two)

Theorem. (Eisenbud, Green, Hulek, Popescu)

The varieties of regularity two are precisely the "linear joins" of varieties of minimal degree.

Varieties of regularity two interpolate between "Chordal graphs" and "varieties of minimal degree".

Theorem. (Blekherman, Plaumann, Sinn, Vinzant)

For varieties of regularity two $\Pi(X) := \dim(X) + 1$

Upper bounds for Pythagoras numbers from inclusions

Suppose $X, Y \subseteq \mathbb{P}^n$ are real varieties.

Lemma.

If $X \subseteq Y$ then $\Pi(X) \leq \Pi(Y)$.

This would be very useful if we could compute, or even just bound, the right hand side...

Algebraic treewidth and an Upper Bound Theorem

Definition.

The algebraic treewidth of a variety $X \subseteq \mathbb{P}^n$, denoted $\mathrm{tw}(X)$ is the smallest dimension of a variety Y of regularity 2 with $X \subseteq Y$.

Theorem. (Blekherman,Sinn,Smith,-)

The inequality $\Pi(X) \leq \mathrm{tw}(X) + 1$ holds.

Theorem. (Blekherman,Sinn,Smith,-)

Suppose Y is a variety of minimal degree and $X \subseteq Y$. If $\mathrm{qp(X)} = \mathrm{qp(Y)}$ then:

1. $\mathrm{tw}(X) = \mathrm{dim}(Y)$
2. $b(X) = b(Y)$
3. $\Pi(X) = 1 + \mathrm{tw}(X) = n + 1 - \mathrm{qp}(X)$

Example:

Let P be a lattice polytope and let $C_k := P \times [0, k]$. For all sufficiently large k the equality $\Pi(X(Q)) = 1 + \#P$ holds.

An open problem.

Theorem. (Scheiderer)

For all $d \geq 2$ the following inequalities

$$d + 1 \leq \Pi\left(\nu_d(\mathbb{P}^2)\right) \leq d + 2$$

Moreover:

d	Π
2	3
3	4
4	**??**
5	**??**
6	**??**
$\vdots$	$\vdots$