Nonlinear algebra and matrix completion

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- ICERM

Motivation

Problem

Let $\Omega \subseteq [m] \times [n]$. For a given Ω-partial matrix $X \in \mathbb{C}^{\Omega}$, the low-rank matrix completion problem is

$$\text{Minimize } \operatorname{rank}(M) \quad \text{subject to} \quad M_{ij} = X_{ij} \text{ for all } (i,j) \in \Omega$$

Example

Let $\Omega = \{(1,1),(1,2),(2,1)\}$ and consider the following Ω-partial matrix

$$X = \begin{pmatrix} 1 & 2 \\ 3 & \cdot \end{pmatrix}.$$

Some applications:

- Collaborative filtering (e.g. the "Netflix problem")
- Computer vision
- Existence of MLE in Gaussian graphical models (Uhler 2012)

State of the art: nuclear norm minimization

The ***nuclear norm*** of a matrix, denoted $\|\cdot\|_*$, is the sum of its singular values

Theorem (Candès and Tao 2010)

Let $M \in \mathbb{R}^{m\times n}$ be a fixed matrix of rank r that is sufficiently "incoherent." Let $\Omega \subseteq [m]\times[n]$ index a set of k entries of M chosen uniformly at random. Then with "high probability," M is the unique solution to

$$\begin{aligned} &\text{minimize} && \|X\|_* \\ &\text{subject to} && X_{ij} = M_{ij} \quad \text{for all } (i,j)\in\Omega. \end{aligned}$$

The upshot: the minimum rank completion of a partial matrix can be recovered via semidefinite programming if:

- the known entries are chosen uniformly at random
- the completed matrix is sufficiently "incoherent"

Goal: use algebraic geometry to understand the structure of low-rank matrix completion and develop methods not requiring above assumptions

The algebraic approach

Some subsets of entries of a rank-r matrix satisfy nontrivial polynomials.

Example

If the following matrix has rank 1, then the bold entries must satisfy the following polynomial

$$\begin{pmatrix} \mathbf{x_{11}} & \mathbf{x_{12}} & x_{13} \\ \mathbf{x_{21}} & x_{22} & \mathbf{x_{23}} \\ x_{31} & \mathbf{x_{32}} & \mathbf{x_{33}} \end{pmatrix} \qquad x_{12}x_{21}x_{33} - x_{13}x_{31}x_{11} = 0$$

Király, Theran, and Tomioka propose using these polynomials to:

- Bound rank of completion of a partial matrix from below
- Solve for missing entries

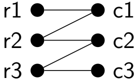
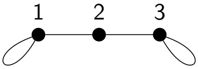
Question

Which subsets of entries of an $m \times n$ matrix of rank r satisfy nontrivial polynomials?

Graphs and partial matrices

Subsets of entries of a matrix can be encoded by graphs:

- non-symmetric matrices $\to$ bipartite graphs
- symmetric matrices $\to$ semisimple graphs

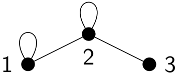
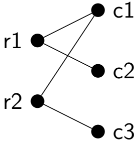
$\mathrm{Mat}_r^{m\times n}$	$m \times n$ matrices of rank $\leq r$	$\begin{pmatrix} 5 & \cdot & \cdot \\ -4 & -2 & \cdot \\ \cdot & 8 & 3 \end{pmatrix}$	
$\mathrm{Sym}_r^{n\times n}$	$n \times n$ symmetric matrices of rank $\leq r$	$\begin{pmatrix} 7 & 4 & \cdot \\ 4 & \cdot & 9 \\ \cdot & 9 & 5 \end{pmatrix}$	

- A ***G-partial matrix*** is a partial matrix whose known entries lie at the positions corresponding to the edges of G.
- A ***completion*** of a G-partial matrix M is a matrix whose entries at positions corresponding to edges of G agree with the entries of M.

Generic completion rank

Definition

Given a (bipartite/semisimple) graph G, the ***generic completion rank of*** G, denoted $\mathrm{gcr}(G)$, is the minimum rank of a **complex** completion of a G-partial matrix **with generic entries**.

type	G	pattern	$\mathrm{gcr}(G)$
symm		$\begin{pmatrix} a_{11} & ? \\ ? & a_{22} \end{pmatrix}$	1
symm		$\begin{pmatrix} a_{11} & a_{12} & ? \\ a_{12} & a_{22} & a_{23} \\ ? & a_{23} & ? \end{pmatrix}$	2
non		$\begin{pmatrix} a_{11} & a_{12} & ? \\ a_{21} & ? & a_{23} \end{pmatrix}$	1

Generic completion rank

Problem

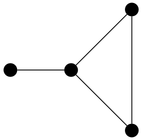
Gain a combinatorial understanding of generic completion rank - how can one use the combinatorics of G to infer $\operatorname{gcr}(G)$?

Proposition (Folklore)

Given a bipartite graph G, $\operatorname{gcr}(G) \leq 1$ iff G has no cycles.

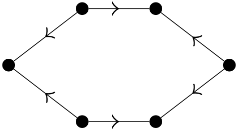
Proposition (Folklore)

Given a semisimple graph G, $\operatorname{gcr}(G) \leq 1$ iff G has no even cycles, and every connected component has at most one odd cycle.

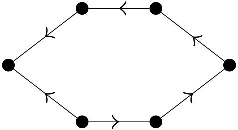


Generic completion rank 2 - nonsymmetric case

A cycle in a directed graph is ***alternating*** if the edge directions alternate.



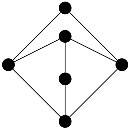
Alternating cycle



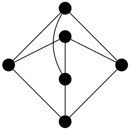
Non-alternating cycle

Theorem (B.-, 2016)

Given a bipartite graph G, $\mathrm{gcr}(G) \leq 2$ if and only if there exists an acyclic orientation of G that has no alternating cycle.



$\mathrm{gcr}(G) = 2$



$\mathrm{gcr}(G) = 3$

Generic completion rank 2 - nonsymmetric case

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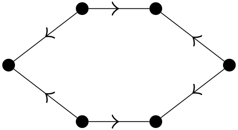
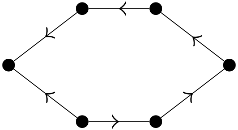


Alternating cycle



Non-alternating cycle

Theorem (B.-, 2016)

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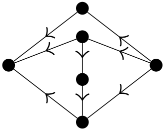
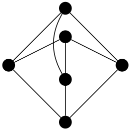


$\mathrm{gcr}(G) = 2$



$\mathrm{gcr}(G) = 3$

Proof sketch

Theorem (B.-, 2016)

Given a bipartite graph G, $\mathrm{gcr}(G) \leq 2$ if and only if there exists an acyclic orientation of G that has no alternating cycle.

- Rephrase the question: describe the independent sets in the algebraic matroid underlying the variety of $m \times n$ matrices of rank at most 2
- This algebraic matroid is a restriction of the algebraic matroid underlying a Grassmannian $\mathrm{Gr}(2, N)$ of affine planes
- Algebraic matroid structure is preserved under tropicalization
- Apply Speyer and Sturmfels' result characterizing the tropicalization of $\mathrm{Gr}(2, N)$ in terms of tree metrics to reduce to an easier combinatorial problem

Open question

Does there exist a polynomial time algorithm to check the combinatorial condition in the above theorem, or is this decision problem NP-hard?

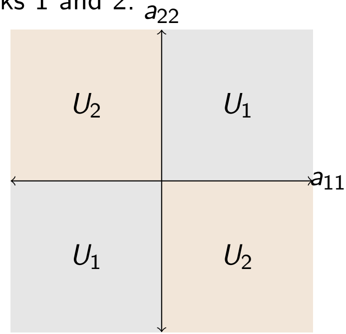
Issue: real vs complex

What happens when you only want to consider ***real*** completions?

Definition

Given a bipartite or semisimple graph G, there may exist multiple open sets $U_1, \ldots, U_k$ in the space of ***real*** G-partial matrices such that the minimum rank of a completion of a partial matrix in U_i is r_i. We call the r_is the ***typical ranks of*** G.

The graph  has typical ranks 1 and 2.



$$\begin{pmatrix} a_{11} & \cdot \\ \cdot & a_{22} \end{pmatrix}$$

In a completion to rank 1, the missing entry t must satisfy $a_{11}a_{22} - t^2 = 0$.

Facts about typical ranks

Proposition (B.-Blekherman-Sinn 2018)

Let G be a bipartite or semisimple graph.

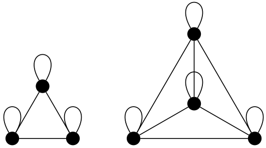
1. *The minimum typical rank of G is $\operatorname{gcr}(G)$.*
2. *The maximum typical rank of G is at most $2\operatorname{gcr}(G)$.*
3. *All integers between $\operatorname{gcr}(G)$ and the maximum typical rank of G are also typical ranks of G.*

See also Bernardi, Blekherman, and Ottaviani 2015 and Blekherman and Teitler 2015.

Case study: disjoint union of cliques

Let $K_m \sqcup K_n$ denote the disjoint union of two cliques with all loops

$K_3 \sqcup K_4 =$



$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & ? & ? & ? & ? \\ a_{12} & a_{22} & a_{23} & ? & ? & ? & ? \\ a_{13} & a_{23} & a_{33} & ? & ? & ? & ? \\ ? & ? & ? & a_{44} & a_{45} & a_{46} & a_{47} \\ ? & ? & ? & a_{45} & a_{55} & a_{56} & a_{57} \\ ? & ? & ? & a_{46} & a_{56} & a_{66} & a_{67} \\ ? & ? & ? & a_{47} & a_{57} & a_{67} & a_{77} \end{pmatrix}$$

Proposition (B.-Blekherman-Lee)

The generic completion rank of $K_m \sqcup K_n$ is $\max\{m, n\}$. The maximum typical rank of $K_m \sqcup K_n$ is $m + n$.

Case study: disjoint union of cliques

Proposition (B.-Blekherman-Lee)

The generic completion rank of $K_m \sqcup K_n$ is $\max\{m, n\}$. The maximum typical rank of $K_m \sqcup K_n$ is $m + n$.

A $(K_m \sqcup K_n)$-partial matrix looks like:

$$M = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}.$$

By Schur complements:

$$\text{rank}(M) = \text{rank}(A) + \text{rank}(B - X^T A^{-1} X).$$

If $A \prec 0$ and $B \succ 0$, then $\det(B - X^T A^{-1} X) > 0$ for real X.

Corollary

Every integer between $\max\{m, n\}$ and $m + n$ is a typical rank of $K_m \sqcup K_n$.

Case study: disjoint union of cliques

Given real symmetric matrices A and B of full rank, of possibly different sizes:

- p_A (p_B) denotes the number of positive eigenvalues of A (B)
- n_A (n_B) denotes the number of negative eigenvalues of A (B)
- the ***eigenvalue sign disagreement of*** A ***and*** B is defined as:

$$\mathrm{esd}(A, B) := \begin{cases} 0 & \text{if } (p_A - p_B)(n_A - n_B) \geq 0 \\ \min\{|p_A - p_B|, |n_A - n_B|\} & \text{otherwise} \end{cases}$$

Theorem (B.-Blekherman-Lee)

Let $M = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}$ *be a generic real* $K_m \sqcup K_n$*-partial matrix. Then* M *is minimally completable to rank* $\max\{m, n\} + \mathrm{esd}(A, B)$.

When full rank is typical

Theorem (B.-Blekherman-Lee)

Let G be a semisimple graph on n vertices. Then n is a typical rank of G if and only if the complement graph of G is bipartite.

If the complement is bipartite, then n is a typical rank:

$$M = \begin{pmatrix} A & X \\ X^T & B \end{pmatrix}$$

By Schur complements:

$$\text{rank}(M) = \text{rank}(A) + \text{rank}(B - X^T A^{-1} X),$$

so if $A \prec 0$ and $B \succ 0$, then $\det(B - X^T A^{-1} X)$ is strictly positive.

When full rank is typical

Theorem (B.-Blekherman-Lee)

Let G be a semisimple graph on n vertices. Then n is a typical rank of G if and only if the complement graph of G is bipartite.

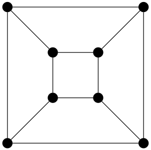
If complement is ***not*** bipartite, then n is ***not*** a typical rank:

- A graph is bipartite if and only if it is free of odd cycles
- If complement graph ***is*** an odd cycle, then determinant of a G-partial matrix, viewed as a polynomial in the unknown entires, has odd degree
- Deleting edges from a graph will not increase maximum typical rank.

$$\begin{pmatrix} a_{11} & \mathbf{x} & a_{13} & a_{14} & \mathbf{t} \\ \mathbf{x} & a_{22} & \mathbf{y} & a_{24} & a_{25} \\ a_{13} & \mathbf{y} & a_{33} & \mathbf{z} & a_{35} \\ a_{14} & a_{24} & \mathbf{z} & a_{44} & \mathbf{w} \\ \mathbf{t} & a_{25} & a_{35} & \mathbf{w} & a_{55} \end{pmatrix}$$

Typical ranks for nonsymmetric matrices: some examples

The following bipartite graph has 2 and 3 as typical ranks.



$$\begin{pmatrix} ? & a_{12} & a_{13} & a_{14} \\ a_{21} & ? & a_{23} & a_{24} \\ a_{31} & a_{32} & ? & a_{34} \\ a_{41} & a_{42} & a_{43} & ? \end{pmatrix}$$

Let $\mathrm{mtr}(G)$ denote the maximum typical rank of G.

Theorem (B.-Blekherman-Sinn)

Let G be obtained by gluing two bipartite graphs G_1 and G_2 along a complete bipartite subgraph $K_{m,n}$. If

$$\max\{\mathrm{mtr}(G_1), \mathrm{mtr}(G_2)\} \geq \max\{m, n\},$$

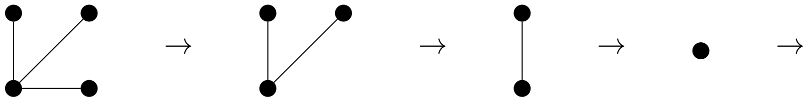
then $\mathrm{mtr}(G) = \max\{\mathrm{mtr}(G_1), \mathrm{mtr}(G_2)\}$. *The same is true for generic completion rank.*

Open question

Does there exist a bipartite graph that has more than two typical ranks?

Empty k-cores

The $k-$core of a graph G is the graph obtained by iteratively removing vertices of degree $k-1$ or less. The 2-core of the graph below is empty.



Theorem (B.-, Blekherman, Sinn)

Let G be bipartite. If the k-core of G is empty, then all typical ranks of G are at most $k-1$.

Corollary

Let G be bipartite. Then the maximum typical rank of G is $2\,\mathrm{gcr}(G)-1$.

Open question

Which bipartite graphs of generic completion rank 2 also have 3 as a typical rank?

Conclusion

- All generic G-partial matrices can be completed to rank $\mathrm{gcr}(G)$ over $\mathbb{C}$
- We can characterize all the bipartite graphs with generic completion rank ≤ 2 (semisimple case is still open)
- Over the reals, a graph can have many typical ranks

Open problems:

- Find a polynomial-time algorithm to decide if a given bipartite graph has an acyclic orientation with no alternating cycle, or prove that this decision problem is NP-hard
- Find a bipartite graph that exhibits three or more typical ranks
- Characterize the graphs with generic completion rank 2 that also exhibit 3 as a typical rank

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