Webs, foams, knot invariants, and representation theory

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Illustrating Number Theory and Algebra
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Overview

1. Knots and their (polynomial) invariants
2. Webs and representation theory
3. Knot homologies and foams
4. Some illustrative consequences

Knots and topology

- A knot is precisely what you think it is: a flexible, closed, knotted piece of string in three-dimensional space.

- Although the study of knots at first appears to be a rather niche problem in topology (i.e. the study of how 1d spaces embed in a certain 3d space), knots provide a means to study 3d topology.
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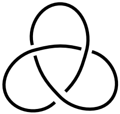

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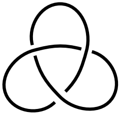

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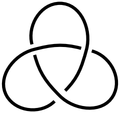

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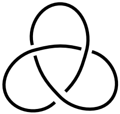

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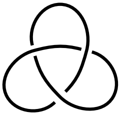

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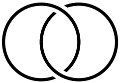
Diagrams for knots

- Despite knots (and links) being inherently 3-dimensional objects, they can be studied via their 2-dimensional diagrams:



Theorem (Reidemeister, 1927)

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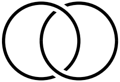

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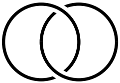

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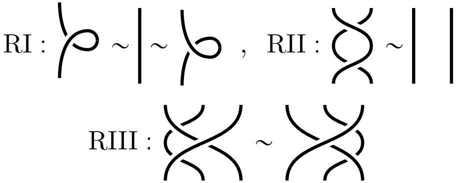

Knot invariants and the Jones polynomial

- Until the 1980's, (most) invariants of knots did not make use of the diagrammatic description afforded by the Reidemeister Theorem, instead being given in terms of "classical" constructions in algebraic topology (e.g. fundamental group, covering spaces, homology).
- In 1985, Jones introduced a polynomial invariant $V_q(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$ for knots $\mathcal{K} \subset S^3$ using algebraic methods (braid group representations).
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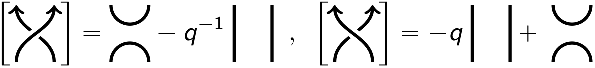

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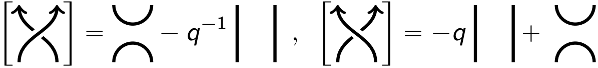

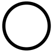 $= [2] := q + q^{-1}$

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The Kauffman bracket

How to interpret $[\mathcal{D}]$?

- As a rule for a "state sum" expansion for the Jones polynomial:

$= [2](q + q^{-3})$

- As a functor from the category of oriented tangles (knot pieces) to the category $\mathcal{TL}$ of $\mathbb{Z}[q, q^{-1}]$-linear combinations of planar curves (modulo the "circle relation"):

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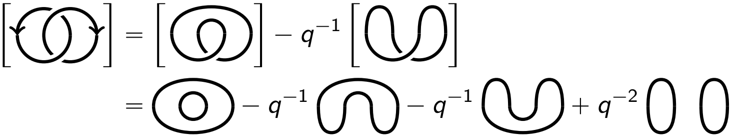

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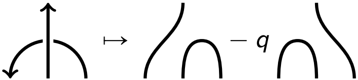

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Q: What is the category $\mathcal{TL}$?

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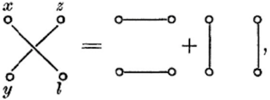
(3) 

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Theorem (Folklore, Rummer-Teller-Weyl)

The category $\mathcal{TL}$ with objects $n \in \mathbb{N}$ and morphisms $n \to m$ consisting of $\mathbb{Z}[q, q^{-1}]$-linear combinations of (m, n) planar curves, modulo the circle relation, is equivalent to the full subcategory of $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$ tensor generated by the standard representation.

- This is a diagrammatic incarnation of results of Reshetikhin-Turaev, that build a knot invariant $P_{\mathfrak{g}}(\mathcal{K}) \in \mathbb{Z}[q, q^{-1}]$ for each simple Lie algebra $\mathfrak{g}$.

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- Following work of Kuperberg, Murakami-Ohtsuki-Yamada (1998) give an analogue of the Kauffman bracket for the $\mathfrak{sl}_n$ knot polynomials:

- Cautis-Kamnitzer-Morrison define a category $n\mathrm{Web}$ presented by generators and relations:

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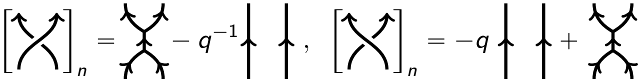

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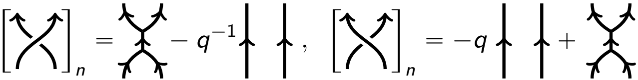

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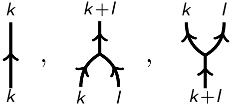

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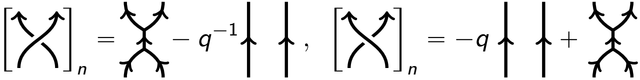

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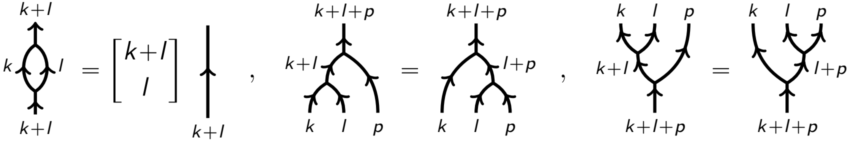

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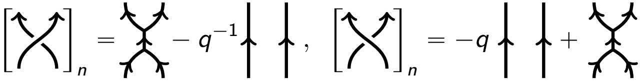

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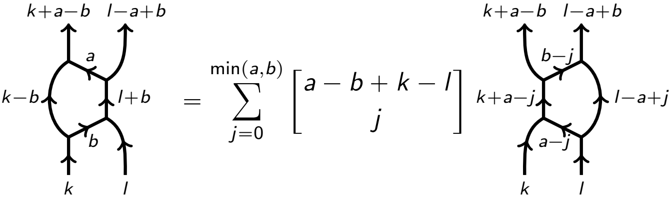

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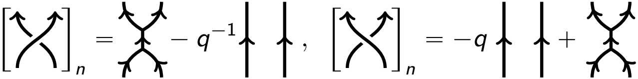

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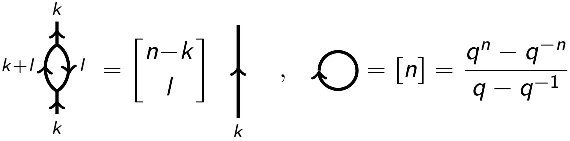

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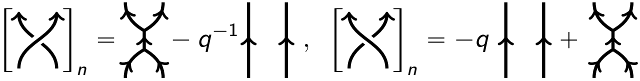

Theorem (Cautis-Kamnitzer-Morrison, 2012)

$n\mathrm{Web}$ is equivalent to the subcategory of $\mathrm{Rep}(U_q(\mathfrak{gl}_n))$ tensor generated by the fundamental representations $\wedge^k V$.

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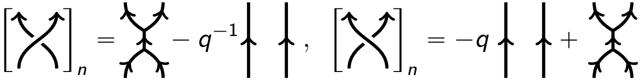

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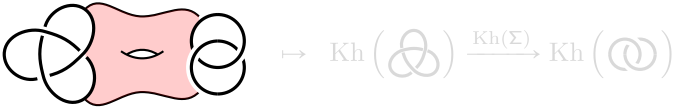

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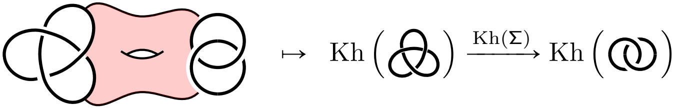

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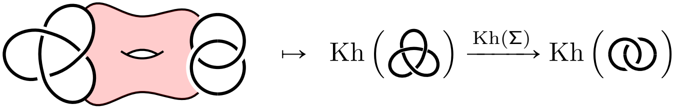

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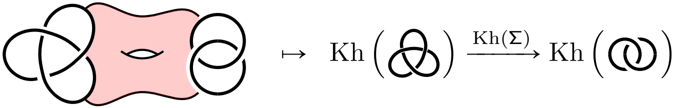

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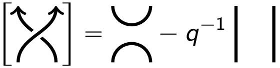

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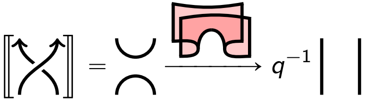

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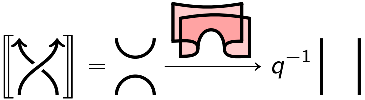

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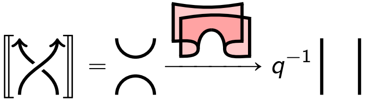

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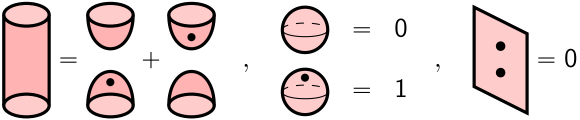

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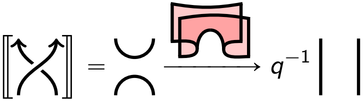

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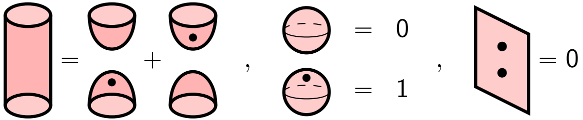

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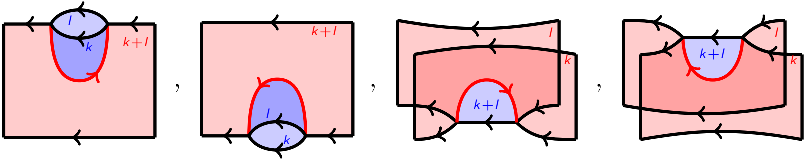

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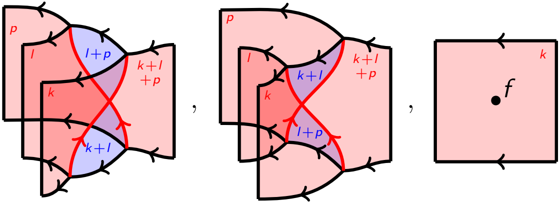

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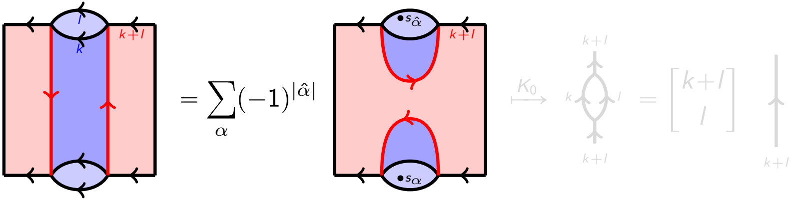

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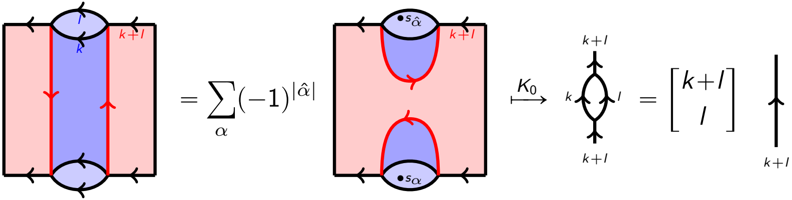

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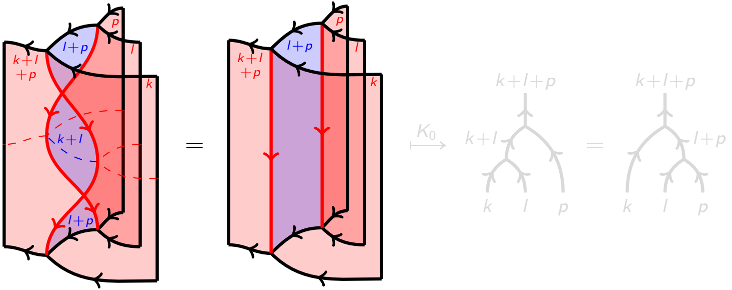

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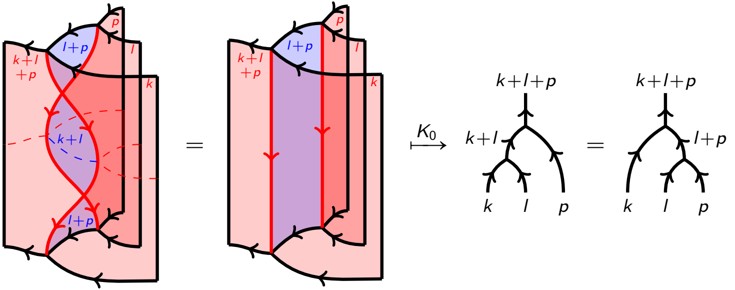

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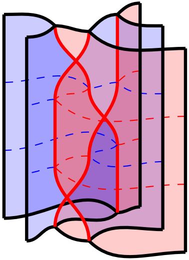
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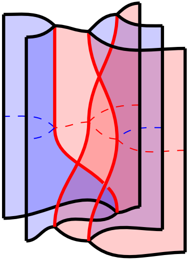
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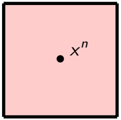

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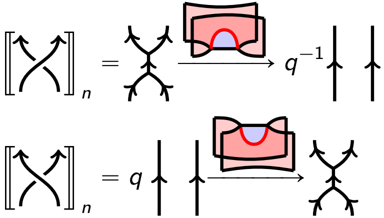

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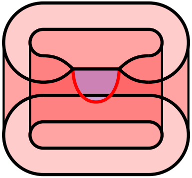

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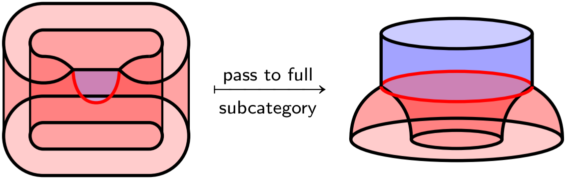

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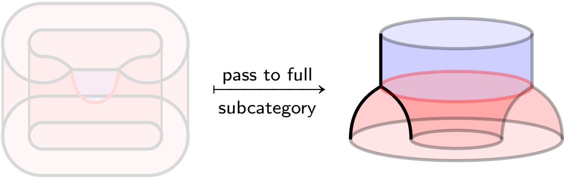

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- On the one hand, $n\mathrm{Foam}$ (and its annular variant) are the setting for various knot homology theories.
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- Results of Rouquier assign a complex of (singular) Soergel bimodules to any braid, and Khovanov builds triply-graded (Khovanov-Rozansky) knot homology using the Hochschild cohomology of these bimodules.

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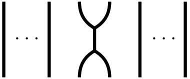
- On the one hand, $n\mathrm{Foam}$ (and its annular variant) are the setting for various knot homology theories.
- On the other, taking the $n \to \infty$ limit gives a 2-category that is a 3d diagrammatic model for the 2-category of singular Soergel bimodules:

 $\leftrightarrow \mathbb{C}[x_1, \ldots, x_m] \otimes_{\mathbb{C}[x_1, \ldots, x_i + x_{i+1}, x_i x_{i+1}, \ldots, x_m]} \mathbb{C}[x_1, \ldots, x_m]$

  The latter is a certain 2-category of bimodules over polynomial rings $\mathbb{C}[x_1, \ldots, x_m]^{W_J}$ equivalent to a 2-category of perverse sheaves on products of partial flag varieties (the “Hecke category” is a full 2-subcategory).
- Results of Rouquier assign a complex of (singular) Soergel bimodules to any braid, and Khovanov builds triply-graded (Khovanov-Rozansky) knot homology using the Hochschild cohomology of these bimodules.

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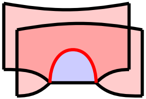 $\leftrightarrow \quad f \otimes g \mapsto fg$

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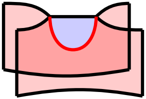 $\leftrightarrow \quad f \mapsto x_1 f \otimes 1 - f \otimes x_2$

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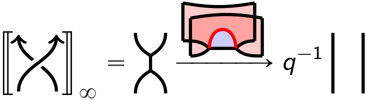

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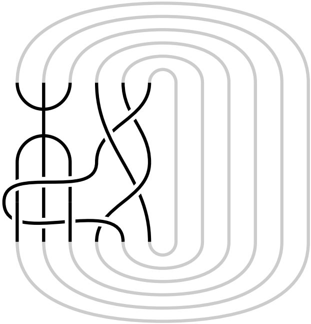
An application: knot homology in a handlebody

- We obtain a complex of singular Soergel bimodules associated to any braided web, and can apply Hochschild cohomology:



- The complement of the web is a handlebody $\mathcal{H}_g$ (i.e. "the inside" of a genus g surface), so this diagram describes a knot $\mathcal{K} \subset \mathcal{H}_g$.

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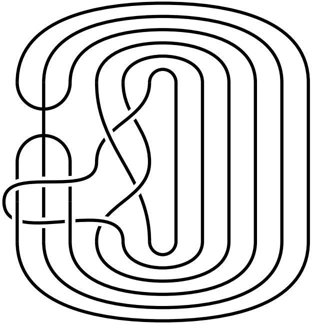

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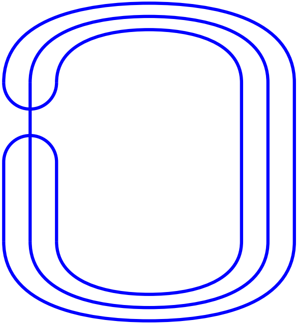

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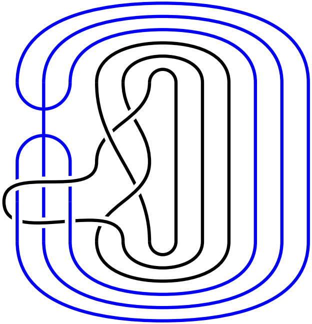

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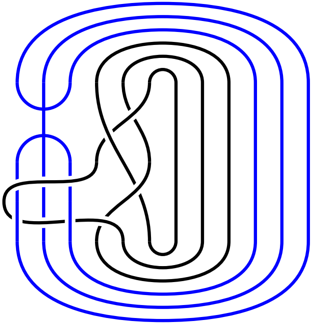

Theorem (R.-Tubbenhauer)

There exists a homology theory for knots in genus g handlebodies, that extends triply-graded (Khovanov-Rozansky) knot homology.

Thanks!