Challenges in the Self-consistent Evolution of Extreme Mass Ratio Inspirals.

Peter Diener[1]
in collaboration with
Barry Wardell[2] and Niels Warburton[2]

[1]Louisiana State University
[2]University College Dublin

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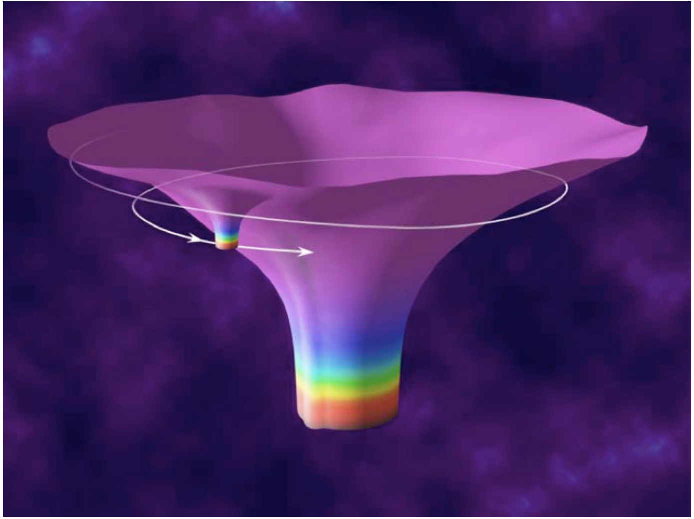
The physics of Extreme Mass Ratio Inspirals.

- Supermassive black holes are surrounded by a cluster of stars.
- Some of these will be black holes or neutron stars.
- Can be brought onto highly eccentric orbits by two-body interactions.
- Energy and angular momentum losses through gravitational wave emission shrinks the orbit until the small object plunges into the supermassive black hole.
- Eccentricity will decrease over time but will most likely still be significant just before the plunge.
- Such systems are expected to be very important events for the space based gravitational wave detector LISA.
- The system can be treated as a background spacetime (with the mass and spin of the large black hole) and with the orbiting small object represented as a point particle.
- The particle perturbs the spacetime and interacts with it's own perturbations (the self-force) to accelerate the orbit thereby causing the inspiral.

The Physics of Extreme Mass Ratio Inspirals.



Artist's impression of an EMRI. Credit NASA

Decomposition into Singular and Regular Fields.

- Treating the small object as a point particle, in practice means using a delta-function source.
- This means that perturbation fields diverge when approaching the particle.
- The diverging retarded field can be decomposed into a singular and a regular piece: $\Psi^{\mathrm{ret}} = \Psi^{\mathrm{S}} + \Psi^{\mathrm{R}}$.
- Only the regular piece contributes to the self-force.
- After decomposing into spherical harmonics, however, each ℓm-mode $\Psi^{\mathrm{S}}_{\ell m}$ is finite.
- It is then possible to extract the singular piece from each mode leaving the regular piece that is solely responsible for the self-force acting on the particle.
- That is: $\Psi^{\mathrm{R}} = \sum_{\ell m} \left(\Psi^{\mathrm{ret}}_{\ell m} - \Psi^{\mathrm{S}}_{\ell m}\right)$.

Decomposition into Singular and Regular Fields.

- Alternatively one can use the effective source approach.
- For simplicity the scalar charge case is considered here, in which case the basic equation is: $\Box\Psi^{\rm ret} = \Box\left(\Psi^{\rm S} + \Psi^{\rm R}\right) = -4\pi q \int \delta(x^\mu, z^\mu(\tau)) d\tau$.
- Inserting a local expansion of the Detweiler-Whiting singular field, $\tilde{\Psi}^{\rm S}$ an equation for $\Psi^{\rm R}$ can be obtained: $\Box\Psi^{\rm R} = -\Box(\mathcal{W}\tilde{\Psi}^{\rm S}) - 4\pi q \int \delta(x^\mu, z^\mu(\tau)) d\tau = S_{\rm eff}$, where $S_{\rm eff}$ is regular (but with limited smoothness) at the location of the particle.
- In this approach we regularize the source and hence do not need to perform mode sum regularization.

Computational Approaches to the Self-force.

- The most accurate calculations of the self-force are done in the frequency domain for prescribed geodesic motion.
- With the field decomposed into spherical harmonics and the time dependence decomposed into Fourier modes what is left to do is to solve ODE's in the radial direction.
- If a delta function source is used then mode sum regularization can be used to calculate the self-force.
- If the effective source approach is used, the resulting regular field can be used to directly calculate the self-force.
- Note: lots of complications have been left out here.
- The self-force can also be calculated in the time domain using either a delta-function source or effective source approach.
- Self-force in the time domain is typically more expensive and less accurate but can also be calculated along accelerated non-periodic orbits.

Approaches to Orbital Evolution.

- Several approaches rely on the radiation reaction time scale being much longer than the orbital time scale for most of the inspiral.
- The simplest approach is kludge waveforms where the radiation and/or orbit may only be treated semi-relativistically.
- The adiabatic approach relies on the sourced Teukolsky equation to calculate gravitational radiation fluxes that can be used to calculate the change of the energy, angular momentum and Carter constant. This takes into account the dissipative part, but ignores the conservative piece of the self-force.
- Geodesic evolution uses both the dissipative and conservative self-force calculated on geodesics and use that to evolve to a neighboring geodesic. This ignores the contribution to the self-force that arises from the fact that the orbit was not geodesic in the path.
- Self-consistent evolution evolves the field and the orbit at the same time. I.e. the self-force is extracted from the current field (consistent with the prior history) and is used to evolve the accelerated orbit that in turn sources the field.

Early Self-consistent Evolution Attempt.

PD, Vega, Wardell, Detweiler, Phys.Rev.Lett. 108 (2012) 191102.

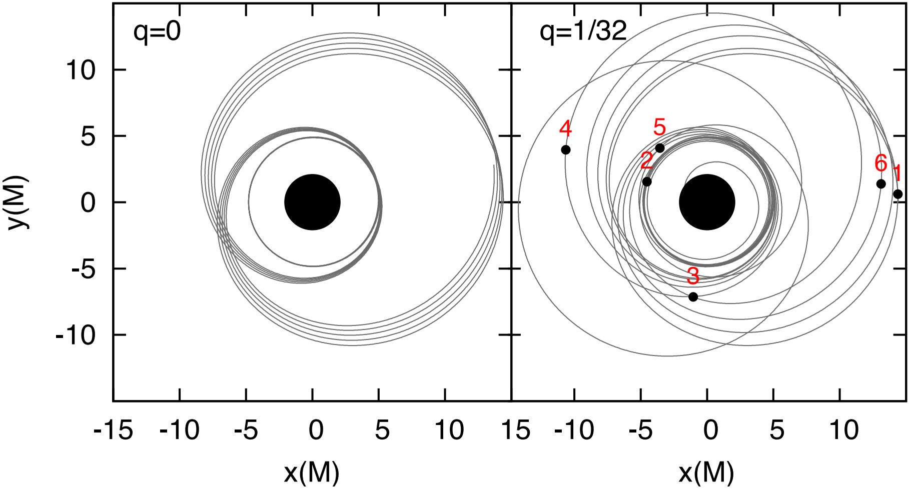
- A 3D multi-block scalar wave equation code.
- Kerr background spacetime in Kerr-Schild coordinates ($a = 0$).
- Spherical inner boundary placed inside the black hole.

Equations:

$$\begin{aligned} \Box\psi^{\mathrm{R}} &= S(x|z^\alpha(\tau), u^\alpha(\tau)), \\ \frac{Du^\alpha}{d\tau} &= \frac{q}{m(\tau)}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\nabla_\beta\psi^{\mathrm{R}}, \\ \frac{dm}{d\tau} &= -qu^\beta\nabla_\beta\psi^{\mathrm{R}}. \end{aligned}$$

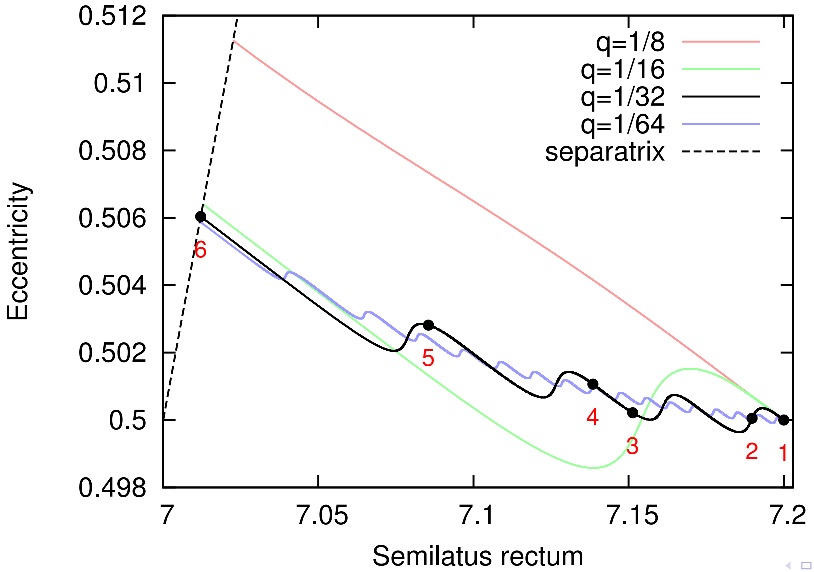
- Spherical outer boundary placed at $\mathscr{I}^+$ using hyperboloidal slicings.
- The field and the particle are evolved together.
  - The particle location $z^\alpha(\tau)$ and four-velocity $u^\alpha(\tau)$ gives the C^0 effective source that determines ψ^{R}.
  - $\nabla_\beta\psi^{\mathrm{R}}$ at the location of the particle in turn affects the orbit.
- We used 8th order summation by parts finite differencing and used penalty boundary conditions at patch boundaries.
- Orbit evolution using the geodesic equations directly as well as using the osculating orbits framework.

Early Self-consistent Evolution Attempt.



$$p = 7.2, e = 0.5, \ r_1 = 4.8M, \ r_2 = 14.4M$$

Early Self-consistent Evolution Attempt.



Problems with Early Self-consistent Evolution Attempt.

- The evolutions initially showed good agreement with the geodesic evolutions provided by Niels Warburton.
- Due to the C^0 effective source, lots of high frequency noise was generated, limiting the accuracy of the extracted self-force to about 0.1 to 1%.
- Attempts to improve on the accuracy by making the effective source smoother ($C^0 \to C^2$) revealed differences in the self-force when back-reaction was turned on.
- We now suspect that this was caused by the effective source not taking into account acceleration information.
- The simulations were very expensive and required super computers.
- Hence it was decided to completely change the computational infrastructure.

SelfForce-1D

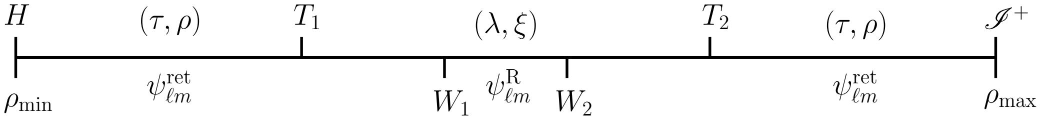
`SelfForce-1D` is an open source code for performing time domain self-force computations.

- Evolves the scalar wave equation (metric perturbation equations are being added) in a Schwarzschild space-time (Kerr is being added).
- Fields are decomposed into Spherical Harmonics resulting in 1+1 dimensional PDE's to be solved using the Method of Lines.
- Nodal Discontinuous Galerkin method being used to discretize the PDE's in the radial direction. Can handle non-smooth features easily.
- Point particle treatment through the Effective Source for generic orbits (including acceleration information) in a world-tube approach (retarded field outside, regular field inside).
- Different coordinate systems are used in different parts of the domain: Hyperboloidal near horizon and $\mathscr{I}^+$, time dependent or Tortoise in between. That is, the computational domain covers everything between the horizon and $\mathscr{I}^+$.

SelfForce-1D (cont)

- Generic orbits evolved using direct geodesic integration (with forces) or through the osculating orbits framework (also with forces).
- Runge-Kutta and Adams-Bashford-Moulton multi-value methods can be used for time integration.
- Self-force can be extracted from the regular field at the particle location.
- Other observers can extract the fields at the horizon and at $\mathscr{I}^+$.
- Can use initial data calculated in the frequency domain for eccentric geodesics for low ℓ-modes in order to avoid having to evolve for a long time before initial transient leaves the computational domain.
- Written mostly in object oriented Fortran with the effective source in C++ (Barry) and initial data in Python (Niels).

The Computational Setup

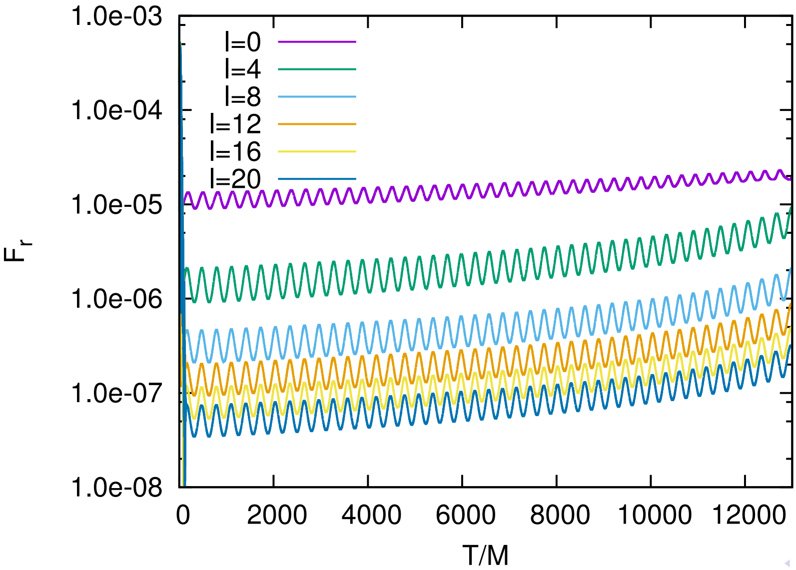


Between H and T_1 ingoing hyperboloidal coordinates (τ, ρ) are used to evolve the retarded field $\psi_{\ell m}^{\mathrm{ret}}$.
Between T_1 and W_1 time dependent coordinates (λ, ξ) are used to evolve the retarded field $\psi_{\ell m}^{\mathrm{ret}}$.
Between W_1 and W_2 time dependent coordinates (λ, ξ) are used to evolve the regular field $\psi_{\ell m}^{\mathrm{R}}$.
Between W_2 and T_2 time dependent coordinates (λ, ξ) are used to evolve the retarded field $\psi_{\ell m}^{\mathrm{ret}}$.
Between T_2 and $\mathscr{I}^+$ outgoing hyperboloidal coordinates (τ, ρ) are used to evolve the retarded field $\psi_{\ell m}^{\mathrm{ret}}$.

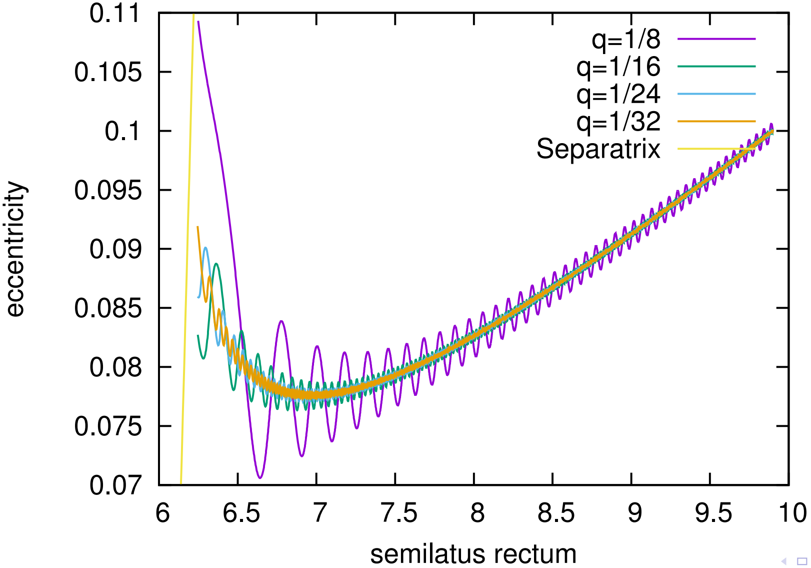
Is the New Code Better?

- In the geodesic phase the relative errors on the individual modes of the self-force are typically better than 10^{-8} to 10^{-6} (depending on ℓ).
- As we have a limited number of modes available, the errors in the total self-force error ($\approx 10^{-5}$) is typically dominated by fitting the expected high $\ell-$mode behavior and extrapolating the mode sum to ∞.
- Much better accuracy is obtained with a code that can run on a workstation.
- With the effective source including acceleration terms, it is formally dependent on the first and second time derivative of the acceleration: $S_{\text{eff}} = S_{\text{eff}}(x|z, u, a, \dot{a}, \ddot{a})$.
- We have to determine $\dot{a}$ and $\ddot{a}$ by one sided finite difference approximations of a.
- When using $\dot{a}$ and/or $\ddot{a}$ information, instabilities gets excited at high $\ell-$modes when back-reaction is turned on and quickly crashes the simulation.
- When only a is used in the effective source we can evolve stably with back reaction.

Self-consistent Evolution of $p = 9.9, e = 0.1, q = 1/8, a, \dot{a}, \ddot{a}$.



Self-consistent Evolution of $p = 9.9, e = 0.1, a, \dot{a}, \ddot{a}$.



Outlook.

- Not known at this point what the error is when we leave out $\dot{a}$ and $\ddot{a}$.
- Attempt to shrink the world-tube to zero size ($\dot{a}$ and $\ddot{a}$ would no longer be needed), resulted in loss of exponential convergence. Need to revisit that.
- Other systems of equations are in the pipeline:
  1. Teukolsky in both Schwarzschild (REU student Skinner, 2019) and Kerr (REU student Sho Gibbs, 2020).
  2. Metric perturbations in Lorenz gauge (Samuel Cupp)
  3. Regge-Wheeler-Zerilli metric perturbations (Samuel Cupp, me)
- Checkpointing/restart (REU student Mary Ogborn, 2020).
- The code is part of both the Einstein Toolkit and the Black Hole Perturbation Toolkit.
- Hopefully the code will be a useful community resource that will inspire new developments that will be contributed back to the toolkits.
- Available at: `https://bitbucket.org/peterdiener/selfforce-1d`

Extra 1: Non-uniformly Accelerated Circular Orbit

$p = 6.7862,\ e = 0.0,\ A = 0.05,\ \sigma = 1.8$