Cosmological Initial Data for Numerical Relativity

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Outline

Why use numerical relativity in cosmology?

To study conditions for inflation

To study conditions for alternatives to inflation

To study spacetime singularities

To test the accuracy of Newtonian cosmological simulations

Initial Data

Initial data for general relativity consists of a spatial metric γ_{ij} and an extrinsic curvature K_{ij} satisfying constraint equations, called the momentum constraint

$$D^i K_{ij} - D_j K = -\gamma^i{}_j T_{i\mu} n^\mu$$

and the Hamiltonian constraint

$${}^{(3)}R + K^2 - K^{ij} K_{ij} = 2 T_{\mu\nu} n^\mu n^\nu$$

Decompose the extrinsic curvature into its trace K and a trace-free part A_{ij} given by

$$A_{ij} = K_{ij} - \frac{1}{3} K \gamma_{ij}$$

Then the constraint equations become

$$D^i A_{ij} - \frac{2}{3} D_j K = -\gamma^i{}_j T_{i\mu} n^\mu$$

$${}^{(3)}R + \frac{2}{3} K^2 - A^{ij} A_{ij} = 2 T_{\mu\nu} n^\mu n^\nu$$

York method

Introduce rescaled quantities $\tilde{\gamma}_{ij}$ and $\tilde{A}_{ij}$ given by

$$\tilde{\gamma}_{ij} = \psi^{-4}\gamma_{ij}$$

and $\tilde{A}_{ij} = \psi^2 A_{ij}$. The quantity $\tilde{A}_{ij}$ is then expressed as

$$\tilde{A}_{ij} = X_{ij} + \tilde{D}_i W_j + \tilde{D}_j W_i - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{\gamma}^{mn}\tilde{D}_m W_n$$

It seems odd to introduce these new quantities ψ and W_i. However, they are essentially "correction terms" to be used to convert an initial guess for a solution of the constraint equations into an actual solution.

Constraint equations in terms of the York variables

$$\tilde{D}^i\left(\tilde{D}_i W_j + \tilde{D}_j W_i - \tfrac{2}{3}\tilde{\gamma}_{ij}\tilde{D}^k W_k\right) + \tilde{D}^i X_{ij} - \tfrac{2}{3}\psi^6 D_j K = -\psi^6 \gamma^i{}_j T_{i\mu} n^\mu$$

$$\tilde{D}^i \tilde{D}_i \psi \, - \, \frac{1}{8}({}^{(3)}\tilde{R})\psi \, - \, \frac{1}{12} K^2 \psi^5 \, + \, \frac{1}{8}\tilde{A}^{ij}\tilde{A}_{ij}\psi^{-7} = - \, \frac{1}{4} T_{\mu\nu} n^\mu n^\nu \psi^5$$

Decouple the equations

Choose constant K (gauge choice)

define the quantity $\tilde{J}_j$ by $\tilde{J}_j = \psi^6 \gamma^i{}_j T_{i\mu} n^\mu$ Then the momentum equation becomes

$$\tilde{D}^i \left(\tilde{D}_i W_j + \tilde{D}_j W_i - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}^k W_k \right) = -\tilde{D}^i X_{ij} \; - \; \tilde{J}_j$$

which we can solve first for W_i and then plug the result into the Hamiltonian constraint equation which we solve for ψ.

Does this equation have unique solutions? Yes, unless there is a conformal Killing field.

$$\tilde{D}_i V_j + \tilde{D}_j V_i - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}^k V_k = 0$$

(Mathematicians declare victory!)

Cosmological Exceptional Case

DG and L. Mead, Phys. Rev. D 04022 (2020) and arXiv:2006.16360

Cosmological scalar perturbations are conformally flat. $\tilde{\gamma}_{ij} = \delta_{ij}$.

Flat space has conformal Killing fields.

Specialize to this conformally flat case, and for simplicity to dependence on only one coordinate

$$\tfrac{4}{3}\frac{d^2 W_x}{dx^2} = -\frac{dX_{xx}}{dx} - \tilde{J}_x$$

Kernel and Fredholm alternative

Kernel is $W_x = const$. Because there is a Kernel, the operator cannot be inverted.

Fredholm alternative: If the source has a piece in the kernel, there is no solution. If the source has no piece in the kernel, there are multiple solutions.

Which solution should we choose? It doesn't matter, because they all give the same $\tilde{A}_{ij}$.

Finite difference version:

$$\frac{d^2 W_x}{dx^2} \to \frac{W_{i+1} + W_{i-1} - 2W_i}{(\Delta x)^2}$$

Solution for W_x

If the source can be expressed in Fourier components, we can express W_x analytically.

Otherwise, we have a numerical method that is a variation of the usual LU decomposition method to solve for W_i.

Cosmological case 1: Single scalar field comology

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi \; - \; g_{\mu\nu}\left(\tfrac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi + V\right)$$

Scalar perturbation mode: ϕ is a background plus a single Fourier mode. Solve for W_x analytically. This gives second order corrections to the first order perturbations

Cosmological case 2: Ekpyrotic two-field comology

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi \; - \; g_{\mu\nu}\left(\tfrac{1}{2}\nabla^\alpha\phi\nabla_\alpha\phi + V\right)$$
$$+ \, \kappa(\phi)\left[\nabla_\mu\chi\nabla_\nu\chi \; - \; \tfrac{1}{2}\, g_{\mu\nu}\,\nabla^\alpha\chi\nabla_\alpha\chi\right]$$

Presence of $\kappa(\phi)$ means neither the source nor W_x can be expressed in terms of a finite number of Fourier modes. Instead we use our numerical method.

Numerical solution for W_x (weak field)

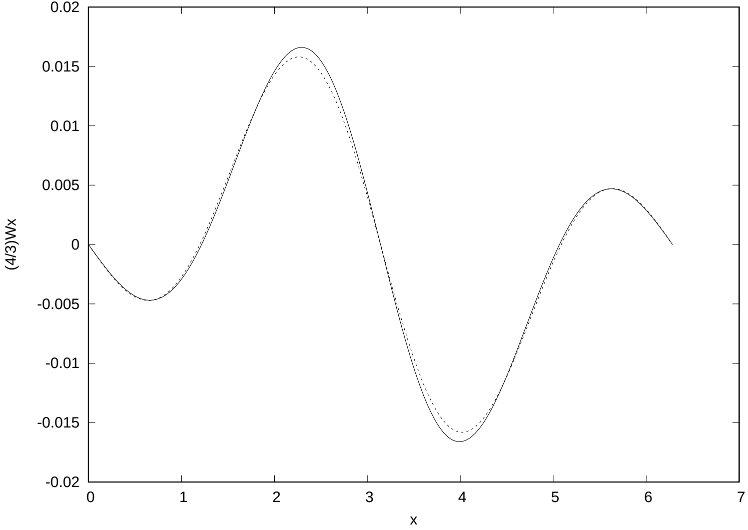

Figure : $(4/3)W_x$ vs. x for the numerical method (solid line) and perturbative method (dashed line) for weak initial data

Numerical solution for W_x (strong field)

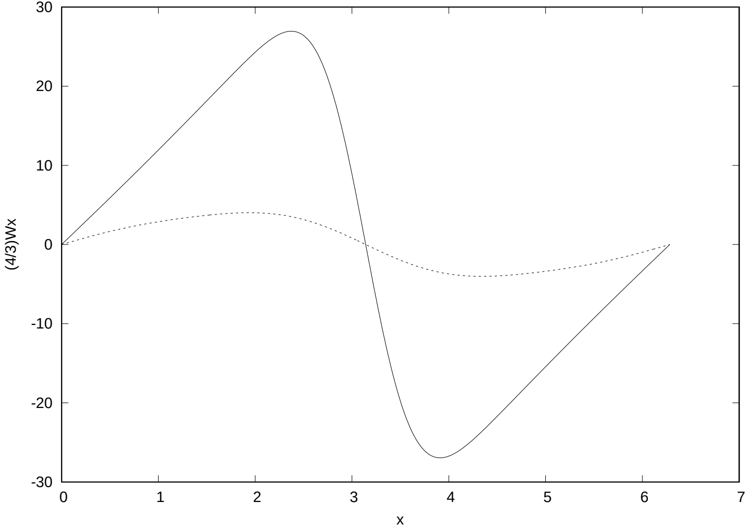

Figure : $(4/3)W_x$ vs. x for the numerical method (solid line) and perturbative method (dashed line) for strong initial data

Conclusions

We now have a way of finding “sufficiently general” initial data for cosmological simulations

The sort of studies mentioned in the motivation section could be extended by using this more general data to strengthen the genericity of their conclusions.