A New Formulation of Relativistic Euler Flow: Miraculous Geo-Analytic Structures and Applications

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Main themes of the talk

- Solutions without symmetry
- We derived a new, geometric way of formulating relativistic Euler flow (joint with Disconzi)
- Key point: non-zero vorticity/entropy allowed
- Motivation: Christodoulou's work on irrotational shock formation and my previous non-relativistic work (with Luk in barotropic case)
- Potential applications: stable shock formation, low regularity, long-time behavior of solutions, dynamics with shocks, numerical simulations?

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Relativistic Euler flow in Minkowski space

$$\boxed{A^{\alpha}(\vec{\Psi})\partial_{\alpha}\vec{\Psi} = 0}$$

- $\vec{\Psi} = (h, u^0, u^1, u^2, u^3, s)$
- $h = \ln H$ with $H =$ enthalpy; $u =$four-velocity; $s =$entropy
- The system is quasilinear hyperbolic
- $\eta_{\alpha\beta}u^{\alpha}u^{\beta} = -1$, $\eta =$ Minkowski metric
- Equation of state $p = p(\varrho, s)$ closes the system ($p =$pressure, $\varrho =$energy density)
- We assume $c =$ sound speed $:= \sqrt{\frac{\partial p}{\partial \varrho}} > 0$
- Two propagation phenomena: sound waves and transporting of vorticity/entropy
- Neither the phenomena nor their coupling are visible
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Geometric tensors associated to the flow

The four-velocity transports vorticity and entropy.

Definition (The four-velocity vectorfield)

$$\boxed{u^\alpha \partial_\alpha}$$

The acoustical metric is tied to sound wave propagation.

Definition (The acoustical metric and its inverse)

$$\boxed{\begin{aligned} \mathbf{g}_{\alpha\beta}(\vec{\Psi}) &:= c^{-2}\eta_{\alpha\beta} + (c^{-2}-1)u_\alpha u_\beta, \\ (\mathbf{g}^{-1})^{\alpha\beta}(\vec{\Psi}) &= c^2(\eta^{-1})^{\alpha\beta} + (c^2-1)u^\alpha u^\beta \end{aligned}}$$

u is $\mathbf{g}$-timelike and thus transverse to acoustically null hypersurfaces:

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Covariant wave operator

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For scalar-valued functions ϕ, we define (as usual)

$$\square_{\mathbf{g}}\phi := \frac{1}{\sqrt{|\det \mathbf{g}|}}\partial_\alpha \left\{ \sqrt{|\det \mathbf{g}|}(\mathbf{g}^{-1})^{\alpha\beta}\partial_\beta\phi \right\}$$

Additional fluid variables

Definition (The *u*-orthogonal vorticity of a one-form)

$$\boxed{\mathrm{vort}^{\alpha}(V) := -\epsilon^{\alpha\beta\gamma\delta} u_{\beta} \partial_{\gamma} V_{\delta}}$$

Definition (Vorticity vectorfield)

$$\boxed{\varpi^{\alpha} := \mathrm{vort}^{\alpha}(Hu) = -\epsilon^{\alpha\beta\gamma\delta} u_{\beta} \partial_{\gamma}(H u_{\delta})}$$

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Modified fluid variables

- Exhibit improved regularity
- Solve PDEs with good quasilinear null structure with respect to g

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\begin{aligned}
\mathcal{C}^{\alpha} := {} & \mathrm{vort}^{\alpha}(\varpi) + c^{-2}\epsilon^{\alpha\beta\gamma\delta}u_{\beta}(\partial_{\gamma}h)\varpi_{\delta} \\
& + (\theta - \theta_{;h})S^{\alpha}(\partial_{\kappa}u^{\kappa}) + (\theta - \theta_{;h})u^{\alpha}(S^{\kappa}\partial_{\kappa}h) \\
& + (\theta_{;h} - \theta)S^{\kappa}((\eta^{-1})^{\alpha\lambda}\partial_{\lambda}u_{\kappa}), \\
\mathcal{D} := {} & \frac{1}{n}(\partial_{\kappa}S^{\kappa}) + \frac{1}{n}(S^{\kappa}\partial_{\kappa}h) - \frac{1}{n}c^{-2}(S^{\kappa}\partial_{\kappa}h)
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- Temperature $\theta(h, s)$ and number density $n(h, s)$ determined by equation of state
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$$\mathcal{Q}_{(\alpha\beta)}(\partial\phi, \partial\widetilde{\phi}) := \partial_\alpha\phi\partial_\beta\widetilde{\phi} - \partial_\alpha\widetilde{\phi}\partial_\beta\phi$$

Purpose of new formulation

The new formulation allows for the application of geometric techniques from mathematical GR and nonlinear wave equations.

Big new issue compared to waves:

- The interaction of wave and transport phenomena, especially from the perspective of regularity and decay.
  "multiple characteristic speeds"

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A new formulation of relativistic Euler

Theorem (JS with M. Disconzi)

For $\Psi \in \vec{\Psi} := (h, u^0, u^1, u^2, u^3, s)$, $\mathfrak{Q} :=$ *combinations of null forms, regular solutions satisfy, up to lower-order terms:*

$$\begin{aligned} \square_{\mathbf{g}(\vec{\Psi})} \Psi &= \mathcal{C} + \mathcal{D} + \mathfrak{Q}(\partial\vec{\Psi}, \partial\vec{\Psi}), \\ u^\kappa \partial_\kappa \varpi^\alpha &= \partial\vec{\Psi}, \\ u^\kappa \partial_\kappa S^\alpha &= \partial\vec{\Psi} \end{aligned}$$

- *Formally,* $\mathcal{C}, \mathcal{D} \sim \partial\partial\vec{\Psi}$, *but they are actually better from various points of view. In fact,* $\partial\varpi, \partial S$ *are better:*

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L^2 regularity via div-curl-transport

- In non-relativistic flow, the div-curl part is along Σ_t.
- In contrast, the relativistic equations $\partial_\alpha \varpi^\alpha = RHS$ and $u^\kappa \partial_\kappa \mathcal{C}^\alpha = RHS$ are **spacetime** div-curl-transport systems for $\partial \varpi$.
- In practice, one needs L^2 regularity for $\partial \varpi$ along Σ_t.
- To achieve this, one also considers the PDEs $u^\kappa \partial_\kappa \varpi^\alpha = RHS$ and $u_\alpha \varpi^\alpha = 0$ (and thus $u_\alpha \partial \varpi^\alpha = -(\partial u_\alpha) \varpi^\alpha$).
- The latter two equations allow one to independently control "timelike parts" of $\partial \varpi$.
- Then the "timelike part" of $\partial \varpi$ can be "excised" from the spacetime div-curl-transport systems to derive a **spatial** div-curl-transport system along Σ_t.
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L^2 regularity via div-curl-transport

- In non-relativistic flow, the div-curl part is along Σ_t.
- In contrast, the relativistic equations $\partial_\alpha \varpi^\alpha = RHS$ and $u^\kappa \partial_\kappa \mathcal{C}^\alpha = RHS$ are **spacetime** div-curl-transport systems for $\partial \varpi$.
- In practice, one needs L^2 regularity for $\partial \varpi$ along Σ_t.
- To achieve this, one also considers the PDEs $u^\kappa \partial_\kappa \varpi^\alpha = RHS$ and $u_\alpha \varpi^\alpha = 0$ (and thus $u_\alpha \partial \varpi^\alpha = -(\partial u_\alpha) \varpi^\alpha$).
- The latter two equations allow one to independently control "timelike parts" of $\partial \varpi$.
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Some potential applications

The new formulation opens the door for several key applications with vorticity and entropy, some of which have been achieved in the non-relativistic case:

- Stable shock formation without symmetry (à la Christodoulou and my work with Luk in the non-relativistic case). Null structure is crucial.
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Nonlinear geometric optics

- Potential applications would require nonlinear geometric optics.
- New formulation allows for sharp implementation of nonlinear geometric optics.
- Implemented via an acoustic eikonal function U:

$$(\mathbf{g}^{-1})^{\alpha\beta}(\vec{\Psi})\partial_\alpha U \partial_\beta U = 0, \qquad \partial_t U > 0$$

- Level sets $\mathcal{C}_U$ of U are $\mathbf{g}$-null hypersurfaces.
- Play a critical role in many delicate local and global results for wave equations.
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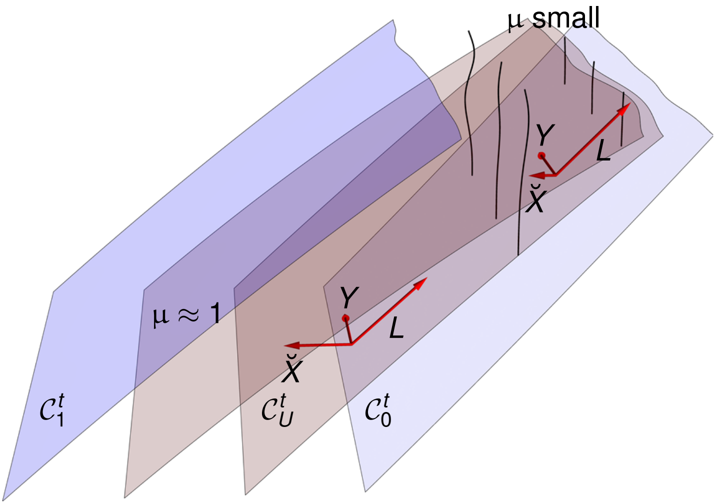
g-null hypersurfaces close to plane symmetry



Acoustic null frame

An acoustic null frame $\{L, \underline{L}, e_1, e_2\}$:

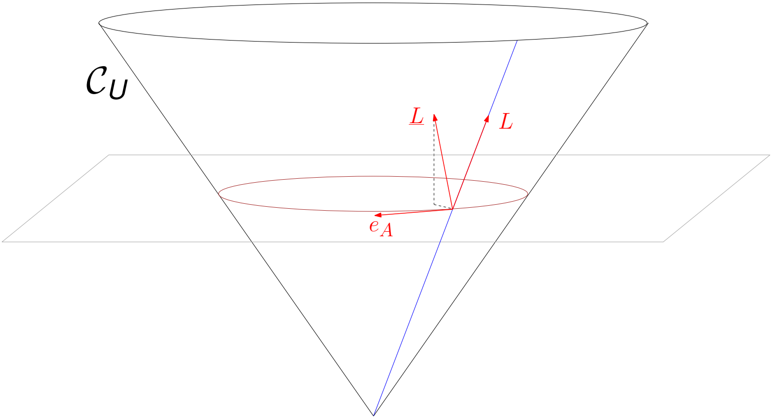


Figure: Null (with respect to **g**) frame

Christodoulou's sharp picture of relativistic Euler shock formation (irrotational case)

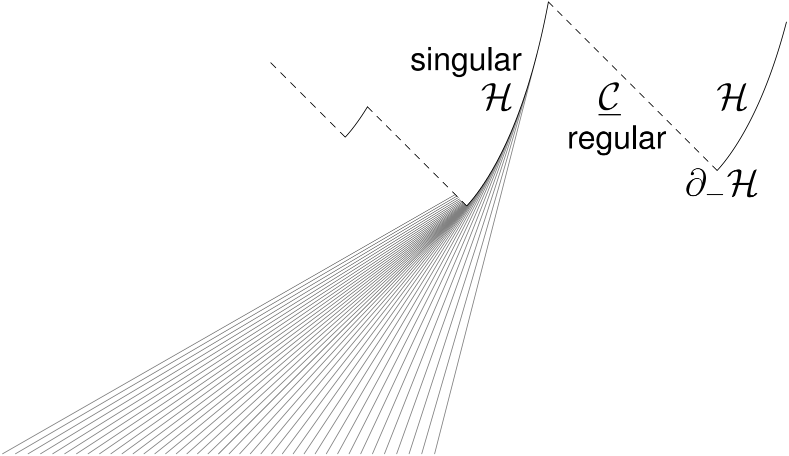


Figure: The maximal development

Model problem

$$\mathbf{g}(\Psi) = -dt \otimes dt + (1+\Psi)^{-2} \sum_{a=1}^{3} dx^a \otimes dx^a$$

$$\boxed{\square_{\mathbf{g}(\Psi)} \Psi = 0}$$

In (t, x^1) plane symmetry, define null vectorfields

$$L := \partial_t + (1+\Psi)\partial_1, \qquad \underline{L} := \partial_t - (1+\Psi)\partial_1.$$

The wave equation can be expressed as:

$$L(\underline{L}\Psi) = \underbrace{\frac{1}{2(1+\Psi)}(\underline{L}\Psi)^2}_{\text{causes Riccati-type blowup}} + \frac{5}{2(1+\Psi)}(\underline{L}\Psi)L\Psi,$$

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Eikonal functions regularize the problem

Define eikonal functions U, $\underline{U}$ by:

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LU &= 0, & \underline{L}\underline{U} &= 0, \\
U(0, x^1) &= -x^1, & \underline{U}(0, x^1) &= x^1.
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Then in $(U, \underline{U})$ coordinates, the wave equation becomes

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\frac{\partial}{\partial \underline{U}} \frac{\partial}{\partial U} \Psi = \frac{2}{(1+\Psi)} \frac{\partial}{\partial \underline{U}} \Psi \cdot \frac{\partial}{\partial U} \Psi
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Singularity is visible in standard coordinates

Set $\mu := \frac{1}{\underline{L}U}$ so that $\mu \underline{L} = \frac{\partial}{\partial U}$. Set $\underline{\mu} := \frac{1}{L\underline{U}}$ so that $\underline{\mu} L = \frac{\partial}{\partial \underline{U}}$.

$\mu \downarrow 0 \implies$ integral curves of L intersect $\implies$ shock

Evolution equations for $\mu, \underline{\mu}$:

$$\frac{\partial}{\partial \underline{U}}\mu = \underbrace{-\frac{\underline{\mu}}{(1+\Psi)}\frac{\partial}{\partial U}\Psi}_{\text{Can drive } \mu \downarrow 0} - \frac{\mu}{(1+\Psi)}\frac{\partial}{\partial \underline{U}}\Psi,$$

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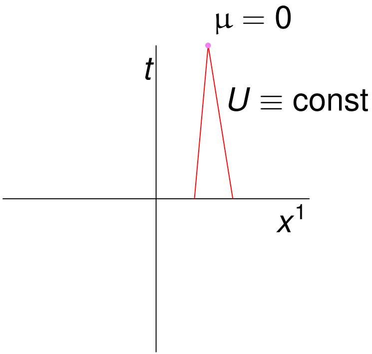
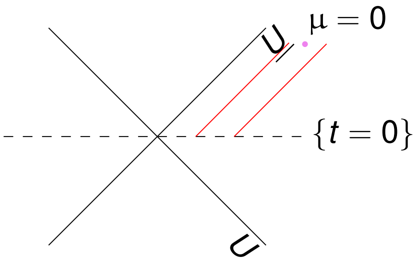
Regularizing the singularity





Significance of null forms

For null forms $\mathcal{Q}^{Null}$, in plane symmetry, $\square_{\mathbf{g}(\Psi)}\Psi = \mathcal{Q}^{Null}(\partial\Psi, \partial\Psi)$ can be written as

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This equation can be treated as before.

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Basic proof strategy in $1 + 3$ dimensions

- Supplement t and U with geometric angular coordinates $\vartheta \in \mathbb{S}^2$
- Prove that the solution remains smooth relative to (t, U, ϑ) coordinates
- Recover the blowup as a degeneracy between (t, U, ϑ) and rectangular coordinates

The degeneracy is signified by the vanishing of the *inverse foliation density*:

$$\mu = -\frac{1}{(g^{-1})^{\alpha\beta}\partial_\alpha t \partial_\beta U} > 0$$

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Some key difficulties in $1+3$ dimensions

- All known well-posedness results rely on L^2-based Sobolev spaces; i.e., one must derive energy estimates
- Energy estimates are very difficult in regions where $\mu \downarrow 0$
- High-order geometric energies can blow up: $\mathbb{E}^{High}(t) \lesssim (\min_{\Sigma_t} \mu)^{-P}$, $P \approx 10$
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Directions to consider

- Does Einstein–Euler exhibit similar good structures?
- Shock formation for Einstein–Euler
- Same questions for MHD, viscous relativistic Euler
- Same questions for more complicated multiple speed systems: elasticity, crystal optics, nonlinear electromagnetism,..., which take the form:

$$h_{AB}^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi^B = 0$$

  Would require the development of new geometry.
- Solve past the shock, locally (shock development problem à la Christodoulou)
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