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Entropy in local algebraic dynamics

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Abstract We introduce and study a new form of entropy, algebraic entropy, for self-maps of finite length of Noetherian local rings. We establish a number of its properties and find various analogies with topological entropy. For finite self-maps we find the expected (from topology) relation between degree and algebraic entropy, over Cohen-Macaulay domains. We use algebraic entropy to extend numerical conditions in Kunz' Regularity Criterion to all characteristics and give a characteristic-free interpretation of the definition of Hilbert-Kunz multiplicity. We find that the generalized Hilbert-Kunz multiplicity of regular local rings in any characteristic is 1. We also show that every self-map of finite length of a complete Noetherian local ring of equal characteristic can be lifted to a self-map of finite length of a complete regular local ring.

Keywords Local algebraic dynamics · Algebraic entropy · Self-maps of finite length · Kunz' Regularity criterion · Generalized Hilbert-Kunz multiplicity

Mathematics Subject Classification (2000) 13B10 · 13B40 · 13D40 · 13H05 · 14B25 · 37P99

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0 Introduction and notations

In dynamical systems theory, iterating a map from a space to itself generates a discrete-time dynamical system. One way to measure the complexity of such a system is by using the notion of entropy. According to [39, p. 313], entropy in dynamical systems is a notion that measures the rate of increase in dynamical complexity as the system evolves with time.

The various existing forms of entropy in dynamical systems theory are each suitable for use in a certain category. For instance, topological entropy was introduced by Adler, Konheim, and McAndrew in [1] for dynamics in the category of compact topological spaces with continuous morphisms. Similarly, measure-theoretic entropy was introduced by Kolmogorov in [22] and later improved by Sinai in [37], for dynamics in the category of probability spaces with measure-preserving morphisms.

Our primary objective in this paper is to introduce and develop a new form of entropy, *algebraic entropy*, that can be used as a tool in studying homological properties of Noetherian local rings. To describe our main results we need two definitions.

Definition 1 A homomorphism $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ of Noetherian local rings is said to be *of finite length*, if it is local and $f(\mathfrak{m})S$ is $\mathfrak{n}$-primary. In this case we define the length of f, $\lambda(f) \in [1, \infty)$ as $\lambda(f) := \ell_S\big(S/f(\mathfrak{m})S\big)$. We say f is *contracting*, if for every $x \in \mathfrak{m}$ the sequence $\{f^n(x)\}_{n\geqslant 1}$ converges to 0 in the $\mathfrak{n}$-adic topology of S.

Remark 1 **a)** For local homomorphisms of Noetherian local rings, finite $\Rightarrow$ integral $\Rightarrow$ finite length, and finite $\Rightarrow$ quasi-finite $\Rightarrow$ finite length. **b)** In [4, Lemma 12.1.4] it was shown that a local endomorphism φ of a Noetherian local ring $(R, \mathfrak{m})$ is contracting if and only if $\varphi^e(\mathfrak{m}) \subset \mathfrak{m}^2$, where e is the embedding dimension of R.

Definition 2 A *local algebraic dynamical system* is a discrete-time dynamical system that is generated by iterating an endomorphism of finite length φ of a Noetherian local ring R. If (R, φ) and (S, ψ) are two local algebraic dynamical systems, a *morphism* $f : (R, \varphi) \to (S, \psi)$ between these two dynamical systems is a local homomorphism $f : R \to S$ such that $\psi \circ f = f \circ \varphi$.

In this paper we study the category of local algebraic dynamical systems. Our main result in Section 1 is:

Theorem 1 *Let (R, φ) be a local algebraic dynamical system. Suppose R is of dimension d and embedding dimension e. Let λ be as defined in Definition 1.*
a) *The sequence $\{(\log \lambda(\varphi^n))/n\}_{n \geqslant 1}$ converges to its infimum that is finite. We define the* algebraic entropy *$h_{\mathrm{alg}}(\varphi, R)$ of φ as this limit.*
b) *If φ is in addition contracting, then $e \cdot h_{\mathrm{alg}}(\varphi, R) \geqslant d \cdot \log 2$.*
c) *If R is of prime characteristic $p > 0$, the algebraic entropy of the Frobenius endomorphism is equal to $d \cdot \log p$.*

Remark 2 **a)** Calling a quantity entropy requires justification. The analogies between $h_{\mathrm{alg}}(\varphi, R)$ and topological entropy serve to justify our terminology. We will show a number of such analogies in this paper. **b)** The definition of algebraic entropy can be stated for *graded* self-maps of finite length of graded rings over a field. Thus, algebraic entropy can also be defined for such maps.

We prove Theorem 1 in Section 1.3. We also provide lower and upper bounds v_h and w_h for algebraic entropy. These bounds are inspired by a work of Samuel in [34, p. 11]. The lower bound v_h for algebraic entropy has also been studied by Favre and Jonsson in a different context, in [12]. In [12, Theorem A] they prove that if k is an arbitrary field and φ is a self-map of the ring $k[\![X, Y]\!]$, then $v_h(\varphi)$ is a quadratic algebraic integer.

In Sections 1.4 and 1.8 we develop the properties of algebraic entropy. A remarkable feature of algebraic entropy is that it shares standard properties of topological entropy. Indeed, writing $h(\varphi)$ for entropy of a self-map φ of a space X, algebraic and topological entropies both satisfy conditions of following type:
1) $h(\varphi^t) = t \cdot h(\varphi)$ for all $t \in \mathbb{N}$, where $\varphi^t = \varphi \circ \varphi \circ \cdots \circ \varphi$ (t copies).
2) If $Y \subset X$ is a closed φ-invariant subspace, then $h(\varphi\restriction_Y) \leqslant h(\varphi)$.
3) If $f : X \to X'$ is an isomorphism, then $h(\varphi) = h(f \circ \varphi \circ f^{-1})$.
4) If $X = \bigcup Y_i$, $i = 1, \ldots, m$, where the Y_i are closed φ-invariant subspaces, then $h(\varphi) = \max\{h(\varphi\restriction_{Y_i}) : 1 \leqslant i \leqslant m\}$.

These conditions were proved in [1] for topological entropy. We will establish them for algebraic entropy in Section 1.4. Some other important results in Sections 1.4 and 1.5 are invariance of algebraic entropy under flat morphisms of finite length between two local algebraic dynamical systems, and the possibility of computing algebraic entropy in mixed characteristic by reducing to equal characteristic $p > 0$.

When two or more forms of entropy can be used to study the complexity of a system, often interesting relations emerge between them. These relations have been studied intensively. For a survey of these studies and some open questions, the interested reader can consult [25]. In Section 1.6 we deal with *finite* self-maps of local *domains* and explore the connection between degree and algebraic entropy of these maps. In particular, for local Cohen-Macaulay domains we establish a formula relating degree and algebraic entropy, that is expected from topology.

In Section 1.8 we consider local algebraic dynamical systems (R, φ) in which φ is *integral*. Denoting the self-map induced by φ on $\operatorname{Spec} R$ by ${}^a\varphi$, we show that when $\operatorname{Spec} R = V(\ker \varphi)$, ${}^a\varphi$ permutes irreducible components of $\operatorname{Spec} R$. As a result, irreducible components of $\operatorname{Spec} R$ are invariant under some iteration of φ.

In Section 2 we have two important results. First, using algebraic entropy we extend numerical conditions of Kunz' Regularity Criterion to arbitrary characteristic. To be more precise, in Section 2.1 we prove:

Theorem 2 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system of arbitrary characteristic. Set $d := \dim R$. Let $h_{\mathrm{alg}}(\varphi, R)$ be the algebraic entropy of this system. Define $q(\varphi) := \exp(h_{\mathrm{alg}}(\varphi, R)/d)$ and consider the conditions:*
a) *R is regular.*
b) *$\varphi : R \to R$ is flat.*
c) *$\lambda(\varphi) = q(\varphi)^d$.*
d) *$\lambda(\varphi^n) = q(\varphi)^{nd}$ for some $n \in \mathbb{N}$.*

Then **a**) $\Rightarrow$ **b**) $\Rightarrow$ **c**) $\Rightarrow$ **d**). *If in addition φ is contracting,* **d**) $\Rightarrow$ **b**) $\Rightarrow$ **a**). *That is, when φ is contracting all above conditions are equivalent.*

We should note that Avramov, Iyengar and Miller have proved the equivalence of conditions **a**) and **b**) (and more) in [4] using different methods. In our proof, we will use Herzog's proof in [19, Satz 3.1] to prove the implication **b**) $\Rightarrow$ **a**). He originally wrote it for the Frobenius endomorphism. This part of our proof, however, is not new and has also appeared in [9, Lemma 3].

In Section 2.2 we propose a characteristic-free definition for the Hilbert-Kunz multiplicity in terms of algebraic entropy. From Theorem 2 it quickly follows that the generalized Hilbert-Kunz multiplicity of a regular local ring with respect to an arbitrary self-map of finite length is 1. This is a well-known fact in the case of the Frobenius endomorphism.

Section 2.3 is inspired by a result of Fakhruddin on lifting polarized self-maps of projective varieties over an infinite field to an ambient projective space. Here we consider the analogous lifting problem for self-maps of finite length of equicharacteristic complete Noetherian local rings, and prove a Structure Theorem for them. As an improvement over Fakhruddin's result, we do not assume our fields to be infinite. Our second main result in Section 2 is:

Theorem 3 (Cohen-Fakhruddin) *Suppose in a local algebraic dynamical system $(A, \mathfrak{n}, \varphi)$, A is a homomorphic image $\pi : R \to A$ of an equicharacteristic complete regular local ring $(R, \mathfrak{m})$. Then φ can be lifted to a self-map of finite length ψ of R such that $\pi \circ \psi = \varphi \circ \pi$, i.e., $\pi : (R, \psi) \to (A, \varphi)$ is a morphism between two local algebraic dynamical systems.*

0.1 Notations and terminology

All rings in this paper are assumed to be Noetherian, commutative and with identity element. By a self-map of a ring we mean an endomorphism of that ring. For a self-map φ of a ring we will write φ^n for the n-fold composition of φ with itself.

If M is an R-module of finite length, we will denote its length by $\ell_R(M)$. If M is a finitely generated R-module, we will denote its minimum number of generators over R by $\mu(M)$. Given a ring homomorphism $f : R \to S$ and an S-module N, we will denote by $f_* N$ the R-module obtained by restriction of scalars. That is, $f_* N$ is the R-module whose underlying abelian group is N and whose R-module structure is given by $r \cdot x = f(r)x$, for $r \in R$ and $x \in f_* N$. Similarly, we will denote by $f_* S$ the ring S considered as an R-algebra via f. This notation is consistent with the notation used in [7].

The set of all minimal prime ideals of a ring R will be denoted by $\mathrm{Min}(R)$. If φ is a self-map of a ring R, we will denote the self-map induced by φ on $\operatorname{Spec} R$ by ${}^a\varphi$.

1 Algebraic entropy

1.1 Preliminaries

In this section we gather some preliminary material that we will refer to throughout the paper. We have omitted the majority of proofs, because they are fairly elementary and the reader can either produce them easily, or find them in the literature.

Proposition 1 *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of finite length of Noetherian local rings.*

a) *If $\mathfrak{p}$ is a prime ideal of S such that $f^{-1}(\mathfrak{p}) = \mathfrak{m}$, then $\mathfrak{p} = \mathfrak{n}$.*
b) *If $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of R, then $f(\mathfrak{q})S$ is $\mathfrak{n}$-primary.*

Corollary 1 *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $g : (S, \mathfrak{n}) \to (T, \mathfrak{p})$ be two local homomorphisms of Noetherian local rings. If f and g are both of finite length, then $g \circ f$ is also of finite length.*

Corollary 2 *Let (R, φ) be a a local algebraic dynamical system. Then φ^n is of finite length for all $n \geqslant 1$.*

Proposition 2 *Let $f : R \to S$ be a local homomorphism of Noetherian local rings with residue fields k_R and k_S and assume $[f_* k_S : k_R] < \infty$. If N is an S-module of finite length, then $f_* N$ is an R-module of finite length, and $\ell_R(f_* N) = [f_* k_S : k_R] \cdot \ell_S(N)$.*

Corollary 3 *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and let φ be a* finite *local self-map of R. Then $\mu(\varphi^n_* R) = [\varphi_* k : k]^n \cdot \lambda(\varphi^n)$.*

Proof By Nakayama's Lemma $\mu(\varphi_*^n R) = \dim_k(\varphi_*^n R/\mathfrak{m}\,\varphi_*^n R)$. Furthermore

$$\dim_k(\varphi_*^n R/\mathfrak{m}\,\varphi_*^n R) = \ell_R(\varphi_*^n R/\mathfrak{m}\,\varphi_*^n R) = \ell_R\left(\varphi_*(R/\varphi^n(\mathfrak{m})R)\right).$$

The result follows from Proposition 2 if we note $[\varphi_*^n k : k] = [\varphi_* k : k]^n$.

Definition 3 Let (R, φ) be a local algebraic dynamical system. An ideal $\mathfrak{a}$ of R is called φ-invariant, if $\varphi(\mathfrak{a})R \subseteq \mathfrak{a}$.

Proposition 3 *Let $(R, \mathfrak{m}), \varphi$ be a local algebraic dynamical system. Suppose $\mathfrak{a}$ is a φ-invariant ideal of R, and let $\overline{\varphi}$ be the local self-map induced by φ on $R/\mathfrak{a}$. Then $\overline{\varphi}$ is of finite length, and for all $n \in \mathbb{N}$:*

$$\lambda(\overline{\varphi}^{\,n}) = \ell_{R/\mathfrak{a}}\Big(\frac{R/\mathfrak{a}}{[\varphi^n(\mathfrak{m})R + \mathfrak{a}]/\mathfrak{a}}\Big) = \ell_R\Big(\frac{R}{\varphi^n(\mathfrak{m})R + \mathfrak{a}}\Big).$$

Proposition 4 *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of finite length of Noetherian local rings. Let M be an R-module of finite length. Then*

a) *$M \otimes_R S$ is of finite length as an S-module.*
b) *In general $\ell_S(M \otimes_R S) \leqslant \lambda(f) \cdot \ell_R(M)$.*
c) *If in addition f is* flat, *then $\ell_S(M \otimes_R S) = \lambda(f) \cdot \ell_R(M)$.*

Corollary 4 *Suppose $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $g : (S, \mathfrak{n}) \to (T, \mathfrak{p})$ are two local homomorphisms of finite length of Noetherian local rings. Then:*

a) *In general $\lambda(g) \leqslant \lambda(g \circ f) \leqslant \lambda(g) \cdot \lambda(f)$.*
b) *If in addition g is* flat, *then $\lambda(g \circ f) = \lambda(g) \cdot \lambda(f)$.*

Proof **a**) By Corollary 1, $\lambda(g \circ f) < \infty$. Since f is local, $g(f(\mathfrak{m})S)T \subset g(\mathfrak{n})T$. Thus $\ell_T(T/g(\mathfrak{n})T) \leqslant \ell_T(T/g(f(\mathfrak{m})S)T)$. This means $\lambda(g) \leqslant \lambda(g \circ f)$. For the second inequality use the canonical T-module isomorphism

$$T/g\left(f(\mathfrak{m})S\right)T \cong (S/f(\mathfrak{m})S) \otimes_S T$$

(see, e.g., [6, Chap. II, § 3.6, Coroll. 2 and 3, pp. 253-254]). By part **b**) of Proposition 4

$$\begin{aligned} \lambda(g \circ f) = \ell_T((S/f(\mathfrak{m})S) \otimes_S T) &\leqslant \lambda(g) \cdot \ell_S(S/f(\mathfrak{m})S) \\ &= \lambda(g) \cdot \lambda(f). \end{aligned} \tag{1}$$

b) If g is flat, then by part **c**) of Proposition 4 the inequality in Equation 1 turns into an equality, and the result follows immediately.

Corollary 5 *Let (R, φ) be a local algebraic dynamical system. Then*

a) *In general $\lambda(\varphi^n) \leqslant \lambda(\varphi)^n$ for all $n \in \mathbb{N}$.*
b) *If in addition φ is flat, then $\lambda(\varphi^n) = \lambda(\varphi)^n$ for all $n \in \mathbb{N}$.*

Proof By induction on n and using Corollary 4.

1.2 Examples of self-maps of finite length

In this section we provide some examples of self-maps of finite length.

Example 1 If R is a local ring of positive prime characteristic p, then the Frobenius endomorphism $x \mapsto x^p$ is a contracting self-map of finite length.

Example 2 A power series ring $R := k[\![X_1, \ldots, X_n]\!]$ over a field k has lots of self-maps of finite length. If elements $f_1, \ldots, f_n$ of R generate an ideal of height n in R, then we obtain a self-map of finite length by setting $X_i \mapsto f_i$ for $1 \leqslant i \leqslant n$. By Theorem 3, every self-map of finite length of a complete equicharacteristic local ring is induced by a self-map described in this example.

Example 3 Let $R := k[\![X_1, \ldots, X_n]\!]$ be a power series ring over a field k, and let φ be a self-map of finite length of R, e.g., as defined in Example 2. Let $z \neq 0$ be an arbitrary element of the maximal ideal of R. Then the ideal $\mathfrak{a}$ generated by $z, \varphi(z), \varphi^2(z), \ldots$ (orbit of z under φ) is φ-invariant. Thus φ induces a self-map of finite length $\overline{\varphi}$ on $R/\mathfrak{a}$. Moreover, if φ is contracting, then so is $\overline{\varphi}$. *Macaulay 2* can be used to generate concrete examples of this type. We mention a few such examples here. Let k be a field of characteristic zero, and let R and $\mathfrak{a}$ be as above.

a) $n = 5$, $z = X_1X_2 + X_3^3 + X_4^5 + X_5^2$. Define φ as $X_i \mapsto X_i^2$, for $1 \leqslant i \leqslant 4$, and $X_5 \mapsto X_5^4$. Then $\mu(\mathfrak{a}) = 5$ and $\dim R/\mathfrak{a} = 2$.

b) $n = 6$, $z = X_1^2 + X_2^3 + X_3^5 + X_4^7 + X_5^{11} + X_6^{13}$. Define φ as $X_i \mapsto X_i^2$, for $1 \leqslant i \leqslant 6$. Then $\mu(\mathfrak{a}) = 5$ and $\dim R/\mathfrak{a} = 2$.

c) $n = 7$, $z = X_1X_2X_3 + X_4^3 + X_5^2X_6 + X_7^3$. Define φ as $X_i \mapsto X_i^2$, for $2 \leqslant i \leqslant 6$ and $X_1 \mapsto X_7^2$, $X_7 \mapsto X_1^2$. Then $\mu(\mathfrak{a}) = 5$ and $\dim R/\mathfrak{a} = 3$.

d) $n = 8$, $z = X_1X_4^5X_8^2 + X_3X_5^4 + X_2X_6^3 + X_7$. Define φ as $X_i \mapsto X_i^2$, for $3 \leqslant i \leqslant 8$ and $X_1 \mapsto X_2^2$, $X_2 \mapsto X_1^2$. Then $\mu(\mathfrak{a}) = 5$ and $\dim R/\mathfrak{a} = 4$.

Example 4 Let $R := k[\![X_1, \ldots, X_n]\!]$ be a power series ring over a field k, and let $\mathfrak{a}$ be an ideal of R with *homogeneous* generators that can be expressed in the form monomial = monomial. Then the self-map of R given by $X_i \mapsto X_i^d$ for some integer $d > 1$, induces a contracting self-map of finite length on $R/\mathfrak{a}$.

1.3 Existence and estimates for algebraic entropy

In this section we prove Theorem 1. We also provide a lower bound v_h and an upper bound w_h for algebraic entropy. The lower bound v_h for algebraic entropy has also been studied by Favre and Jonsson in a different context, in [12]. In [12, Theorem A] they prove that if k is an arbitrary field and φ is a self-map of the ring $k[\![X, Y]\!]$, then $v_h(\varphi)$ is a quadratic algebraic integer.

We begin with an example.

Example 5 Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension zero, and let φ be a local self-map of R. Then R is Artinian and $1 \leqslant \lambda(\varphi^n) \leqslant \ell(R) < \infty$. Apply logarithm, divide by n and let n approach infinity to get $h_{\mathrm{alg}}(\varphi, R) = 0$.

Thus, the algebraic entropy of any local self-map of a Noetherian local ring of dimension zero is 0.

The lemma that follows is fairly well-known in dynamical systems.

Lemma 1 (Fekete) *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers that satisfy the following conditions:*

a) *$\{a_n/n\}$ is bounded above, $a_n \geqslant 0$ and $b_n \geqslant 0$ for all $n \in \mathbb{N}$.*
b) *For all $n, m \in \mathbb{N}$, $a_{n+m} \geqslant a_n + a_m$ and $b_{n+m} \leqslant b_n + b_m$, respectively.*

Then the sequences $\{a_n/n\}$ and $\{b_n/n\}$ are both convergent. In fact

$$\{a_n/n\} \to \sup_n\{a_n/n\} \quad \text{and} \quad \{b_n/n\} \to \inf_n\{b_n/n\}.$$

Proof For a proof of $\{b_n/n\} \to \inf_n\{b_n/n\}$ see, for example [38, Theorem 4.9]. We should note that since the terms b_n of the sequence are non-negative, $\inf_n\{b_n/n\}$ is a non negative real number. For a proof of $\{a_n/n\} \to \sup_n\{a_n/n\}$ let $\alpha := \sup_n \{a_n/n\}$. By assumption (a), α is a non negative real number. For every $\varepsilon > 0$ there exists n_0 such that $a_{n_0}/n_0 \geqslant \alpha - \varepsilon$ Given an integer $n > n_0$, let us write $n = n_0 q + r$, with $0 \leqslant r < n_0$. Then using (a) and (b)

$$a_n \geqslant a_{n_0 q} + a_r \geqslant a_{n_0 q} \geqslant q \cdot a_{n_0}.$$

From these inequalities we deduce

$$\frac{\log a_n}{n} \geqslant \frac{qn_0}{n} \cdot \frac{\log a_{n_0}}{n_0} \geqslant \frac{qn_0}{n} \cdot (\alpha - \varepsilon) = \frac{n_0}{n_0 + r/q} \cdot (\alpha - \varepsilon).$$

Thus, if we take n large enough so that $n_0/(n_0 + r/q) \geqslant (\alpha - 2\varepsilon)/(\alpha - \varepsilon)$ then we obtain $(\alpha - 2\varepsilon) \leqslant a_n/n \leqslant \alpha$. The result follows.

The following definition is inspired by a definition in [34, p. 11].

Definition 4 Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local homomorphism of finite length of Noetherian local rings. We define

$$\begin{aligned} v(f) &= \max\{k \in \mathbb{N} \mid f(\mathfrak{m})S \subset \mathfrak{n}^k\}, \\ w(f) &= \min\{k \in \mathbb{N} \mid \mathfrak{n}^k \subset f(\mathfrak{m})S\}. \end{aligned}$$

Remark 3 It quickly follows from this definition that $\mathfrak{n}^{w(f)} \subset f(\mathfrak{m})S \subset \mathfrak{n}^{v(f)}$. Thus, we always have $v(f) \leqslant w(f)$.

Lemma 2 *Let $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $g : (S, \mathfrak{n}) \to (T, \mathfrak{p})$ be local homomorphisms of finite length of Noetherian local rings. Then*

$$\begin{aligned} v(g \circ f) &\geqslant v(g) \cdot v(f), \\ w(g \circ f) &\leqslant w(g) \cdot w(f). \end{aligned}$$

Proof First note that for an ideal $\mathfrak{a}$ of S, $g(\mathfrak{a}^n)T = [g(\mathfrak{a})T]^n$ for all $n \in \mathbb{N}$. (see [2, Exercise 1.18, p. 10]). We can write

$$\begin{aligned}[(g \circ f)(\mathfrak{m})]T &= g(f(\mathfrak{m})S)T \subset g(\mathfrak{n}^{v(f)})T \\ &= [g(\mathfrak{n})T]^{v(f)}\, R \subset \mathfrak{p}^{v(g)v(f)}.\end{aligned}$$

Thus, by definition of $v(g \circ f)$ we must have $v(g \circ f) \geqslant v(g) \cdot v(f)$. Similarly

$$\begin{aligned}\mathfrak{p}^{w(g)w(f)} &\subset [g(\mathfrak{n})T]^{w(f)} = g(\mathfrak{n}^{w(f)})T \\ &\subset g(f(\mathfrak{m})S)T = [(g \circ f)(\mathfrak{m})]T.\end{aligned}$$

Again, by definition of $w(g \circ f)$ we must have $w(g \circ f) \leqslant w(g) \cdot w(f)$.

Corollary 6 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Then for all $m, n \in \mathbb{N}$ the following inequalities hold:*

$$\begin{aligned}v(\varphi^{n+m}) &\geqslant v(\varphi^n) \cdot v(\varphi^m), \\ w(\varphi^{n+m}) &\leqslant w(\varphi^n) \cdot w(\varphi^m).\end{aligned}$$

Proof Apply Lemma 2 taking φ^n as g and φ^m as f.

Proposition 5 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Then the sequences $\{(\log v(\varphi^n))/n\}$ and $\{(\log w(\varphi^n))/n\}$ converge to their supremum and infimum, respectively. We will denote these limits by $v_h(\varphi)$ and $w_h(\varphi)$.*

Proof We will apply Lemma 1, taking $\{\log v(\varphi^n)\}$ and $\{\log w(\varphi^n)\}$ as $\{a_n\}$ and $\{b_n\}$ in the lemma, respectively. We verify that the conditions of the lemma are satisfied. By Corollary 6 and Remark 3, for every $n \in \mathbb{N}$

$$1 \leqslant [v(\varphi)]^n \leqslant v(\varphi^n) \leqslant w(\varphi^n) \leqslant [w(\varphi)]^n.$$

Thus, condition **a**) of Lemma 1 is satisfied. Moreover, Corollary 6 shows that condition **b**) of Lemma 1 is also satisfied. Hence the sequences $\{\log(v(\varphi^n))/n\}$ and $\{\log(w(\varphi^n))/n\}$ converge to their supremum and infimum, respectively.

Theorem 4 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system, and let $d := \dim R$. Then*

$$d \cdot v_h(\varphi) \leqslant h_{\mathrm{alg}}(\varphi, R) \leqslant d \cdot w_h(\varphi).$$

Proof By Definition 4, $\mathfrak{m}^{w(\varphi^n)} \subset \varphi^n(\mathfrak{m})R \subset \mathfrak{m}^{v(\varphi^n)}$. Thus

$$\ell_R(R/\mathfrak{m}^{v(\varphi^n)}) \leqslant \lambda(\varphi^n) \leqslant \ell_R(R/\mathfrak{m}^{w(\varphi^n)}).$$

We consider two cases: $v(\varphi^n) \to \infty$ and $v(\varphi^n) \nrightarrow \infty$. In the first case by Remark 3 $w(\varphi^n) \to \infty$, as well, and for large n, the lengths $\ell_R(R/\mathfrak{m}^{v(\varphi^n)})$ and $\ell_R(R/\mathfrak{m}^{w(\varphi^n)})$ are polynomials in $v(\varphi^n)$ and $w(\varphi^n)$, respectively, of precise degree d, with highest degree terms $e(\mathfrak{m})(v(\varphi^n))^d/d!$ and $e(\mathfrak{m})(w(\varphi^n))^d/d!$. Thus, for large n we obtain

$$\frac{e(\mathfrak{m})}{d!}\left(v(\varphi^n)\right)^d \leqslant \lambda(\varphi^n) \leqslant \frac{e(\mathfrak{m})}{d!}\left(w(\varphi^n)\right)^d.$$

Applying logarithm, dividing by n and letting n approach infinity, we see that

$$0 \leqslant d \cdot v_h(\varphi) \leqslant h_{\mathrm{alg}}(\varphi, R) \leqslant d \cdot w_h(\varphi) < \infty.$$

In the second case, when $v(\varphi^n) \nrightarrow \infty$, the sequence $\{v(\varphi^n)\}$ must be bounded. Hence, there is a constant c such that $1 \leqslant v(\varphi^n) \leqslant c$. Applying logarithm, dividing by n and letting n approach infinity, we see that $v_h(\varphi) = 0$. Now, if $w(\varphi^n) \to \infty$, then starting with the inequality

$$1 \leqslant \lambda(\varphi^n) \leqslant \ell_R(R/\mathfrak{m}^{w(\varphi^n)})$$

and repeating the same argument as before, we arrive at the desired inequality

$$v_h(\varphi) = 0 \leqslant h_{\mathrm{alg}}(\varphi) \leqslant d \cdot w_h(\varphi).$$

Finally if $w(\varphi^n) \nrightarrow \infty$, then the sequence $\{w(\varphi^n)\}$ is also bounded and there exists a constant c' such that $1 \leqslant w(\varphi^n) \leqslant c'$. After applying logarithm, dividing by n and letting n approach infinity, we see that $w_h(\varphi) = 0$. Since $v_h(\varphi) = 0$ as well, the proof will be completed by showing $h_{\mathrm{alg}}(\varphi, R) = 0$. This follows from the inequality

$$1 \leqslant \lambda(\varphi^n) \leqslant \ell_R(R/\mathfrak{m}^{w(\varphi^n)}) \leqslant \ell_R(R/\mathfrak{m}^{c'})$$

by applying logarithm, dividing by n and letting n approach infinity.

Proof (of Theorem 1) **a)** We apply Lemma 1, taking $b_n = \log \lambda(\varphi^n)$. We verify the conditions of this lemma. By Corollary 4,

$$\log \lambda(\varphi^{m+n}) \leqslant \log \lambda(\varphi^m) + \log \lambda(\varphi^n).$$

The condition $\log \lambda(\varphi^n) \geqslant 0$ is clear. By Lemma 1 the sequence $\{(\log \lambda(\varphi^n))/n\}$ converges to its infimum, which is a real number.

b) If $e = 0$ then there is nothing to prove. Assume $e > 0$. Since φ is contracting, by Remark 1, $\varphi^e(\mathfrak{m})R \subseteq \mathfrak{m}^2$. Hence

$$\varphi^{ne}(\mathfrak{m})R \subseteq \mathfrak{m}^{2^n}.$$

By definition of $v(\,\cdot\,)$ in Definition 4, $v(\varphi^{ne}) \geqslant 2^n$. Thus

$$(\log v(\varphi^{ne}))/(ne) \geqslant (n \log 2)/ne.$$

Letting n approach infinity we obtain $v_h(\varphi) \geqslant \log 2/e$. Now using Theorem 4,

$$h_{\mathrm{alg}}(\varphi, R) \geqslant d \cdot v_h(\varphi) \geqslant (d \cdot \log 2)/e.$$

c) If R is of characteristic p and φ is its Frobenius endomorphism, then by [23, Proposition 3.2]

$$p^{nd} \leqslant \lambda(\varphi^n) \leqslant \min_{\{y_1,\ldots,y_d\}} \left[\ell_R\left(R/(y_1,\ldots,y_d)R\right)\right] \cdot p^{nd},$$

where $\{y_1,\ldots,y_d\}$ runs over all systems of parameters of R. Apply logarithm, divide by n and let n approach infinity. We see $h_{\mathrm{alg}}(\varphi, R) = d \cdot \log p$.

The following corollary can be thought of as the converse of Example 5.

Corollary 7 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system and suppose φ is contracting. If $h_{\mathrm{alg}}(\varphi, R) = 0$, then* $\dim R = 0$.

1.4 Properties of algebraic entropy

We establish a number of important properties of algebraic entropy in this section. As mentioned in the introduction, some of these properties are in common between algebraic and topological entropies.

Proposition 6 *Let (R, φ) be a local algebraic dynamical system. Then for all $t \in \mathbb{N}$: $h_{\mathrm{alg}}(\varphi^t, R) = t \cdot h_{\mathrm{alg}}(\varphi, R)$.*

Proof By definition of algebraic entropy

$$\begin{aligned} h_{\mathrm{alg}}(\varphi^t, R) &= \lim_{n\to\infty} (1/n) \cdot \log \lambda(\varphi^{tn}) \\ &= t \cdot \lim_{n\to\infty} (1/(tn)) \cdot \log \lambda(\varphi^{tn}) \\ &= t \cdot h_{\mathrm{alg}}(\varphi, R). \end{aligned}$$

Proposition 7 *Let $f : (R, \mathfrak{m}, \varphi) \to (S, \mathfrak{n}, \psi)$ be a morphism between two local algebraic dynamical systems. Assume that f is of finite length. Then*

a) *In general $h_{\mathrm{alg}}(\psi, S) \leqslant h_{\mathrm{alg}}(\varphi, R)$.*
b) *If in addition f is flat, then $h_{\mathrm{alg}}(\psi, S) = h_{\mathrm{alg}}(\varphi, R)$.*

Proof **a**) By Corollary 2 and our assumptions, φ^n and ψ^n are also of finite length. Noting that $\psi^n \circ f = f \circ \varphi^n$ for all $n \in \mathbb{N}$ and using Corollary 4

$$\lambda(\psi^n) \leqslant \lambda(\psi^n \circ f) = \lambda(f \circ \varphi^n) \leqslant \lambda(f) \cdot \lambda(\varphi^n). \tag{2}$$

We obtain the result by applying logarithm to either side of this inequality, then dividing by n and taking limits as n approaches infinity.
b) If f is flat, then using Corollary 4 we compute

$$\begin{aligned} \lambda(\varphi^n) &= \lambda(f) \cdot \lambda(\varphi^n)/\lambda(f) = \lambda(f \circ \varphi^n)/\lambda(f) \\ &= \lambda(\psi^n \circ f)/\lambda(f) \leqslant \lambda(\psi^n) \cdot \lambda(f)/\lambda(f) = \lambda(\psi^n). \end{aligned}$$

Thus, using Inequality 2, $\lambda(\varphi^n) \leqslant \lambda(\psi^n) \leqslant \lambda(f) \cdot \lambda(\varphi^n)$. The result follows quickly by taking logarithms, dividing by n, and taking limits as n approaches infinity.

Corollary 8 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. If $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R then $h_{\mathrm{alg}}(\varphi, R) = h_{\mathrm{alg}}(\widehat{\varphi}, \widehat{R})$.*

Proof We have a flat morphism of finite length $\widehat{\ } : (R, \varphi) \to (\widehat{R}, \widehat{\varphi})$.

Corollary 9 *Consider homomorphisms of finite length $f : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ and $g : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ of Noetherian local rings. Then*

$$h_{\mathrm{alg}}(g \circ f, R) = h_{\mathrm{alg}}(f \circ g, S).$$

Proof $f : (R, g\circ f) \to (S, f\circ g)$ and $g : (S, f\circ g) \to (R, g\circ f)$ are morphisms between local algebraic dynamical systems. By Proposition 7

$$h_{\mathrm{alg}}(f\circ g, S) \leqslant h_{\mathrm{alg}}(g\circ f, R) \quad \text{and} \quad h_{\mathrm{alg}}(g\circ f, R) \leqslant h_{\mathrm{alg}}(f\circ g, S).$$

The result follows immediately.

Corollary 10 (Invariance) *Let $(R,\mathfrak{m})$ and $(S,\mathfrak{n})$ be Noetherian local rings. Suppose $f : R \to S$ is an isomorphism, and let φ be a self-map of of finite length of R . Then $h_{\mathrm{alg}}(f\circ\varphi\circ f^{-1}, S) = h_{\mathrm{alg}}(\varphi, R)$.*

Proof Apply Corollary 9 to homomorphisms $f\circ\varphi : R \to S$ and $f^{-1} : S \to R$.

Corollary 11 *Let (R,φ) be a local algebraic dynamical system and let $\mathfrak{a}$ be a φ-invariant ideal of R. Write $\overline{\varphi}$ for both self-maps induced by φ on $R/\mathfrak{a}$ and $R/\varphi(\mathfrak{a})R$. Then $h_{\mathrm{alg}}(\overline{\varphi}, R/\mathfrak{a}) = h_{\mathrm{alg}}(\overline{\varphi}, R/\varphi(\mathfrak{a})R)$.*

Proof Let $\varphi' : R/\mathfrak{a} \to R/\varphi(\mathfrak{a})R$ and $\overline{\mathrm{id}} : R/\varphi(\mathfrak{a})R \to R/\mathfrak{a}$ be homomorphisms induced by φ and identity map of R. Apply Corollary 9 to φ' and $\overline{\mathrm{id}}$.

We will need the following two lemmas in the proof of Proposition 8.

Lemma 3 *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers not less than 1 such that $\lim_{n\to\infty}(\log a_n)/n = \alpha$ and $\lim_{n\to\infty}(\log b_n)/n = \beta$ exist. Then*

$$\lim_{n\to\infty} \log(a_n + b_n)/n = \max\{\alpha, \beta\}.$$

Proof See [1, p. 312].

Lemma 4 *Let $(R,\mathfrak{m},\varphi)$ be a local algebraic dynamical system. Let $\mathfrak{a}_1,\dots,\mathfrak{a}_s$ be a collection of not necessarily distinct φ-invariant ideals of R. Let $\overline{\varphi}$ and $\overline{\varphi}_i$ be the self-maps induced by φ on $R/\prod_i \mathfrak{a}_i$ and $R/\mathfrak{a}_i$, respectively. Then*

$$h_{\mathrm{alg}}(\overline{\varphi}, R/\textstyle\prod_i \mathfrak{a}_i) = \max\{h_{\mathrm{alg}}(\overline{\varphi}_i, R/\mathfrak{a}_i) \mid 1 \leqslant i \leqslant s\}.$$

Proof We proceed by induction on s, the number of ideals, counting possible repetitions. There is nothing to prove if $s = 1$, so suppose $s = 2$. Without loss of generality we may assume

$$h_{\mathrm{alg}}(\overline{\varphi}_1, R/\mathfrak{a}_1) = \max\{h_{\mathrm{alg}}(\overline{\varphi}_1, R/\mathfrak{a}_1), h_{\mathrm{alg}}(\overline{\varphi}_2, R/\mathfrak{a}_2)\}.$$

Since $\mathfrak{a}_1\mathfrak{a}_2 \subset \mathfrak{a}_1$, we have $\mathfrak{a}_1 \cap (\mathfrak{a}_1\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R) = \mathfrak{a}_1\mathfrak{a}_2 + (\mathfrak{a}_1 \cap \varphi^n(\mathfrak{m})R)$. Thus, if we apply the Second Isomorphism Theorem to make the identification

$$\frac{\mathfrak{a}_1 + \varphi^n(\mathfrak{m})R}{\mathfrak{a}_1\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R} \cong \frac{\mathfrak{a}_1}{\mathfrak{a}_1\mathfrak{a}_2 + (\mathfrak{a}_1 \cap \varphi^n(\mathfrak{m})R)},$$

then we can write an exact sequence

$$0 \to \frac{\mathfrak{a}_1}{\mathfrak{a}_1\mathfrak{a}_2 + (\mathfrak{a}_1 \cap \varphi^n(\mathfrak{m})R)} \to \frac{R}{\mathfrak{a}_1\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R} \to \frac{R}{\mathfrak{a}_1 + \varphi^n(\mathfrak{m})R} \to 0.$$

From this exact sequence

$$
\begin{aligned}
\ell_R(R/[\mathfrak{a}_1 + \varphi^n(\mathfrak{m})R]) \leqslant{}& \ell_R(R/[\mathfrak{a}_1\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R]) \\
&= \ell_R(\mathfrak{a}_1/[\mathfrak{a}_1\mathfrak{a}_2 + (\mathfrak{a}_1 \cap \varphi^n(\mathfrak{m})\, R)]) \\
&+ \ell_R(R/[\mathfrak{a}_1 + \varphi^n(\mathfrak{m})R]).
\end{aligned} \tag{3}
$$

Since in the quotient ring $R/(\mathfrak{a}_1\mathfrak{a}_2)$ the ideal $\mathfrak{a}_2/(\mathfrak{a}_1\mathfrak{a}_2)$ annihilates $\mathfrak{a}_1/(\mathfrak{a}_1\mathfrak{a}_2)$, we can consider $\mathfrak{a}_1/(\mathfrak{a}_1\mathfrak{a}_2)$ as a finite $[(R/(\mathfrak{a}_1\mathfrak{a}_2))\,/\,(\mathfrak{a}_2/(\mathfrak{a}_1\mathfrak{a}_2))]$-module and as such, there is a surjection

$$
\left(\frac{R/(\mathfrak{a}_1\mathfrak{a}_2)}{\mathfrak{a}_2/(\mathfrak{a}_1\mathfrak{a}_2)}\right)^t \to \frac{\mathfrak{a}_1}{(\mathfrak{a}_1\mathfrak{a}_2)} \to 0.
$$

If we tensor this surjection over the quotient ring $R/(\mathfrak{a}_1\mathfrak{a}_2)$ with

$$
\frac{R/(\mathfrak{a}_1\mathfrak{a}_2)}{[\mathfrak{a}_1\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R]/(\mathfrak{a}_1\mathfrak{a}_2)}
$$

and then compare the lengths in the resulting surjection, by Proposition 3, Proposition 2 and the Third Isomorphism Theorem, we can quickly see

$$
\begin{aligned}
\ell_R(\mathfrak{a}_1/[\mathfrak{a}_1\mathfrak{a}_2 + \mathfrak{a}_1 \cdot \varphi^n(\mathfrak{m})R]) \leqslant{}& \ell_R(\mathfrak{a}_1/[\mathfrak{a}_1^2\mathfrak{a}_2 + \mathfrak{a}_1 \cdot \varphi^n(\mathfrak{m})R]) \\
&\leqslant t \cdot \ell_R(R/[\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R]).
\end{aligned}
$$

Since $\ell_R(\mathfrak{a}_1/[\mathfrak{a}_1\mathfrak{a}_2+(\mathfrak{a}_1\cap\varphi^n(\mathfrak{m})R)]) \leqslant \ell_R(\mathfrak{a}_1/[\mathfrak{a}_1\mathfrak{a}_2+\mathfrak{a}_1\cdot\varphi^n(\mathfrak{m})R])$, the previous inequality together with Inequality 3 yield

$$
\begin{aligned}
\ell_R(R/[\mathfrak{a}_1 + \varphi^n(\mathfrak{m})R]) \leqslant{}& \ell_R(R/[\mathfrak{a}_1\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R]) \\
&\leqslant \ell_R(R/[\mathfrak{a}_1 + \varphi^n(\mathfrak{m})R]) + t \cdot \ell_R(R/[\mathfrak{a}_2 + \varphi^n(\mathfrak{m})R]).
\end{aligned}
$$

Apply logarithm, divide by n, and let n approach infinity. By Lemma 3 and Proposition 3

$$
h_{\mathrm{alg}}(\overline{\varphi}_1, R/\mathfrak{a}_1) \leqslant h_{\mathrm{alg}}(\overline{\varphi}, R/\mathfrak{a}_1\mathfrak{a}_2) \leqslant \max\{h_{\mathrm{alg}}(\overline{\varphi}_1, R/\mathfrak{a}_1), h_{\mathrm{alg}}(\overline{\varphi}_2, R/\mathfrak{a}_2)\}.
$$

This establishes the result for $s = 2$. Now we assume the statement holds for all s with $2 \leqslant s \leqslant n_0$, and we show it also holds for $s = n_0 + 1$. To this end, we can write the product $\prod_{i=1}^{n_0+1} \mathfrak{a}_i$ of our ideals in the form $(\prod_{i=1}^{n_0} \mathfrak{a}_i)(\mathfrak{a}_{n_0+1})$ and then apply the case $s = 2$ followed by the case $s = n_0$ to establish the result for $s = n_0 + 1$, using the induction hypothesis.

Our next result shows that if all minimal prime ideals of a Noetherian local ring R are invariant under a self-map of the ring, then the algebraic entropy is equal to the maximum algebraic entropy of the self-maps induced on irreducible components of $\operatorname{Spec} R$.

Proposition 8 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Suppose all minimal prime ideal of R are φ-invariant and for each $\mathfrak{p}_i \in \operatorname{Min}(R)$, let $\overline{\varphi}_i$ be the self-map induced by φ on $R/\mathfrak{p}_i$. Then*

$$
h_{\mathrm{alg}}(\varphi, R) = \max\{h_{\mathrm{alg}}(\overline{\varphi}_i, R/\mathfrak{p}_i) \mid \mathfrak{p}_i \in \operatorname{Min}(R)\}. \tag{4}
$$

Proof Let $\operatorname{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$ and let $\mathfrak{a} = \prod_i \mathfrak{p}_i$. Then $\mathfrak{a}$ is contained in the nilradical of R, hence $\mathfrak{a}^N = (0)$ for some N. Therefore it is clear that $h_{\text{alg}}(\varphi, R) = h_{\text{alg}}(\overline{\varphi}, R/\mathfrak{a}^N)$. But by Lemma 4

$$h_{\text{alg}}(\overline{\varphi}, R/\mathfrak{a}^N) = \max\{h_{\text{alg}}(\overline{\varphi}_i, R/\mathfrak{p}_i) \mid \mathfrak{p}_i \in \operatorname{Min}(R)\}.$$

Remark 4 As we shall see in Proposition 13, under certain conditions, when a self-map is *integral*, minimal prime ideals are invariant under some power of the self-map. As a result, we can apply Proposition 8 to a power of our self-map in this case. We will obtain formulas similar to Formula 4 in Corollary 14 and Proposition 15, below.

1.5 Reduction to equal characteristic

In this section we show that any self-map of a local ring of mixed characteristic naturally induces a self-map of another local ring of equal characteristic $p > 0$ with the same algebraic entropy. Using this result, computing algebraic entropy in mixed characteristic can be reduced to the case of equal characteristic $p > 0$.

For a given local algebraic dynamical system $(R, \mathfrak{m}, \varphi)$, we define

$$S := \textstyle\bigcap_{n=1}^{\infty} \varphi^n(R) \quad \text{and} \quad \mathfrak{n} := \textstyle\bigcap_{n=1}^{\infty} \varphi^n(\mathfrak{m}). \tag{5}$$

Lemma 5 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Let S and $\mathfrak{n}$ be as defined in Equation 5, and let $\mathfrak{a}$ be the ideal generated by $\mathfrak{n}$ in R. Then*

a) *S is a local subring of R with maximal ideal $\mathfrak{n}$.*
b) *$\mathfrak{a}$ is a φ-invariant ideal of R.*
c) *If φ is in addition injective, then $\varphi(\mathfrak{a})R = \mathfrak{a}$.*

Proof **a**) It is immediately clear that S is a subring of R and that $\mathfrak{n}$ is an ideal of S. To show that $\mathfrak{n}$ is the (only) maximal ideal of S, consider an element $s \in S \backslash \mathfrak{n}$. Since $s \notin \mathfrak{n}$, there is an n_0 such that $s \notin \varphi^{n_0}(\mathfrak{m})$. In fact, since for $n \geqslant n_0$, $\varphi^n(\mathfrak{m}) \subseteq \varphi^{n_0}(\mathfrak{m})$, we see that $s \notin \varphi^n(\mathfrak{m})$ for all $n \geqslant n_0$. Hence, there are units $y_n \in R \backslash \mathfrak{m}$ such that $s = \varphi^n(y_n)$ for all $n \geqslant n_0$. Since s is clearly a unit in R, it has a unique multiplicative inverse s^{-1} in R. From uniqueness of multiplicative inverse it immediately follows that we must have $s^{-1} = \varphi^n(y_n^{-1})$, for all $n \geqslant n_0$. Hence, $s^{-1} \in S$, that is, s is also a unit in S.
b) Note that by its definition, $\mathfrak{a}$ has a set of generators $x_1, \ldots, x_g \in \mathfrak{n}$. So $\varphi(\mathfrak{a})R$ can be generated by $\varphi(x_1), \ldots, \varphi(x_g)$ and it suffices to show that each $\varphi(x_i)$ is in $\mathfrak{a}$. Since $x_i \in \mathfrak{n}$, there is a sequence of element $y_{i,n} \in \mathfrak{m}$ such that $x_i = \varphi(y_{i,1}) = \ldots = \varphi^n(y_{i,n}) = \ldots$. Thus, $\varphi(x_i) = \varphi^2(y_{i,1}) = \ldots = \varphi^{n+1}(y_{i,n}) = \ldots$, showing that $\varphi(x_i) \in \mathfrak{n} \subset \mathfrak{a}$.
c) Now suppose φ is injective. To show $\varphi(\mathfrak{a})R = \mathfrak{a}$ it suffices to show that each x_i is in $\varphi(\mathfrak{a})$. Since $x_i \in \mathfrak{n}$, there is a sequence of element $y_{i,n} \in \mathfrak{m}$ such that $x_i = \varphi(y_{i,1}) = \ldots = \varphi^n(y_{i,n}) = \ldots$. Since $x_i = \varphi(y_{i,1})$, we will be done by showing that $y_{i,1} \in \mathfrak{n}$. By injectivity of φ, $y_{i,1} = \varphi(y_{i,2}) = \ldots = \varphi^{n-1}(y_{i,n}) = \ldots$, which means $y_{i,1} \in \mathfrak{n}$.

Remark 5 Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system and let $\mathfrak{n}$ be as defined in Equation 5. If $\mathfrak{n} = (0)$, then by Lemma 5 R contains a field and is of equal characteristic. As noted in [3, Remark 5.9, p. 10], this occurs, for example, if φ is a contracting self-map.

Proposition 9 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Let $\mathfrak{a}$ be the ideal of R defined in Lemma 5, and let $\overline{\varphi}$ be the local self-map induced by φ on $R/\mathfrak{a}$. Then*

a) $h_{\mathrm{alg}}(\overline{\varphi}, R/\mathfrak{a}) = h_{\mathrm{alg}}(\varphi, R)$.
b) *If R is of mixed characteristic, then $R/\mathfrak{a}$ is of equal characteristic $p > 0$.*

Proof **a**) Note that $\varphi^n(\mathfrak{m})R \supset \mathfrak{a}$ for all $n \geqslant 1$. Hence, $\varphi^n(\mathfrak{m})R + \mathfrak{a} = \varphi^n(\mathfrak{m})R$. By Proposition 3, $\lambda(\overline{\varphi}^{\,n}) = \lambda(\varphi^n)$. Our claim quickly follows.
b) With reference to Lemma 5, the image of the subring S of R in $R/\mathfrak{a}$ is a field, because it's maximal ideal $\mathfrak{n}$ is contained in $\mathfrak{a}$ and is mapped to 0. Hence $R/\mathfrak{a}$ contains a field and must be a local ring of equal characteristic $p > 0$, as its residue field is of characteristic $p > 0$.

1.6 Algebraic entropy and degree

The analogy between algebraic and topological entropies also extends to their relation to the degree of the self-map. Misiurewicz and Przytycki showed in [28], that if f is a C^1 self-map of a smooth compact orientable manifold M, then $h_{\mathrm{top}}(f, M) \geqslant \log|\deg(f)|$. For a holomorphic self-map f of $\mathbb{CP}^n$, Gromov established the formula $h_{\mathrm{top}}(f, \mathbb{CP}^n) = \log|\deg(f)|$ in [15]. Here $\deg(f)$ is the topological degree of f.

In this section we obtain similar formulas relating algebraic entropy to degree of finite self-maps of local domains. For local Cohen-Macaulay domains we prove an analog of Gromov's formula. But first we shall make it clear what we mean by degree.

Definition 5 Let R be a Noetherian local domain, and let φ be a finite self-map of R. Then by *degree* of φ, $\deg(\varphi)$, we mean the rank of the R-module $\varphi_* R$. Note that the equality $\deg(\varphi^n) = [\deg(\varphi)]^n$ holds for all $n \in \mathbb{N}$.

Proposition 10 *Let (R, φ) be a local algebraic dynamical system, where φ is finite. If we denote the minimum number of generators of the R-module $\varphi^n_* R$ by $\mu(\varphi^n_* R)$, then the sequence $\{(\log \mu(\varphi^n_* R))/n\}$ converges to its infimum. We will denote this limit by by μ_∞.*

Proof We will apply Lemma 1, taking $b_n = \log \mu(\varphi^n_* R)$. To verify conditions of Lemma 1, first note that the inequality $b_{n+m} \leqslant b_n + b_m$ holds because if $\{x_1, \ldots, x_t\}$ and $\{y_1, \ldots, y_s\}$ are sets of generators of $\varphi^m_* R$ and $\varphi^n_* R$ over R, respectively, then $\{\varphi^m(y_j)x_i \mid 1 \leqslant i \leqslant t, 1 \leqslant j \leqslant s\}$ is a set of generators of $\varphi^{n+m}_* R$ over R. Therefore

$$\mu(\varphi^{n+m}_* R) \leqslant \mu(\varphi^n_* R) \cdot \mu(\varphi^m_* R).$$

On the other hand, it is clear that $b_n = \log \mu(\varphi^n_\star R) \geqslant 0$. Hence, by Lemma 1 the sequence $\{(\log \mu(\varphi^n_\star R))/n\}$ converges to its infimum.

Corollary 12 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system, where φ is finite, and let k be the residue field of R. Then $\mu_\infty = \log[\varphi_\star k : k] + h_{\mathrm{alg}}(\varphi, R)$, where μ_∞ is as defined in Proposition 10.*

Proof By Corollary 3 $\mu(\varphi^n_\star R) = [\varphi_\star k : k]^n \cdot \lambda(\varphi^n)$. The result follows by applying logarithm to both sides of this equation, then dividing by n and letting n approach infinity.

Lemma 6 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system, where R is a domain and φ is finite, and let k be the residue field of R. If $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal of R and $n \in \mathbb{N}$, then*

$$e(\varphi^n(\mathfrak{q})R) = \frac{e(\mathfrak{q})\,(\deg(\varphi))^n}{[\varphi_\star k : k]^n}. \tag{6}$$

Proof Let $d = \dim R$. By definition of multiplicity

$$\begin{aligned} e(\mathfrak{q}, \varphi^n_\star R) &= \lim_{m\to\infty} \frac{d!}{m^d} \cdot \ell_R\Big(\frac{\varphi^n_\star R}{\mathfrak{q}^m \cdot \varphi^n_\star R}\Big) \\ &= \lim_{m\to\infty} \frac{d!}{m^d} \cdot \ell_R\Big(\varphi^n_\star\big(\frac{R}{\varphi^n(\mathfrak{q}^m)R}\big)\Big) \\ &= \lim_{m\to\infty} \frac{d!}{m^d} \cdot \ell_R\Big(\varphi^n_\star\big(\frac{R}{(\varphi^n(\mathfrak{q})R)^m}\big)\Big) \end{aligned}$$

(for the last equality, see, e.g., [2, Exercise 1.18, p. 10]). Now by Proposition 2

$$\begin{aligned} \lim_{m\to\infty} \frac{d!}{m^d} \cdot \ell_R\Big(\varphi^n_\star\big(\frac{R}{(\varphi^n(\mathfrak{q})R)^m}\big)\Big) &= \lim_{m\to\infty} \frac{d!}{m^d} \cdot [\varphi^n_\star k : k] \cdot \ell_R\Big(\frac{R}{(\varphi^n(\mathfrak{q})R)^m}\Big) \\ &= [\varphi_\star k : k]^n \cdot \lim_{m\to\infty} \frac{d!}{m^d} \cdot \ell_R\Big(\frac{R}{(\varphi^n(\mathfrak{q})R)^m}\Big) \\ &= [\varphi_\star k : k]^n \cdot e(\varphi^n(\mathfrak{q})R). \end{aligned}$$

So $e(\mathfrak{q}, \varphi^n_\star R) = [\varphi_\star k : k]^n \cdot e(\varphi^n(\mathfrak{q})R)$. On the other hand

$$e(\mathfrak{q}, \varphi^n_\star R) = e(\mathfrak{q}) \cdot \deg(\varphi^n),$$

(see [27, Theorem 14.8]) and Formula 6 quickly follows.

Remark 6 Formula 6 can also be deduced from [40, Corollary 1, Chapter VIII].

Corollary 13 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system, where R is a domain and φ is finite, and let k be the residue field of R. Set $d := \dim R$ and define $q(\varphi) := \exp(h_{\mathrm{alg}}(\varphi, R)/d)$. Let $\chi(x_1, \ldots, x_d; R)$ be the Euler-Poincaré characteristic of the Koszul complex on elements $x_1, \ldots, x_d$. The following conditions are equivalent:*

a) $\log \deg(\varphi) = \log[\varphi_\star k : k] + h_{\mathrm{alg}}(\varphi, R)$

b) *For any system of parameters* $\{x_1, \ldots, x_d\}$ *of* R *and for any* $n \in \mathbb{N}$

$$\chi(\varphi^n(x_1), \ldots, \varphi^n(x_d); R) = q(\varphi)^{nd} \cdot \chi(x_1, \ldots, x_d; R). \tag{7}$$

c) *Equation 7 holds for* some *system of parameters of* R *and* some $n \in \mathbb{N}$.

Proof By [36, Chap. IV, Theorem 1] for any parameter ideal $\mathfrak{q}$ of R generated by a system of parameters $\{y_1, \ldots, y_d\}$ we have $e(\mathfrak{q}) = \chi(y_1, \ldots, y_d; R)$. By Corollary 2 and Proposition 1, $\{\varphi^n(x_1), \ldots, \varphi^n(x_d)\}$ is a system of parameters of R for all $n \in \mathbb{N}$. The result quickly follows from Proposition 6 and Equation 6 in Lemma 6.

Example 6 Let $(R, \mathfrak{m})$ be a Noetherian local domain of prime characteristic p, and let φ be the Frobenius endomorphism of R. Then by [24, Proposition 2.3] condition **a)** of Corollary 13 holds.

Proposition 11 *Let* $(R, \mathfrak{m}, \varphi)$ *be a local algebraic dynamical system, where* R *is a domain and* φ *is finite, and let* k *be the residue field of* R*. Then*

a) $\log \deg(\varphi) \leqslant \log[\varphi_* k : k] + h_{\text{alg}}(\varphi, R)$.

b) *If in addition* R *is Cohen-Macaulay,* $\log \deg(\varphi) = \log[\varphi_* k : k] + h_{\text{alg}}(\varphi, R)$.

Proof **a**) Consider a minimal free presentation of the R-module $\varphi^n_* R$

$$R^s \to R^t \to \varphi^n_* R \to 0.$$

If we localize this presentation at (0) we see $\operatorname{rank} \varphi^n_* R \leqslant t = \mu(\varphi^n_* R)$. On the other hand by Corollary 3, $\mu(\varphi^n_* R) = [\varphi_* k : k]^n \cdot \lambda(\varphi^n)$. Since by definition of degree $\operatorname{rank} \varphi^n_* R = \deg(\varphi^n) = (\deg(\varphi))^n$, we obtain

$$(\deg(\varphi))^n \leqslant [\varphi_* k : k]^n \cdot \lambda(\varphi^n).$$

The desired inequality is obtained by applying logarithm, dividing by n and letting n approach infinity.

b) Let $\mathfrak{q}$ be an arbitrary parameter ideal of R. Then

$$\lambda(\varphi^n) = \ell_R\left(R/\varphi^n(\mathfrak{m})R\right) \leqslant \ell_R\left(R/\varphi^n(\mathfrak{q})R\right).$$

If R is Cohen-Macaulay, then $\ell_R\left(R/\varphi^n(\mathfrak{q})R\right) = e(\varphi^n(\mathfrak{q})R)$ (see, for instance, [27, Theorem 17.11]). Thus

$$\lambda(\varphi^n) \leqslant e(\varphi^n(\mathfrak{q})R) = \frac{e(\mathfrak{q})(\deg(\varphi))^n}{[\varphi_* k : k]^n},$$

where the last equality holds by Lemma 6. Applying logarithm, dividing by n, and letting n approach infinity we obtain

$$h_{\text{alg}}(\varphi, R) \leqslant \log \deg(\varphi) - \log[\varphi_* k : k].$$

This inequality together with the inequality in part **a**) yield the result.

1.7 A note on projective varieties

In this section we will prove a formula similar to the formula in part **b)** of Proposition 11, for finite polarized self-maps of projective varieties over a field. We first establish a lemma.

Lemma 7 *Let $(X, \mathcal{O}_X)$ be a separated Noetherian integral scheme, and let α be an additive non-negative function from coherent $\mathcal{O}_X$-modules to $[0, \infty)$. Then α is a constant multiple of generic rank.*

Proof (due to Angelo Vistoli) By Noetherian induction we can assume that for every proper integral subscheme Y of X, the restriction of α to coherent $\mathcal{O}_Y$-modules is given by a constant multiple c_Y of generic rank at Y. Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules supported on a proper integral subscheme Y of X and let $\mathcal{I}$ be the ideal sheaf of Y in X. Since X is Noetherian, there is a (smallest) integer n such that $\mathcal{I}^n\mathcal{F} = 0$. Thus, $\mathcal{F}$ has a filtration

$$\mathcal{F} \supsetneq \mathcal{I}\mathcal{F} \supsetneq \mathcal{I}^2\mathcal{F} \supsetneq \ldots \supsetneq \mathcal{I}^n\mathcal{F} = (0).$$

So by additivity of α, $\alpha(\mathcal{F}) = \sum_{i=1}^{n} \alpha(\mathcal{I}^{i-1}\mathcal{F}/\mathcal{I}^i\mathcal{F})$. The sheaves $\mathcal{I}^{i-1}\mathcal{F}/\mathcal{I}^i\mathcal{F}$ are coherent sheaves of $\mathcal{O}_Y$-modules. Thus from the above sum we see that $\alpha(\mathcal{F})$ is equal to c_Y times the length of the stalk of $\mathcal{F}$ at the generic point of Y. On the other hand, the length of the stalk of the sheaf $\mathcal{O}_X/\mathcal{I}^n$ at the generic point of Y is unbounded, as $n \to \infty$. However, by additivity and positivity of α, the value of $\alpha(\mathcal{O}_X/\mathcal{I}^n)$ is bounded by $\alpha(\mathcal{O}_X)$. Hence $c_Y = 0$ and α is zero on all coherent $\mathcal{O}_X$-modules supported on a proper integral subscheme of X. Next, we show that α is zero on all coherent torsion sheaves. Let $\mathcal{F}$ be a coherent torsion sheaf of $\mathcal{O}_X$-modules. By [18, Corollary 3.2.8, p. 43] any coherent sheaf $\mathcal{F}$ has a filtration

$$\mathcal{F} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots \supseteq \mathcal{F}_n = (0)$$

consisting of coherent $\mathcal{O}_X$-modules, such that the quotients $\mathcal{F}_i/\mathcal{F}_{i+1}$ are either zero, or $\mathrm{Ass}(\mathcal{F}_i/\mathcal{F}_{i+1})$ is exactly a single point and $\mathrm{Ass}(\mathcal{F}_i/\mathcal{F}_{i+1}) \subset \mathrm{Supp}(\mathcal{F})$. Again by additivity of α, $\alpha(\mathcal{F}) = \sum_{i=0}^{n-1} \alpha(\mathcal{F}_i/\mathcal{F}_{i+1})$. If $\mathrm{Ass}(\mathcal{F}_i/\mathcal{F}_{i+1})$ is exactly a single point, then $\mathrm{Supp}(\mathcal{F}_i/\mathcal{F}_{i+1})$ is an irreducible proper (closed) subset of X (see [18, Corollary 3.1.4, p. 37]). Thus, from the previous part, $\alpha(\mathcal{F}) = 0$. In particular, if $\mathcal{F} \to \mathcal{G}$ is a generic isomorphism of coherent sheaves, then $\alpha(\mathcal{F}) = \alpha(\mathcal{G})$.

Now suppose $\mathcal{F}$ is a coherent torsion-free sheaf on X with generic rank r. Then there is an open affine neighborhood U of the generic point of X with a monomorphism $\mathcal{F}|_U \hookrightarrow \mathcal{O}_U^{\oplus r}$ (see [31, Chap. II, Lemma 1.1.8]). We can extend $\mathcal{F}|_U$ to a coherent sheaf $\mathcal{F}'$ on X with a monomorphism $\eta : \mathcal{F}' \hookrightarrow \mathcal{O}_X^{\oplus r}$ in such a way that $\mathcal{F}'|_U \cong \mathcal{F}|_U$ (see [13, Chap. VI, Lemma 3.5, p. 168]). Since η is a generic isomorphism, $\alpha(\mathcal{F}') = \alpha(\mathcal{O}_X^{\oplus r}) = r \cdot \alpha(\mathcal{O}_X)$. On the other hand, there is a coherent sheaf $\mathcal{G}$ on X with homomorphisms $\mathcal{G} \to \mathcal{F}$ and $\mathcal{G} \to \mathcal{F}'$ that are generic isomorphisms (see [13, Chap. VI, Lemma 3.7, p. 169]). The result follows.

A proof of the next theorem when X is a Kähler manifold appeared in [41, Lemma 1.1.1]. A. Chambert-Loir has also given a proof of this theorem. Here we present a proof using Lemma 7.

Proposition 12 *Let X be an integral projective variety of dimension d over a field k and let $\varphi : X \to X$ be a finite morphism. Assume that (X, φ) is polarized by an ample line bundle $\mathcal{L}$ on X, that is, for some integer $q \geqslant 1$, $\varphi^\star(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$. Then $\deg(\varphi) = q^d$.*

Proof To simplify notations, for any coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and for $n \in \mathbb{Z}$ we set $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$. By Projection Formula and using the assumption that $\mathcal{L}$ is polarized,

$$(\varphi_\star \mathcal{O}_X)(n) \cong \varphi_\star(\mathcal{O}_X \otimes_{\mathcal{O}_X} \varphi^\star(\mathcal{L}^{\otimes n})) \cong \varphi_\star(\mathcal{L}^{\otimes nq}) = \varphi_\star(\mathcal{O}_X(nq)), \quad \text{for } n \in \mathbb{Z}.$$

Since φ is a finite morphism, it is affine. Hence (see [17, Corollary 1.3.3, p. 88])

$$\mathrm{H}^i(X, \varphi_\star(\mathcal{O}_X(nq))) \cong \mathrm{H}^i(X, \mathcal{O}_X(nq)), \quad \text{for } i \geqslant 0.$$

Writing $\chi_k(\,\cdot\,)$ for the Euler-Poincaré characteristic, we obtain

$$\chi_k\left((\varphi_\star \mathcal{O}_X)(n)\right) = \chi_k(\mathcal{O}_X(nq)). \tag{8}$$

Replacing $\mathcal{L}$ with $\mathcal{L}^{\otimes m}$ for large m if necessary, we may assume, without loss of generality, that $\mathcal{L}$ is very ample ([16, Proposition 4.5.10, p. 86]). Then for any coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ and any $n \in \mathbb{Z}$, the value of $\chi_k(\mathcal{F}(n))$ is equal to the value of the Hilbert polynomial of $\mathcal{F}$ at n, and the coefficient of the leading term of the Hilbert polynomial of $\mathcal{F}$ is non negative (see [17, Theorem 2.5.3, p. 109]). Since $\chi_k(\,\cdot\,)$ is an additive function on the category of coherent $\mathcal{O}_X$-modules, we obtain an additive non negative function

$$\alpha(\mathcal{F}) := \lim_{n \to \infty} \frac{\chi_k(\mathcal{F}(n))}{n^d}$$

from the category of coherent $\mathcal{O}_X$-modules to rational numbers. Note that if $\dim \mathrm{Supp}(\mathcal{F}) < d$ then $\alpha(\mathcal{F}) = 0$ (see [18, Proposition 5.3.1, p. 92]). From Equation 8 we quickly obtain $\alpha(\varphi_\star \mathcal{O}_X) = \alpha(\mathcal{O}_X) \cdot q^d$. On the other hand, using Lemma 7

$$\alpha(\varphi_\star \mathcal{O}_X) = \alpha(\mathcal{O}_X) \cdot \deg(\varphi).$$

Hence $\deg(\varphi) = q^d$.

1.8 The case of integral self-maps

In this section we study local algebraic dynamical systems (R, φ) generated by integral self-maps. We show that when $\operatorname{Spec} R = V(\ker \varphi)$, ${}^a\varphi$ permutes the irreducible components of $\operatorname{Spec} R$. Thus, there is a smallest number p such that all irreducible components of $\operatorname{Spec} R$ are φ^p-invariant. We give formulas relating algebraic entropy of φ^p to algebraic entropies of its restrictions to irreducible components of $\operatorname{Spec} R$.

Proposition 13 *Let (R,φ) be a local algebraic dynamical system. Assume that φ is* integral *and* $\operatorname{Spec} R = V(\ker\varphi)$*. Then the restriction of* ${}^a\varphi$ *to* $\operatorname{Min}(R)$ *is a permutation of* $\operatorname{Min}(R)$*.*

Proof Let $\tilde{\varphi} : (R/\ker\varphi) \hookrightarrow R$ be the map induced by φ. We have a commuting diagram

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\pi"'] & R \\ R/\ker\varphi \arrow[ur, hook, "\tilde{\varphi}"'] & \end{tikzcd}$$

Let $\mathfrak{q} \in \operatorname{Min}(R)$. Then by assumption $\ker\varphi \subset \mathfrak{q}$, hence $\pi(\mathfrak{q}) \in \operatorname{Min}(R/\ker\varphi)$. Since φ is integral, there is an element $\mathfrak{p} \in \operatorname{Spec} R$ such that $\pi(\mathfrak{q}) = \tilde{\varphi}^{\,-1}(\mathfrak{p})$. Thus, $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$, or equivalently $\mathfrak{q} = {}^a\varphi(\mathfrak{p})$. We claim that $\mathfrak{p} \in \operatorname{Min}(R)$. If $\mathfrak{p}$ were not a minimal prime ideal of R, then it would contain a minimal prime ideal $\mathfrak{p}'$. In that case $\pi(\mathfrak{q}) = \tilde{\varphi}^{\,-1}(\mathfrak{p}) \supseteq \tilde{\varphi}^{\,-1}(\mathfrak{p}')$ and the minimality of $\pi(\mathfrak{q})$ would force $\tilde{\varphi}^{\,-1}(\mathfrak{p}') = \pi(\mathfrak{q})$. But since φ is integral, there can be no inclusion between prime ideals of R lying over $\pi(\mathfrak{q})$ [27, Theorem 9.3]. This establishes our claim that $\mathfrak{p} \in \operatorname{Min}(R)$. Thus, we see that

$$\operatorname{Min}(R) \subseteq {}^a\varphi\left(\operatorname{Min}(R)\right).$$

Now, since $\operatorname{Min}(R)$ is a finite set, we must have $\operatorname{Min}(R) = {}^a\varphi\left(\operatorname{Min}(R)\right)$. Hence the restriction of ${}^a\varphi$ to $\operatorname{Min}(R)$ is a bijective map of the set $\operatorname{Min}(R)$ to itself.

Corollary 14 *Let (R,φ) be a local algebraic dynamical system. Assume that φ is* integral *and* $\operatorname{Spec} R = V(\ker\varphi)$*. Let p be the smallest integer such that ${}^a\varphi^p$ is the identity map on* $\operatorname{Min}(R)$*. For $\mathfrak{p}_i \in \operatorname{Min}(R)$ let $\overline{\varphi}_i$ be the self-map induced by φ^p on $R/\mathfrak{p}_i$. Then*

$$h_{\mathrm{alg}}(\varphi, R) = \frac{1}{p} \cdot \max\{h_{\mathrm{alg}}(\overline{\varphi}_i, R/\mathfrak{p}_i) \mid \mathfrak{p}_i \in \operatorname{Min}(R)\}.$$

Proof By Proposition 8, $h_{\mathrm{alg}}(\varphi^p, R) = \max\{h_{\mathrm{alg}}(\overline{\varphi}_i, R/\mathfrak{p}_i) \mid \mathfrak{p}_i \in \operatorname{Min}(R)\}$. By Proposition 6, $h_{\mathrm{alg}}(\varphi^p, R) = p \cdot h_{\mathrm{alg}}(\varphi, R)$ and the result follows.

Corollary 15 *Let (R,φ) be a local algebraic dynamical system. Suppose φ is* integral *and* $\operatorname{Spec} R = V(\ker\varphi)$*. Then an element $x \in R$ belongs to a minimal prime ideal of R, if and only if $\varphi(x)$ belongs to a minimal prime ideal of R.*

Proof Let x be an element of R. If $\varphi(x) \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Min}(R)$, then $x \in \varphi^{-1}(\mathfrak{p})$. By Proposition 13, $\varphi^{-1}(\mathfrak{p}) \in \operatorname{Min}(R)$. Conversely, suppose $x \in \mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Min}(R)$. Then by Proposition 13 there is a $\mathfrak{p} \in \operatorname{Min}(R)$ such that $\mathfrak{q} = \varphi^{-1}(\mathfrak{p})$. Hence $\varphi(x) \in \mathfrak{p}$.

Corollary 16 *Let (R,φ) be a local algebraic dynamical system. Assume that φ is* integral *and* $\operatorname{Spec} R = V(\ker\varphi)$*. If $\mathfrak{p} \notin \operatorname{Min}(R)$, then $\varphi^{-1}(\mathfrak{p}) \notin \operatorname{Min}(R)$.*

Proof This follows quickly from the proof of Proposition 13.

Remark 7 If (R, φ) is a local algebraic dynamical system, then for every $n \in \mathbb{N}$, $\varphi(\ker \varphi^n) \subset \ker \varphi^{n-1} \subset \ker \varphi^n$. Hence φ induces a local self-map of $R/\ker \varphi^n$.

Proposition 14 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Let $\overline{\varphi}_n$ be the local self-map induced by φ on $R/\ker \varphi^n$, $n \in \mathbb{N}$. Then*

a) $h_{\mathrm{alg}}(\varphi, R) = h_{\mathrm{alg}}(\overline{\varphi}_n, R/\ker \varphi^n)$.
b) *For large n, $\overline{\varphi}_n : R/\ker \varphi^n \to R/\ker \varphi^n$ is injective.*
c) *If φ is integral, then so is $\overline{\varphi}_n$* (see [7, Chapter V, Proposition 2, p. 305]).

Proof **a**) Apply Corollary 11 to the self-map φ^n of R, taking $\ker \varphi^n$ as the ideal $\mathfrak{a}$ in that corollary. Since $\varphi^n(\ker \varphi^n)R = (0)$

$$h_{\mathrm{alg}}(\overline{\varphi}_n^n, R/\ker \varphi^n) = h_{\mathrm{alg}}(\overline{\varphi}_n^n, R/\varphi^n(\ker \varphi^n)R) = h_{\mathrm{alg}}(\varphi^n, R).$$

The result follows from Proposition 6.
b) R is Noetherian, so the ascending chain $\ker \varphi \subset \ker \varphi^2 \subset \ker \varphi^3 \subset \ldots$ is stationary. Let n_0 be such that $\ker \varphi^n = \ker \varphi^{n+1}$ for $n \geqslant n_0$. We will show that if $n \geqslant n_0$, then $\overline{\varphi}_n : R/\ker \varphi^n \to R/\ker \varphi^n$ is injective. Let $\overline{x} \in R/\ker \varphi^n$. Saying $\overline{\varphi}_n(\overline{x}) = 0$ is equivalent to saying $\varphi(x) \in \ker \varphi^n$, which is equivalent to saying $x \in \ker \varphi^{n+1}$. Since $\ker \varphi^{n+1} = \ker \varphi^n$, we see that $x \in \ker \varphi^n$, or $\overline{x} = 0$ in $R/\ker \varphi^n$. Thus, $\overline{\varphi}_n$ is injective.
c) Let $\pi_n : R \to R/\ker \varphi^n$ be the canonical surjection. Then π_n is in fact a morphism between local dynamical systems $(R, \varphi) \to (R/\ker \varphi^n, \overline{\varphi}_n)$. Let $\pi_n(x) \in R/\ker \varphi^n$. Since φ is integral, x satisfies an equation

$$x^n + \varphi(a_{n-1})x^{n-1} + \ldots + \varphi(a_1)x + \varphi(a_0) = 0, \quad a_i \in R.$$

Apply π_n and note that since π_n is a morphism, $\pi_n \circ \varphi = \overline{\varphi}_n \circ \pi_n$. We obtain

$$(\pi_n(x))^n + \overline{\varphi}_n(\pi_n(a_{n-1}))(\pi_n(x))^{n-1} + \ldots + \overline{\varphi}_n(\pi_n(a_0)) = 0.$$

Thus $\pi_n(x)$ is integral over the subring $\overline{\varphi}_n(R/\ker \varphi^n)$ of $R/\ker \varphi^n$.

Proposition 15 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system, where φ is integral. Let $\overline{\varphi}_n$ be the self-map induced by φ on $R/\ker \varphi^n$, $n \in \mathbb{N}$, and let $\pi_n : R \to R/\ker \varphi^n$ be the canonical surjection. Fix a large enough n for which $\overline{\varphi}_n$ is injective and let p be the smallest integer such that ${}^a\overline{\varphi}_n^p$ is the identity map on $\mathrm{Min}(R/\ker \varphi^n)$. Then*

a) ${}^a\pi_n\big(\mathrm{Min}(R/\ker \varphi^n)\big) = \mathrm{Min}(R) \cap V(\ker \varphi^n)$.
b) *If $\mathfrak{p}_i \in \mathrm{Min}(R) \cap V(\ker \varphi^n)$ then $\mathfrak{p}_i$ is φ^p-invariant.*
c) *For $\mathfrak{p}_i \in \mathrm{Min}(R) \cap V(\ker \varphi^n)$ if $\overline{\varphi}_{\mathfrak{p}_i}$ is the self-map induced by φ^p on $R/\mathfrak{p}_i$*

$$h_{\mathrm{alg}}(\varphi, R) = \frac{1}{p} \cdot \max\left\{h_{\mathrm{alg}}(\overline{\varphi}_{\mathfrak{p}_i}, R/\mathfrak{p}_i) \mid \mathfrak{p}_i \in \mathrm{Min}(R) \cap V(\ker \varphi^n)\right\}.$$

Proof **a**) It is clear that ${}^a\pi_n\big(\operatorname{Min}(R/\ker\varphi^n)\big) \supseteq \operatorname{Min}(R) \cap V(\ker\varphi^n)$. To show the inclusion in the other direction, let $\tilde{\varphi^n} : R/\ker\varphi^n \hookrightarrow R$ be the map induced by φ^n. We have a commuting diagram

$$\begin{tikzcd} R \arrow[r, "\varphi^n"] \arrow[d, "\pi_n"'] & R \\ R/\ker\varphi^n \arrow[ur, hook, "\tilde{\varphi^n}"'] & \end{tikzcd}$$

Let $\mathfrak{p} \in {}^a\pi_n\big(\operatorname{Min}(R/\ker\varphi^n)\big)$. If $\mathfrak{p} \notin \operatorname{Min}(R)$, then it would contain a prime ideal $\mathfrak{p}' \in \operatorname{Min}(R)$. By assumption $\overline{\varphi}_n$ is injective and integral. Thus, ${}^a\overline{\varphi}_n$ must permute elements of $\operatorname{Min}(R/\ker\varphi^n)$ by Proposition 13. In particular, $\overline{\varphi}_n^{\,-n}(\pi_n(\mathfrak{p})) \in \operatorname{Min}(R/\ker\varphi^n)$. Since $(\tilde{\varphi^n})^{-1}(\mathfrak{p}') \subset (\tilde{\varphi^n})^{-1}(\mathfrak{p}) = \overline{\varphi}_n^{\,-n}(\pi_n(\mathfrak{p}))$, we see that $(\tilde{\varphi^n})^{-1}(\mathfrak{p}') \in \operatorname{Min}(R/\ker\varphi^n)$. Thus, $(\tilde{\varphi^n})^{-1}(\mathfrak{p}') = (\tilde{\varphi^n})^{-1}(\mathfrak{p})$. But this is a contradiction, because φ^n is integral, and there can be no inclusion between prime ideals of R lying over $\overline{\varphi}_n^{\,-n}(\pi_n(\mathfrak{p}))$ [27, Theorem 9.3]. Thus, $\mathfrak{p} \in \operatorname{Min}(R)$ as claimed.
b) $\pi_n : (R, \varphi^p) \to (R/\ker\varphi^n, \overline{\varphi}_n^{\,p})$ is a morphism between local dynamical systems. In other words, there is a commutative diagram

$$\begin{tikzcd} R \arrow[r, "\varphi^p"] \arrow[d, "\pi_n"'] & R \arrow[d, "\pi_n"] \\ R/\ker\varphi^n \arrow[r, "\overline{\varphi}_n^{\,p}"] & R/\ker\varphi^n. \end{tikzcd}$$

From this diagram and the assumption that ${}^a\overline{\varphi}_n^{\,p}$ is the identity map on $\operatorname{Min}(R/\ker\varphi^n)$, and by part **a**) it quickly follows that $\varphi^p(\mathfrak{p}_i)R \subset \mathfrak{p}_i$, for all $\mathfrak{p}_i \in \operatorname{Min}(R) \cap V(\ker\varphi^n)$.
c) By Proposition 14-**a** and Proposition 6

$$h_{\mathrm{alg}}(\varphi, R) = \frac{1}{p} \cdot h_{\mathrm{alg}}(\overline{\varphi}_n^{\,p}, R/\ker\varphi^n).$$

Applying Proposition 8 to the local algebraic system $(R/\ker\varphi^n, \overline{\varphi}_n^{\,p})$ we obtain

$$h_{\mathrm{alg}}(\overline{\varphi}_n^{\,p}, R/\ker\varphi^n) = \max\Big\{h_{\mathrm{alg}}\Big(\overline{\varphi}_{\overline{\mathfrak{p}}_i}, \frac{R/\ker\varphi^n}{\mathfrak{p}_i/\ker\varphi^n}\Big) \mid \mathfrak{p}_i \in \operatorname{Min}(R) \cap V(\ker\varphi^n)\Big\},$$

where $\overline{\varphi}_{\overline{\mathfrak{p}}_i}$ is the self-map induced by $\overline{\varphi}_n^{\,p}$ on $(R/\ker\varphi^n)/(\mathfrak{p}_i/\ker\varphi^n)$. To finish the proof, apply Proposition 3 first and then Proposition 2 to obtain

$$h_{\mathrm{alg}}\Big(\overline{\varphi}_{\overline{\mathfrak{p}}_i}, \frac{R/\ker\varphi^n}{\mathfrak{p}_i/\ker\varphi^n}\Big) = h_{\mathrm{alg}}(\overline{\varphi}_{\mathfrak{p}_i}, R/\mathfrak{p}_i).$$

1.9 Alternative methods for computing entropy

In this section we will show that algebraic entropy can be computed using any module of finite length. We begin with a definition.

Definition 6 Let R be a Noetherian local ring, and let φ be a self-map of R. Let R-Mod be the category of R-modules. For every $n \in \mathbb{N}$ we define a functor Φ^n : R-Mod $\to$ R-Mod as follows: if $M \in$ R-Mod, then

$$\Phi^n(M) := M \otimes_R \varphi^n_\star R, \tag{9}$$

where the R-module structure of $\Phi^n(M)$ is defined to be

$$r \cdot x = \sum m_i \otimes r \cdot r_i, \quad \text{if} \quad x = \sum m_i \otimes r_i \in \Phi^n(M) \quad \text{and} \quad r \in R.$$

For the Frobenius endomorphism the functors defined in Definition 6 are known as Frobenius functors. They were first introduced in [32, Definition 1.2]. Important properties of Frobenius functors were established in [32] and [19]. The same proofs can be re-written for the functors Φ^n and will establish the next proposition.

Proposition 16 *Let R be a Noetherian local ring, and let φ be a local self-map of R. The functor Φ^n, $n \in \mathbb{N}$ has the following properties:*
a) *Φ^n is a right-exact functor.*
b) *If R^s is a finitely generated free module, then $\Phi^n(R^s) \cong R^s$.*
c) *Let $R^s \xrightarrow{\alpha} R^t$ be a map of finitely generated free R-modules. Choose bases $\mathscr{B}_s$ and $\mathscr{B}_t$ for R^s and R^t, and let (a_{ij}) be the matrix representation of α in these bases. Then the matrix representation of $\Phi^n(\alpha)$ in the bases of $\Phi^n(R^s)$ and $\Phi^n(R^t)$ obtained from $\mathscr{B}_s$ and $\mathscr{B}_t$ by applying the isomorphism of part* **b**) *is $(\varphi^n(a_{ij}))$.*
d) *If $\mathfrak{a}$ is an ideal of R, then $\Phi^n(R/\mathfrak{a}) \cong R/\varphi^n(\mathfrak{a})R$, as R-modules.*
e) *If M is an R-module of finite length, then $\Phi^n(M)$ is an R-module of finite length, and $\ell_R(\Phi^n(M)) \leqslant \ell_R(M) \cdot \lambda(\varphi^n)$.*

Proof As mentioned above, parts **a**) to **d**) are standard. Part **e**) is restatement of Proposition 4 in terms of Φ^n.

Proposition 17 *Let (R, φ) be a local algebraic dynamical system. If M is a nonzero module of finite length, then*

$$h_{\mathrm{alg}}(\varphi, R) = \lim_{n\to\infty} \frac{1}{n} \cdot \log \ell_R(\Phi^n(M)).$$

Proof By Proposition 16-**d**, $\Phi^n(R/\mathfrak{m}) \cong R/\varphi^n(\mathfrak{m})R$. Thus,

$$\ell_R(\Phi^n(R/\mathfrak{m})) = \ell_R(R/\varphi^n(\mathfrak{m})R) = \lambda(\varphi^n).$$

Since M is of finite length, there is a surjection $M \to R/\mathfrak{m} \to 0$. Apply the functor Φ^n to obtain a surjection $\Phi^n(M) \to \Phi^n(R/\mathfrak{m}) \to 0$. Using this surjection and by Propositiob 16-**e**

$$\lambda(\varphi^n) = \ell_R(\Phi^n(R/\mathfrak{m})) \leqslant \ell_R(\Phi^n(M)) \leqslant \lambda(\varphi^n) \cdot \ell_R(M).$$

The result follows after applying logarithm, dividing by n and letting $n \to \infty$.

Proposition 18 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system. Assume $\varphi(\mathfrak{m})R \neq \mathfrak{m}$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \cdot \log \ell_R \left(\mathfrak{m}/\varphi^n(\mathfrak{m})R\right) = h_{\mathrm{alg}}(\varphi, R).$$

Proof From the exact sequence: $0 \to \mathfrak{m}/\varphi^n(\mathfrak{m})R \to R/\varphi^n(\mathfrak{m})R \to R/\mathfrak{m} \to 0$,

$$\begin{aligned}\ell_R \left(\mathfrak{m}/\varphi^n(\mathfrak{m})R\right) &= \ell_R \left(R/\varphi^n(\mathfrak{m})R\right) - \ell_R \left(R/\mathfrak{m}\right)\\ &= \ell_R \left(R/\varphi^n(\mathfrak{m})R\right) - 1.\end{aligned}$$

Since $\varphi(\mathfrak{m})R \neq \mathfrak{m}$, $\lambda(\varphi^n) = \ell_R \left(R/\varphi^n(\mathfrak{m})R\right) \geqslant 2$. Thus

$$\frac{1}{2}\,\lambda(\varphi^n) \leqslant \lambda(\varphi^n) - 1 = \ell_R \left(\mathfrak{m}/\varphi^n(\mathfrak{m})R\right) \leqslant \lambda(\varphi^n).$$

Apply logarithm, divide by n and let n approach infinity.

2 Regularity and contracting self-maps

Our main objective in this section is to give a proof of Theorems 2 and 3. Let $(R, \mathfrak{m})$ be a Noetherian local ring of positive prime characteristic p and of dimension d, and let φ be the Frobenius endomorphism of R. In [23] Kunz showed that the following conditions are equivalent:

a) R is regular.
b) φ is flat.
c) $\lambda(\varphi) = p^d$.
d) $\lambda(\varphi^n) = p^{nd}$ for some $n \in \mathbb{N}$.

Later Rodicio showed in [33], that these conditions are also equivalent to

e) flat $\dim_R \varphi_\star R < \infty$.

At first glance, Kunz' conditions **c**) and **d**) appear to be stated in terms of the characteristic p of the ring and one may not expect to be able to extend, or even state them in arbitrary characteristic. Nevertheless, algebraic entropy can be used to make sense of Kunz' numerical conditions **c**) and **d**) for all self-maps of finite length in any characteristic. Theorem 2 states that with this new interpretation, all conditions in Kunz' result are still equivalent.

We should also note that in [4, Theorem 13.3] Avramov, Iyengar and Miller have extended the equivalence of conditions **a**) and **b**) of Kunz and **e**) of Rodicio to contracting local self-maps of Noetherian local rings in all characteristics.

We list two results here that we will need in our proof of Theorem 2.

Lemma 8 ([19, Lemma 3.2]) *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and let M be a finitely generated R-module. Consider an ideal $\mathfrak{b} \subseteq \mathfrak{m}$ of R. Then there exists an integer $\mu_0 \geqslant 0$ such that* $\mathrm{depth}(\mathfrak{m}, \mathfrak{b}^\mu M) > 0$ *for all $\mu \geqslant \mu_0$.*

Remark 8 Lemma 8 must be used together with the standard convention that the depth of the zero module is ∞ (see, for example, [20, p. 291]). Otherwise, if M is an R-module of finite length, then for $\mu \gg 0$ we have $\mathfrak{m}^{\mu}M = (0)$, and this would have been a counter-example to Lemma 8.

The next proposition is taken from [8, Chap. 10, § 1, Proposition 1].

Proposition 19 *Let R be a Noetherian ring and let $\mathfrak{a}$ be an ideal of R. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. If we define $d' = \operatorname{depth}(\mathfrak{a}, M')$, $d = \operatorname{depth}(\mathfrak{a}, M)$, and $d'' = \operatorname{depth}(\mathfrak{a}, M'')$, then one of the following mutually exclusive possibilities hold:*

$$d' = d \leqslant d'' \text{ or } d = d'' < d' \text{ or } d'' = d' - 1 < d.$$

2.1 Kunz' Regularity Criterion via algebraic entropy

In order to prove Theorem 2 we first need to establish two lemmas. We begin with a flatness criterion that is due to Nagata. A proof can be found in [30, Chap. II, Theorem 19.1]. See also [27, Ex. 22.1, p. 178].

Theorem 5 (Nagata) *Let $g : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be an injective homomorphism of finite length of Noetherian local rings. Then S is flat over R, if and only if for every $\mathfrak{m}$-primary ideal $\mathfrak{q}$ of R,*

$$\ell_R(R/\mathfrak{q}) \cdot \ell_S(S/g(\mathfrak{m})S) = \ell_S(S/g(\mathfrak{q})S). \tag{10}$$

We need a stronger version of Nagata's theorem that we state and prove here.

Lemma 9 *Let $g : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of finite length of Noetherian local rings. If Equation 10 holds for a family of $\mathfrak{m}$-primary ideals $\{\mathfrak{q}_\alpha\}_{\alpha \in A}$ that define the $\mathfrak{m}$-adic topology, then it holds for all $\mathfrak{m}$-primary ideals.*

Proof Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal. We will show Equation 10 holds for $\mathfrak{q}$. First, using Proposition 4

$$\ell_S(S/g(\mathfrak{q})S) = \ell_S(S \otimes_R R/\mathfrak{q}) \leqslant \lambda(g) \cdot \ell_R(R/\mathfrak{q}).$$

To show the reverse inequality, note that by assumption there is a $\mathfrak{q}_\alpha \subseteq \mathfrak{q}$. The exact sequence $0 \to \mathfrak{q}/\mathfrak{q}_\alpha \to R/\mathfrak{q}_\alpha \to R/\mathfrak{q} \to 0$ yields

$$\ell_R(R/\mathfrak{q}_\alpha) = \ell_R(R/\mathfrak{q}) + \ell_R(\mathfrak{q}/\mathfrak{q}_\alpha). \tag{11}$$

If we tensor the previous exact sequence with S, we obtain an exact sequence of S-modules $\mathfrak{q}/\mathfrak{q}_\alpha \otimes_R S \to S/g(\mathfrak{q}_\alpha)S \to S/g(\mathfrak{q})S \to 0$. Thus

$$\ell_S(S/g(\mathfrak{q}_\alpha)S) \leqslant \ell_S(S/g(\mathfrak{q})S) + \ell_S(\mathfrak{q}/\mathfrak{q}_\alpha \otimes_R S).$$

Since Equation 10 holds for $\mathfrak{q}_\alpha$, and by using Proposition 4 we quickly see

$$\ell_R(R/\mathfrak{q}_\alpha) \cdot \lambda(g) \leqslant \ell_S(S/g(\mathfrak{q})S) + \ell_R(\mathfrak{q}/\mathfrak{q}_\alpha) \cdot \lambda(g).$$

Now using Equation 11 we quickly obtain $\lambda(g) \cdot \ell_R(R/\mathfrak{q}) \leqslant \ell_S(S/g(\mathfrak{q})S)$.

Lemma 10 *Let $(R, \mathfrak{m}, \varphi)$ be a local algebraic dynamical system, and let $\mathfrak{a}$ be a φ-invariant ideal of R. Let $\overline{\varphi}$ be the self-map of $R/\mathfrak{a}$ induced by φ. Set $d := \dim R$ and $\overline{d} := \dim R/\mathfrak{a}$ and let $q(\varphi)$ be as defined in Theorem 2.*
i) *If $\lambda(\varphi^n) = q(\varphi)^{nd}$ for some $n \in \mathbb{N}$, then $\lambda(\varphi^{nt}) = q(\varphi)^{ntd}$ for all $t \in \mathbb{N}$.*
ii) *If in addition to the assumption in* **i**) *we have $h_{\mathrm{alg}}(\overline{\varphi}, R/\mathfrak{a}) = h_{\mathrm{alg}}(\varphi, R)$ and if φ is contracting, then $\mathfrak{a} = (0)$.*

Proof **i**) Let $t \in \mathbb{N}$. As the sequence $\{\log \lambda(\varphi^{nt})/(nt)\}$ converges to its infimum by Theorem 2,

$$h_{\mathrm{alg}}(\varphi, R) \leqslant \log \lambda(\varphi^{nt})/(nt).$$

From this inequality we quickly obtain $q(\varphi)^{ntd} \leqslant \lambda(\varphi^{nt})$. Also, by Corollary 5, $\lambda(\varphi^{nt}) \leqslant \lambda(\varphi^n)^t$. Using assumption **i**) and the previous inequalities we obtain

$$q(\varphi)^{ntd} \leqslant \lambda(\varphi^{nt}) \leqslant \lambda(\varphi^n)^t = q(\varphi)^{ntd}.$$

Hence, $\lambda(\varphi^{nt}) = q(\varphi)^{ntd}$ for all $t \in \mathbb{N}$.
ii) Similar to the previous part, we can write

$$q(\overline{\varphi})^{nt\overline{d}} \leqslant \lambda(\overline{\varphi}^{nt}) \leqslant \lambda(\varphi^{nt}) = q^{ntd}. \tag{12}$$

From assumption **ii**) it follows $q(\overline{\varphi})^{\overline{d}} = q(\varphi)^d$. Then from Equation 12 we conclude $\lambda(\overline{\varphi}^{nt}) = \lambda(\varphi^{nt})$ for all $t \in \mathbb{N}$. Since $\lambda(\overline{\varphi}^{nt}) = \ell_R(R/[\varphi^{nt}(\mathfrak{m})R + \mathfrak{a}])$ by Proposition 3, we obtain

$$\ell_R(R/[\varphi^{nt}(\mathfrak{m})R + \mathfrak{a}]) = \ell_R(R/\varphi^{nt}(\mathfrak{m})R), \quad \forall t \in \mathbb{N}. \tag{13}$$

The surjection $R/\varphi^{nt}(\mathfrak{m})R \to R/[\varphi^{nt}(\mathfrak{m})R + \mathfrak{a}]) \to 0$ and Equation 13 then show

$$R/[\varphi^{nt}(\mathfrak{m})R + \mathfrak{a}] = R/\varphi^{nt}(\mathfrak{m})R, \quad \forall t \in \mathbb{N}.$$

Hence,

$$\mathfrak{a} \subset \textstyle\bigcap_{t \in \mathbb{N}} \varphi^{nt}(\mathfrak{m})R = (0),$$

where the last equality follows from Remark 1 because φ is by assumption, contracting.

Proof (of Theorem 2) **a**) $\Rightarrow$ **b**): To say that φ is of finite length means $\dim R/\varphi(\mathfrak{m})R = 0$. Hence, the following equation holds:

$$\dim R = \dim R + \dim R/\varphi(\mathfrak{m})R.$$

Since R is regular, the result follows from [27, Theorem 23.1].
b) $\Rightarrow$ **c**): This follows from Corollary 5. Since φ is flat by assumption, by that corollary $\lambda(\varphi^n) = \lambda(\varphi)^n$ for all $n \in \mathbb{N}$. Thus, by definition of algebraic entropy

$$\begin{aligned} h_{\mathrm{alg}}(\varphi, R) &= \lim_{n \to \infty} (1/n) \cdot \log \lambda(\varphi^n) \\ &= \lim_{n \to \infty} (1/n) \cdot \log \lambda(\varphi)^n \\ &= \log \lambda(\varphi). \end{aligned}$$

This means $\lambda(\varphi) = q(\varphi)^d$.
c) $\Rightarrow$ **d**): This is clear.
b) $\Rightarrow$ **a**): We use Herzog's proof in [19, Satz 3.1]. We re-write it for an arbitrary self-map here. See also [9, Lemma 3]. To show that R is regular, it suffices to show all finitely generated R-modules have finite projective dimension. So let M be a finitely generated R-module. Suppose M were of infinite projective dimension. Consider a minimal (infinite) free resolution of M

$$L_\bullet \to M \to 0.$$

Let $s := \operatorname{depth}(\mathfrak{m}, R)$, and take an R-regular sequence of elements $\{x_1, \ldots, x_s\}$ in $\mathfrak{m}$. Write $\mathfrak{a}$ for the ideal generated by this regular sequence. (If $s = 0$, take $\mathfrak{a} = (0)$.) Let Φ^n be the functor defined in Definition 6. For every $n \in \mathbb{N}$ we set

$$C^n_\bullet := \Phi^n(L_\bullet) \otimes_R R/\mathfrak{a} \quad \text{ and } \quad B^n_i := \operatorname{image}(C^n_{i+1} \to C^n_i).$$

Using Proposition 16-**b**, we quickly see that $C^n_i \cong L_i/\mathfrak{a}L_i$. This shows that C^n_i is independent of n, and that C^n_i is a nonzero finitely generated module of depth zero for all i. Using Proposition 16-**c**, we can see that $B^n_i \subseteq \varphi^n(\mathfrak{m})C^n_i$. Applying Lemma 8, let μ_{i_0} be such that $\operatorname{depth}(\mathfrak{m}, \mathfrak{m}^{\mu_{i_0}} C^n_i) > 0$. Since φ is contracting by assumption, from Remark 1 it easily follows that if n is large enough, then $\varphi^n(\mathfrak{m})R \subseteq \mathfrak{m}^{\mu_{i_0}}$ and in that case, $B^n_i \subseteq \varphi^n(\mathfrak{m})C^n_i \subseteq \mathfrak{m}^{\mu_{i_0}} C^n_i$. This shows that $\operatorname{depth}(\mathfrak{m}, B^n_i) > 0$ for large n. On the other hand, since φ is flat, $\Phi^n(L_\bullet)$ is exact. Thus, by parts **a**), **b**), and **c**) of Proposition 16

$$\Phi^n(L_\bullet) \to \Phi^n(M) \to 0$$

is a minial (infinite) free resolution of $\Phi^n(M)$. Hence

$$\mathrm{H}_i(C^n_\bullet) = \operatorname{Tor}^R_i(\Phi^n(M), R/\mathfrak{a}) = 0, \quad \text{for} \quad i > s.$$

This shows that if $i > s$, then the sequences

$$0 \to B^n_{i+1} \to C^n_{i+1} \to B^n_i \to 0 \tag{14}$$

are exact for all $n \in \mathbb{N}$. Take $i = s + 1$ in Sequence 14, for instance. By the above argument, if we take n large enough, we will obtain $\operatorname{depth}(\mathfrak{m}, B^n_{s+1}) > 0$ and $\operatorname{depth}(\mathfrak{m}, B^n_{s+2}) > 0$, while $\operatorname{depth}(\mathfrak{m}, C^n_{s+2}) = 0$. By Proposition 19 this is not possible. Hence, the projective dimension of M must be finite.
d) $\Rightarrow$ **b**): We will use Nagata's Flatness Theorem to show that φ^n is flat. We first need to show that φ is injective. Clearly $\ker\varphi$ is φ-invariant. Let $\overline{\varphi}$ be the local self-map induced by φ on $R/\ker\varphi$. Then by Proposition 14, $h_{\mathrm{alg}}(\varphi, R) = h_{\mathrm{alg}}(\overline{\varphi}, R/\ker\varphi)$. By assumption, $\lambda(\varphi^n) = q(\varphi)^{nd}$ for some $n \in \mathbb{N}$. From Lemma 10 it follows that $\ker\varphi = (0)$.

Now since φ is contracting, using Remark 1 we quickly see that the family $\{\varphi^{nt}(\mathfrak{m})R\}_{t\in\mathbb{N}}$ defines the $\mathfrak{m}$-adic topology of R. By Lemma 9 it suffices to verify Equation 10 for this family of $\mathfrak{m}$-primary ideals. We need to show

$$\ell_R\big(R/\varphi^n(\varphi^{nt}(\mathfrak{m}))R\big) = \ell_R\big(R/\varphi^{nt}(\mathfrak{m})R\big) \cdot \ell_R\big(R/\varphi^n(\mathfrak{m})R\big).$$

This equation translates into $\lambda(\varphi^{n(t+1)}) = \lambda(\varphi^{nt}) \cdot \lambda(\varphi^n)$. Using Lemma 10, this equality holds, if and only if

$$q(\varphi)^{n(t+1)d} = q(\varphi)^{ntd} \cdot q(\varphi)^{nd}.$$

Since this equality holds trivially, by Nagata's Flatness Theorem φ^n is flat. The implication **b** $\Rightarrow$ **a**) of Theorem 2 applied to φ^n then tells us that R is regular, and the implication **a** $\Rightarrow$ **b**) of the same theorem shows that φ is flat, as well.

2.2 Generalized Hilbert-Kunz multiplicity

Following ideas of Kunz, Monsky in [29] defined the Hilbert-Kunz multiplicity for the Frobenius endomorphism of Noetherian local rings of positive prime characteristic. He then showed that in this case, Hilbert-Kunz multiplicity always exists. Since then, it has become evident through works of various authors, that the Hilbert-Kunz multiplicity provides a reasonable measure of the singularity of the local ring. Here, inspired by part **c)** of Theorem 1, we propose a characteristic-free interpretation of the definition of Hilbert-Kunz multiplicity associated with a self-map of finite length.

Definition 7 (Hilbert-Kunz multiplicity) Let (R, φ) be a local algebraic dynamical system and set $d := \dim R$. Let $q(\varphi) := \exp(h_{\mathrm{alg}}(\varphi, R)/d)$. The Hilbert-Kunz multiplicity of R with respect to φ is defined as

$$e_{\mathrm{HK}}(\varphi, R) := \lim_{n\to\infty} \frac{\lambda(\varphi^n)}{q(\varphi)^{nd}}, \tag{15}$$

provided that the limit exists.

Remark 9 We do not know whether the limit in Equation 15 always exists or not. Nevertheless, the next corollary shows that in the case of a regular local ring the Hilbert-Kunz multiplicity is precisely what we expect it to be.

Corollary 17 *Let φ be a self-map of finite length of a regular local ring R. Then $e_{\mathrm{HK}}(\varphi, R) = 1$.*

Proof This quickly follows from Theorem 2 and Corollary 5.

We end this section with a note that not all homological properties of the Frobenius endomorphism extend to arbitrary self-maps. For example, in [32, Theorem 1.7, p. 58] Peskine and Szpiro showed that a finite free resolution of a module remains exact after applying the Frobenius functor (see Definition 6). This property may fail in general, for an arbitrary self-map, even in the simple case of a Koszul complex with one element. The image of a non-zerodivisor under an integral self-map could be a zerodivisor, as the next example shows.

Example 7 Consider the polynomial ring $k[x, y, z, w]$ over a field k. Let $\mathfrak{a}$ be the ideal (x^2, xy, xz, zw) and let $A = k[x, y, z, w]/\mathfrak{a}$. Then

$$\mathrm{Ass}(A) = \{(x, z), (x, w), (x, y, z)\}.$$

Define a self-map φ of $k[x, y, z, w]$ as follows $x \overset{\varphi}{\mapsto} x^2$; $y \overset{\varphi}{\mapsto} y$; $z \overset{\varphi}{\mapsto} w$; $w \overset{\varphi}{\mapsto} z$. $\mathfrak{a}$ is φ-invariant. Let $\overline{\varphi}$ be the self-map of A induced by φ. The A-module $\overline{\varphi}_\star A$ is finitely generated. In fact, it is generated by 1 and x as an A-module. Now, $y + w$ is not a zerodivisor in A because it does not belong to any prime ideal in $\mathrm{Ass}(A)$. But $\overline{\varphi}(y + w) = y + z$ is a zerodivisor in A; it is killed by x, for example. On the other hand, $y + z$ is a zerodivisor but is mapped to $y + w$, a non-zerodivisor.

Nonetheless, in the previous example $\overline{\varphi}^2$ sends any A-regular sequence to an A-regular sequence. This motivates the following

Question 1 Let (R, φ) be a local algebraic dynamical system. Does there exist a positive integer n such that φ^n will send any R-regular sequence to an R-regular sequence?

2.3 The Cohen-Fakhruddin Structure Theorem

In this section we will prove Theorem 3. This theorem is inspired by a result of Fakhruddin on lifting polarized self-maps of projective varieties to an ambient projective space. In [11, Corollary 2.2] Fakhruddin showed that given a self-map φ of a projective variety X over an infinite field K and an ample line bundle $\mathcal{L}$ on X with $\varphi^\star(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$ for some $q \geqslant 1$ (polarized condition), there exists an embedding $\imath$ of X in some $\mathbb{P}^N_K$, given by an appropriate tensor power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$, and a self-map ψ of $\mathbb{P}^N_K$ such that $\psi \circ \imath = \imath \circ \varphi$. In [5, Theorem 1] Szpiro and Bhatnagar relaxed some of Fakhruddin's hypotheses and showed that one can keep the same embedding of X given by $\mathcal{L}$, and instead lift an appropriate power φ^r of the self-map to the ambient projective space.

In this section we will consider the analogous lifting problem for self-maps of of finite length of complete Noetherian local rings of equal characteristic. Theorem 3 states that if (A, φ) is a local algebraic dynamical system with A a homomorphic image $\pi : R \twoheadrightarrow A$ of a complete equicharacteristic regular local ring R, then there exists a (non unique) self-map of finite length ψ of R, such that $\pi : (R, \psi) \to (A, \varphi)$ is a morphism of local algebraic dynamical systems. As an improvement over Fakhruddin's result, we do not assume our fields to be infinite.

We begin with a few preparatory results that will be needed in the proof of Theorem 3.

Definition 8 ([35, p. 159]) In a Noetherian local ring R of dimension d and of embedding dimension e, a system of parameters $\{x_1, \ldots, x_d\}$ is called a *strong system of parameters* if it is part of a minimal set of generators $\{x_1, \ldots, x_d, \ldots, x_e\}$ of the maximal ideal.

Lemma 11 *A Noetherian local ring* $(R, \mathfrak{m})$ *has strong systems of parameters.*

Proof Let k be the residue field of R, e the embedding dimension of R, and $d = \dim R$. If $d = 0$ then the statement holds trivially, since every system of parameters is empty. So we assume $d > 0$. We will use the Prime Avoidance Lemma [26, p. 2] to construct a strong system of parameters inductively. It suffices to construct a sequence of elements $x_1, \ldots, x_d \in \mathfrak{m}$ such that

a) $\dim R/\langle x_1, \ldots, x_i \rangle = d - i$, for $1 \leqslant i \leqslant d$, and
b) the images of $x_1, \ldots, x_d$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k.

To choose x_1, let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ be the set of minimal prime ideals of R with the property $\dim R/\mathfrak{p}_i = d$. By the Avoidance Lemma we can choose an element

$$x_1 \in \mathfrak{m} \backslash \left(\mathfrak{m}^2 \cup \mathfrak{p}_1 \cup \ldots \cup \mathfrak{p}_t\right).$$

Then $\dim R/\langle x_1 \rangle = d - 1$ and the image of x_1 in $\mathfrak{m}/\mathfrak{m}^2$ is linearly independent over k. Now let $r - 1 < d$ and suppose we have chosen a sequence of elements $x_1, \ldots, x_{r-1}$ in $\mathfrak{m}$ with desired properties a) and b). To choose the next element x_r, let $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_s\}$ be the set of minimal associated prime ideals of $R/\langle x_1, \ldots, x_{r-1}\rangle$ that satisfy $\dim R/\mathfrak{q}_i = d - r + 1$. Since $r - 1 < d \leqslant e$, we cannot have $\mathfrak{m} = \mathfrak{m}^2 + \langle x_1, \ldots, x_{r-1}\rangle$. Hence, by the Avoidance Lemma there is an element

$$x_r \in \mathfrak{m} \backslash \left(\mathfrak{m}^2 + \langle x_1, \ldots, x_{r-1}\rangle \cup \mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_s\right).$$

Then $\dim R/\langle x_1, \ldots, x_r\rangle = d - r$. To complete the proof we need to show that the images $\overline{x}_1, \ldots, \overline{x}_r$ of $x_1, \ldots, x_r$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k. If not, then since by induction hypothesis $\overline{x}_1, \ldots, \overline{x}_{r-1}$ are linearly independent over k, we must have a dependence relation of the form

$$\alpha_1 \overline{x}_1 + \ldots + \alpha_{r-1}\overline{x}_{r-1} - x_r = 0$$

in $\mathfrak{m}/\mathfrak{m}^2$, with $\alpha_i \in k$. Thus, if for $1 \leqslant i \leqslant r - 1$ we choose elements $a_i \in R$ such that they map to α_i in $R/\mathfrak{m}$, then $a_1 x_1 + \ldots + a_{r-1}x_{r-1} - x_r \in \mathfrak{m}^2$, or $x_r \in \mathfrak{m}^2 + \langle x_1, \ldots, x_{r-1}\rangle$. This contradicts the choice of x_r. Thus, the images of $x_1, \ldots, x_r$ in $\mathfrak{m}/\mathfrak{m}^2$ must be linearly independent over k.

Lemma 12 *Let* $(R, \mathfrak{m})$ *be a complete local ring of equal characteristic and assume that* A *is a homomorphic image* $\pi : R \to A$ *of* R*. If* K *is a subfield of* A*, then there is a subfield* L *of* R *such that* $\pi|_L : L \to K$ *is an isomorphism.*

Proof Let $B = \pi^{-1}(K)$. Then B is a local subring of R with maximal ideal $\mathfrak{q} = \pi^{-1}(0)$. Note that $\mathfrak{q} = \ker \pi$ as subsets of R. Since $B/\mathfrak{q} \cong K$, B is also of equal characteristic. In general B need not be Noetherian. We claim that $B \subseteq R$ is a closed subset in the $\mathfrak{m}$-adic topology of R. To see this, let $\mathfrak{n}$ be the maximal ideal of A and note that the topology induced from the $\mathfrak{n}$-adic topology of A on any subfield of A is the discrete topology. Therefore, any subfield of A is complete with respect to the topology induced from A, and hence is closed in A. Since π is a continuous map and $B = \pi^{-1}(K)$, the claim follows. In particular, B is complete with respect to the topology induced from the $\mathfrak{m}$-adic topology of R.

Denote the $\mathfrak{q}$-adic completion of B by $\widehat{B}$. Since B is a local subring of R and R is complete, we obtain a map $\widehat{i}:\widehat{B}\to R$, where $i:B\hookrightarrow R$ is the inclusion homomorphism. Furthermore, since B is complete with respect to the topology induced from the $\mathfrak{m}$-adic topology of R, we see that $\widehat{i}(\widehat{B})=B$. Let L' be a coefficient field of $\widehat{B}$ (For the existence of coefficient fields in complete local rings that are not necessarily Noetherian, see [30, Theorem 31.1], or [27, Theorem 28.3] or [14, Corollary 2]). Let $L:=\widehat{i}(L')$. Then L is subfield of B that is isomorphic to L'. Furthermore, the following diagram is commutative, and shows that $\pi|_L:L\to K$ is an isomorphism.

$$\begin{tikzcd}
L' \arrow[r, hook] \arrow[rrr, bend left, "\simeq"] \arrow[dr, "\simeq"'] & \widehat{B} \arrow[r, two heads, "\widehat{i}"] \arrow[d] & B \arrow[d, "\pi|_B"'] & L \arrow[l, hook'] \arrow[dl] \\
& \widehat{B}/\widehat{\mathfrak{q}} \arrow[r, "\simeq"] & K &
\end{tikzcd}$$

Proof (of Theorem 3) Let K be an arbitrary coefficient field of R. Then $\varphi(\pi(K))$ is a subfield of A, and can be lifted to a subfield L of R, by Lemma 12, in such a way that $\pi|_L:L\to\varphi(\pi(K))$ is an isomorphism. We will use L at the end of our proof to construct the self-map ψ of R. Let $d=\dim A$ and let e be the embedding dimension of A. By Lemma 11 we can choose a strong system of parameters $\{x_1,\ldots,x_d\}$ of A which is part of a minimal set of generators $\{x_1,\ldots,x_d,\ldots,x_e\}$ of $\mathfrak{n}$. Choose elements $X_1,\ldots,X_e$ in $\mathfrak{m}$ in such a way that $\pi(X_i)=x_i$ for each i. We claim that since the images of $x_1,\ldots,x_e$ in $\mathfrak{n}/\mathfrak{n}^2$ are linearly independent over $A/\mathfrak{n}$, the images $\overline{X}_1,\ldots,\overline{X}_e$ of $X_1,\ldots,X_e$ in $\mathfrak{m}/\mathfrak{m}^2$ are also linearly independent over $R/\mathfrak{m}$. If not, there will be a dependence relation $\alpha_1\overline{X}_1+\ldots+\alpha_e\overline{X}_e=0$ with $\alpha_i\in R/\mathfrak{m}$ not all zero. This means if we choose $a_i\in R$ such that they map to α_i in $R/\mathfrak{m}$ for $1\leqslant i\leqslant e$, then

$$a_1X_1+\ldots+a_eX_e\in\mathfrak{m}^2.$$

If we apply π to this relation, we obtain $\pi(a_1)x_1+\ldots+\pi(a_e)x_e\in\mathfrak{n}^2$. But then the image in $\mathfrak{n}/\mathfrak{n}^2$ would provide a nontrivial dependence relation

$$\pi(a_1)\overline{x}_1+\ldots+\pi(a_e)\overline{x}_e=0,$$

contradicting the linear independence of $\overline{x}_1,\ldots,\overline{x}_e$ in $\mathfrak{n}/\mathfrak{n}^2$ over $A/\mathfrak{n}$. Our claim follows. Hence, we can extend $\{\overline{X}_1,\ldots,\overline{X}_e\}$ to a basis $\{\overline{X}_1,\ldots,\overline{X}_e,\ldots,\overline{X}_n\}$ of $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$, where $n=\dim R$. If we choose elements $X_i\in\mathfrak{m}$ such that they map to $\overline{X}_i$ in $\mathfrak{m}/\mathfrak{m}^2$ for $e+1\leqslant i\leqslant n$, then by Nakayama's Lemma $\{X_1,\ldots,X_n\}$ is a minimal set of generators of $\mathfrak{m}$. Furthermore, it follows from the Cohen Structure Theorem that $R=K[\![X_1,\ldots,X_n]\!]$.

Now consider elements $\varphi(\pi(X_i))$ in A and for $1\leqslant i\leqslant d$ choose $f_i\in\mathfrak{m}$ such that $\pi(f_i)=\varphi(\pi(X_i))$. We claim that the ideal $\langle f_1,\ldots,f_d\rangle$ of R has height

d. First, by Krull's Theorem $\operatorname{ht}\langle f_1,\ldots,f_d\rangle \leqslant d$. For inequality in the other direction, we show the ideal $\mathfrak{b} := \langle \varphi(\pi(X_1)),\ldots,\varphi(\pi(X_d))\rangle$ is $\mathfrak{n}$-primary. This follows from Proposition 1 because φ is of finite length and $\{x_1,\ldots,x_d\}$ is a system of parameters of A. Hence the ideal $\pi^{-1}(\mathfrak{b}) = \langle f_1,\ldots,f_d\rangle + \ker\pi$ is $\mathfrak{m}$-primary in R. Since R is regular, by Serre's Intersection Theorem [36, Chap. V, Theorem 1]

$$\dim R/\ker\pi + \dim R/\langle f_1,\ldots,f_d\rangle \leqslant \dim R,$$

or, $d + \dim R/\langle f_1,\ldots,f_d\rangle \leqslant n$. But $\dim R/\langle f_1,\ldots,f_d\rangle = n - \operatorname{ht}\langle f_1,\ldots,f_d\rangle$ as R is regular. We obtain $\operatorname{ht}\langle f_1,\ldots,f_d\rangle \geqslant d$ and our claim follows.

Next, we will choose elements $f_{d+1},\ldots,f_n \in \mathfrak{m}$ inductively, making sure at each step that $\pi(f_t) = \varphi(\pi(X_t))$ and that $\dim R/\langle f_1,\ldots,f_t\rangle = n-t$. Assume $d \leqslant t < n$ and that $f_1,\ldots f_t$ have been chosen with desired properties. To choose f_{t+1} we use the *coset version* of the Prime Avoidance Lemma due to E. Davis (see [21, Theorem 124] or [27, Exercise 16.8]), that can be stated as follows: *let I be an ideal of a commutative ring R and $x \in R$ be an element. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_s$ be prime ideals of R none of which contain I. Then*

$$x + I \nsubseteq \textstyle\bigcup_{i=1}^{s} \mathfrak{p}_i.$$

Choose an element $u \in \mathfrak{m}$ such that $\pi(u) = \varphi(\pi(X_{t+1}))$. If

$$\dim R/\langle f_1,\ldots f_t, u\rangle = n - t - 1,$$

then set $f_{t+1} = u$. If not, let $\{\mathfrak{p}_1,\ldots,\mathfrak{p}_s\}$ be the set of minimal associated prime ideals of $R/\langle f_1,\ldots,f_t\rangle$ that satisfy $\dim R/\mathfrak{p}_i = \dim R/\langle f_1,\ldots,f_t\rangle$. Since $\langle f_1,\ldots,f_t\rangle + \ker\pi$ is an $\mathfrak{m}$-primary ideal in R, none of these $\mathfrak{p}_i$'s can contain $\ker\pi$. Therefore by the coset version of the Prime Avoidance Lemma there exists an element $a \in \ker\pi$ such that

$$u + a \notin \textstyle\bigcup_{i=1}^{s} \mathfrak{p}_i.$$

Setting $f_{t+1} = u + a$ we see $\dim R/\langle f_1,\ldots,f_{t+1}\rangle = n - t - 1$ and $\pi(f_{t+1}) = \varphi(\pi(X_{t+1}))$, as desired. After choosing $\{f_1,\ldots,f_n\}$ as described, we define a self-map ψ of $R = K[\![X_1,\ldots,X_n]\!]$ as follows. For each $1 \leqslant i \leqslant n$, we define $\psi(X_i)$ to be f_i and for every element α of K we define $\psi(\alpha)$ to be $(\pi|_L)^{-1}(\varphi(\pi(\alpha)))$. Then we extend the definition of ψ to all elements of R by continuity. Since $\psi(\mathfrak{m})R = \langle f_1,\cdots,f_n\rangle$ is $\mathfrak{m}$-primary by construction of the f_i's, ψ is of finite length. Moreover, it is clear from the construction that $\varphi \circ \pi = \pi \circ \psi$, that is, $\pi : (R,\psi) \to (A,\varphi)$ is a morphism of local algebraic dynamical systems.

Corollary 18 *If in Theorem 3 φ is finite, then so is ψ.*

Proof This follows from [10, Theorem 8]: a local homomorphism $f : S \to T$ of complete Noetherian local rings is finite if and only if f is of finite length, and $[f_\star k_T : k_S]$ is a finite (algebraic) field extension, where k_S and k_T are residue fields of S and T.

Remark 10 Let X be a projective variety over a field K with a self-map φ, and let $\mathcal{L}$ be an ample line bundle on X such that $\varphi^\star(\mathcal{L}) \cong \mathcal{L}^{\otimes q}$ for some $q \geqslant 1$. Then some appropriate tensor power $\mathcal{L}^{\otimes n}$ of $\mathcal{L}$ is very ample and can be used to embed X in some projective space $\mathbb{P}_K^N$, realizing X as $\operatorname{Proj} K[X_1, \cdots, X_N]/\mathfrak{a}$ for some graded ideal $\mathfrak{a}$. Let

$$\pi : K[X_1, \ldots, X_N] \to K[X_1, \cdots, X_N]/\mathfrak{a}$$

be the canonical surjection, $\mathfrak{m} = \langle X_1, \cdots, X_N \rangle$ and $\mathfrak{m}_X = \langle \pi(X_1), \cdots, \pi(X_N) \rangle$ be the corresponding irrelevant maximal ideals. Then φ will induce a graded K-self-map of finite length of $K[X_1, \cdots, X_N]/\mathfrak{a}$, which we will also denote by φ. The proof of Theorem 3 can be re-written in this setting, keeping careful track of grading, to lift φ to a graded K-self-map of finite length ψ of $K[X_1, \ldots, X_N]$. This shows the assumption in [11, Corollary 2.2], that K is infinite can be avoided.

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ALMOST NEWTON, SOMETIMES LATTÈS

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1. Introduction

Given a morphism $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ we can iterate ϕ to create a (discrete) dynamical system. We denote the n^{th} iterate of ϕ as $\phi^n = \phi(\phi^{n-1})$. Calculus students are exposed to dynamical systems through the iterated root finding method known as Newton's Method where given a differentiable function $f(x)$ and an initial point x_0 one constructs the sequence

$$x_{n+1} = \phi(x_n) = x_n - \frac{f(x_n)}{f'(x_n)}.$$

In general, this sequence converges to a root of $f(x)$. In terms of dynamics, we would say that the roots of $f(x)$ are attracting fixed points of $\phi(x)$. More generally, one says that P is a *periodic point of period* n for ϕ if $\phi^n(P) = P$.

A common example of a dynamical system with periodic points is to take an endomorphism of an elliptic curve $[m] : E \to E$ and project onto the first coordinate. This construction induces a map on $\mathbb{P}^1$ called a *Lattès map*, and for $m \in \mathbb{Z}$ its degree is m^2 and its periodic points are the torsion points of the elliptic curve.

Denote Hom_d as the set of degree d morphisms on $\mathbb{P}^1$. There is a natural action on $\mathbb{P}^1$ by PGL_2 through conjugation that induces an action on Hom_d. We take the quotient as $M_d = \mathrm{Hom}_d / \mathrm{PGL}_2$. By [6], the moduli space M_d is a geometric quotient. We say that $\gamma \in \mathrm{PGL}_2$ is an *automorphism* of $\phi \in \mathrm{Hom}_d$ if $\gamma^{-1} \circ \phi \circ \gamma = \phi$. We denote the (finite [4]) group of automorphisms as $\mathrm{Aut}(\phi)$.

In this note, we examine a family of morphisms on $\mathbb{P}^1$ with connections to Newton's method, Lattès maps, and automorphisms. Let K be a number field and $F \in K[X,Y]$ be a homogeneous polynomial of degree d with distinct roots. Define

$$\phi_F(X,Y) = [F_Y, -F_X] : \mathbb{P}^1 \to \mathbb{P}^1.$$

In Section 2 we examine the dynamical properties of these maps.

Theorem. *The fixed points of $\phi_F(X,Y)$ are the solutions to $F(X,Y) = 0$, and the multipliers of the fixed points are $1-d$.*

Theorem. *The family of maps of the form $\phi_F = (F_Y, -F_X) : \mathbb{P}^1 \to \mathbb{P}^1$ is invariant under the conjugation action by* PGL_2.

We also give a description of the higher order periodic points and a recursive definition of the polynomial whose roots are the n-periodic points. We also examine related, more general Newton-Raphson maps and, finally, recall the connection to invariant theory and maps with automorphisms.

In Section 3 we explore the connection with Lattès maps.

Theorem. *Maps of the form*

$$\tilde{\phi}(x) = x - 3\frac{f(x)}{f'(x)}$$

are the Lattès maps from multiplication by $[2]$ *and $f(x) = \prod(x - x_i)$ where x_i are the x-coordinates of the* 3*-torsion points.*

Finally, when E has complex multiplication ($m \notin \mathbb{Z}$) the associated ϕ_F can have a non-trivial automorphism group.

Theorem. *If E has $\mathrm{Aut}(E) \supsetneq \mathbb{Z}/2\mathbb{Z}$ and the zeros of $F(X,Y)$ are torsion points of E, then an induced map ϕ_F has a non-trivial automorphism group.*

2. Almost Newton Maps

Let K be a field and consider a two variable homogeneous polynomial $F(X,Y) \in K[X,Y]$ of degree d with no multiple roots. Consider the degree $d-1$ map

$$\begin{aligned} \phi_F : \mathbb{P}^1 &\to \mathbb{P}^1 \\ (X,Y) &\mapsto (F_Y(X,Y), -F_X(X,Y)). \end{aligned}$$

In particular, $F_X = F_Y = 0$ has no nonzero solutions and so ϕ_F is a morphism. We will make frequent use of the Euler relation for homogeneous polynomials, so we recall it here for the convenience of the reader.

Lemma 1 (Euler Relation)**.** *Let $F(X_1,\dots,X_n)$ be a homogeneous polynomial of degree d, then*

$$\sum_i X_i \frac{\partial F}{\partial X_i} = dF.$$

Label $x = \frac{X}{Y}$ and consider

$$f(x) = \frac{F(X,Y)}{Y^d}$$

and notice that

$$f'(x) = \frac{F_X(X,Y)}{Y^{d-1}}.$$

Lemma 2. *The map induced on affine space by ϕ_F is given by*

$$\tilde{\phi}_F(x) = x - d\frac{f(x)}{f'(x)}$$

Proof.

$$\tilde{\phi}_F(x) = -\frac{F_Y(X,Y)}{F_X(X,Y)} = -\frac{YF_Y(X,Y)}{YF_X(X,Y)} = \frac{XF_X(X,Y) - dF(X,Y)}{YF_X(X,Y)} = x - d\frac{f(x)}{f'(x)}.$$

□

Definition 3. Let $\phi = (\phi_1, \phi_2) : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map on $\mathbb{P}^1$. Define $\mathrm{Res}(\phi) = \mathrm{Res}(\phi_1, \phi_2)$, the *resultant* of the coordinate functions of ϕ. For a homogeneous polynomial F, denote $\mathrm{Disc}(F)$ for the *discriminant* of F.

Proposition 4. *Let $F(X,Y)$ be a homogeneous polynomial of degree d with no multiple roots. Then,*

$$\mathrm{Res}(\phi_F(X,Y)) = (-1)^{d(d-1)/2} d^{d-2}\,\mathrm{Disc}(F(X,Y)).$$

Proof. Denote $F(X,Y) = a_d X^d + a_{d-1}X^{d-1}Y + \cdots + a_0 Y^d$. Then we have

$$\begin{aligned} F_X(X,Y) &= da_d X^{d-1} + \cdots + a_1 Y^{d-1} \\ F_Y(X,Y) &= a_{d-1}X^{d-1} + \cdots + da_0 Y^{d-1}. \end{aligned}$$

From standard properties of resultants and discriminants we have

$$\begin{aligned}
a_d D(F(X,Y)) &= (-1)^{d(d-1)/2} R(F(X,Y), F_X(X,Y)) \\
&= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} R(dF(X,Y), -F_X(X,Y)) \\
&= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} R(XF_X(X,Y) + YF_Y(X,Y), -F_X(X,Y)) \\
&= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} R(YF_Y(X,Y), -F_X(X,Y)).
\end{aligned}$$

Now we see that

$$R(YF_Y, -F_X) = \begin{vmatrix} 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 & 0 \\ 0 & 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 \\ \vdots & & & & \vdots & \\ -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 & 0 \\ 0 & -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 \\ \vdots & & & & \vdots & \end{vmatrix}.$$

Expanding down the first column we have

$$\begin{aligned}
R(YF_Y(X,Y), -F_X(X,Y)) &= -da_n(-1)^{d+1} \begin{vmatrix} a_{d-1} & 2a_{d-2} & \cdots & da_1 & 0 & 0 \\ 0 & a_{d-1} & 2a_{d-2} & \cdots & da_1 & 0 \\ \vdots & & & & \vdots & \\ -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 & 0 \\ 0 & -da_d & -(d-1)a_{d-1} & \cdots & -a_1 & 0 \\ \vdots & & & & \vdots & \end{vmatrix} \\
&= da_d(-1)^{d+2} R(F_Y(X,Y), -F_X(X,Y)).
\end{aligned}$$

Thus, we compute

$$\begin{aligned}
a_d D(F(X,Y)) &= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} R(YF_Y(X,Y), -F_X(X,Y)) \\
&= (-1)^{d(d-1)/2} \frac{(-1)^d}{d^{d-1}} (-1)^{d+2} da_n R(F_Y(X,Y), -F_X(X,Y)) \\
&= (-1)^{d(d-1)/2} \frac{a_d}{d^{d-2}} R(F_Y(X,Y), -F_X(X,Y)).
\end{aligned}$$

□

Definition 5. Let P be a periodic point of period n for $\tilde{\phi}$, then the *multiplier* at P is the value $(\tilde{\phi}^n)'(P)$. If P is the point at infinity, then we can compute the multiplier by first changing coordinates.

Theorem 6. *The fixed points of $\phi_F(X,Y)$ are the solutions to $F(X,Y) = 0$, and the multipliers of the fixed points are $1-d$.*

Proof. The projective equality

$$\phi(X,Y) = (X,Y)$$

is equivalent to

$$YF_Y(X,Y) = -XF_X(X,Y).$$

Using the Euler relation with then have

$$XF_X(X,Y) + YF_Y(X,Y) = dF(X,Y) = 0.$$

Since d is a nonzero integer the fixed points satisfy $F(X,Y)=0$.

To calculate the multipliers, we first examine the affine fixed points. We take a derivative evaluated at a fixed point to see

$$\tilde{\phi}'_F(x) = 1 - d\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} = 1 - d\frac{f'(x)f'(x)}{(f'(x))^2} = 1 - d.$$

If a fixed point has multiplier one, then it would have multiplicity at least 2 and, hence, would be at least a double root of F. Since F has no multiple roots, every multiplier is not equal to one. Thus, to see that the multiplier at infinity (when it is fixed) is also $1-d$ we may use the relation [7, Theorem 1.14]

$$\sum_{i=1}^{d} \frac{1}{1-\lambda_i} = 1. \tag{1}$$

□

Remark. If $\operatorname{char} K \mid d$, then ϕ_F is the identity map. Let $F(X,Y) = a_d X^d + a_{d-1}X^{d-1}Y + \cdots + a_0 Y^d$. Then we have

$$\begin{aligned}
F_X(X,Y) &= (d-1)a_{d-1}X^{d-1}Y + \cdots a_1 Y^{d-1} = Y((d-1)a_{d-1}X^{d-1} + \cdots a_1 Y^{d-2}) \\
F_Y(X,Y) &= a_{d-1}X^{d-1} + \cdots + (d-1)a_1 Y^{d-2}X = X(a_{d-1}X^{d-1} + \cdots + (d-1)a_1 Y^{d-2}).
\end{aligned}$$

Since $-i \equiv d-i \pmod{d}$ we have that

$$\phi_F(X,Y) = (F_Y, -F_X) = (XP(X,Y), YP(X,Y)) = (X,Y),$$

where $P(X,Y)$ is a homogeneous polynomial.

We next show that maps of the form ϕ_F form a family in the moduli space of dynamical systems. In other words, for every $\gamma \in \mathrm{PGL}_2$ and ϕ_F, there exists a $G(X,Y)$ such that $\gamma^{-1} \circ \phi_F \circ \gamma = \phi_G$. In fact, $G(X,Y)$ is the polynomial resulting from allowing γ^{-1} to act on F.

Theorem 7. *Every rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d-1$ whose fixed points are $\{(a_1,b_1),\ldots,(a_d,b_d)\}$ all with multiplier $(1-d)$ is a map of the form $\phi_F(X,Y) = (F_Y(X,Y), -F_X(X,Y))$ for*

$$F(X,Y) = (b_1X - a_1Y)(b_2X - a_2Y)\cdots(b_dX - a_dY).$$

Proof. Let $(a_1,b_1),\ldots,(a_d,b_d)$ be the collection of fixed points for the map $\psi(X,Y) : \mathbb{P}^1 \to \mathbb{P}^1$ whose multiplies are $1-d$. Then on $\mathbb{A}^1$ we may write the map of degree $d-1$ as

$$\tilde{\psi}(x) = x - \frac{P(x)}{Q(x)}$$

for some pair of polynomials $P(x)$ and $Q(x)$ with no common zeros. Let $\tilde{\phi}_F(x)$ be the affine map associated to $F(X,Y) = (b_1X - a_1Y)\cdots(b_dX - a_dY)$ and we can write

$$\tilde{\phi}_F(x) = x - d\frac{f(x)}{f'(x)}$$

where

$$f(x) = \frac{F(X,Y)}{Y^d}.$$

The fixed points of $\tilde{\psi}(x)$ are the points where $\frac{P(x)}{Q(x)} = 0$ and, hence, where $P(x) = 0$. The fixed points of $\tilde{\psi}(x)$ are the same as for $\tilde{\phi}_F(x)$, so we must have $P(x) = cf(x)$ for some nonzero constant c. Using the fact that the multipliers are $1-d$ we get

$$\tilde{\psi}'(x) = 1 - \frac{cf'Q - cQ'}{(Q')^2} = 1 - \frac{cf'}{Q} = 1 - d.$$

Therefore we know that
$$\frac{c}{d}f'(x_i) = Q(x_i)$$
where $x_1, \ldots, x_d$ are the fixed points (or $x_1, \ldots, x_{d-1}$ if $(1,0) \in \mathbb{P}^1$ is a fixed point). Since $f'(x)$ and $Q(x)$ are both degree $d-1$ polynomials (or $d-2$), so this is a system of d (or $d-1$) equations in the d (or $d-1$) coefficients of $Q(x)$. Since the values x_i are distinct (since the multipliers are $\neq 1$) the Vandermonde matrix is invertible and we get a unique solution for $Q(x)$. In particular, we must have
$$\frac{c}{d}f'(x) = Q(x)$$
and thus
$$\tilde{\psi}(x) = \tilde{\phi}(x).$$
$\square$

Corollary 8. *The family of maps of the form $\phi_F(X,Y) = (F_Y(X,Y), -F_X(X,Y)) : \mathbb{P}^1 \to \mathbb{P}^1$ is invariant under the conjugation action by PGL_2. In particular, the family of ϕ_F where $\deg F(X,Y) = d$ is isomorphic to an arbitrary choice of $d-3$ distinct points in $\mathbb{P}^1$.*

Proof. Conjugation fixes the multipliers and moves the fixed points, so by Theorem 7 the conjugated map is of the same form.

A map of degree $d-1$ on $\mathbb{P}^1$ has d fixed points. The action by PGL_2 can move any 3 distinct points to any 3 distinct points. Thus, the choice of the remaining $d-3$ fixed points determines ϕ_F. $\square$

2.1. **Extended Example.**

Proposition 9. *Let $F(X,Y)$ be a degree 4 homogeneous polynomial with no multiple roots with associated morphism $\phi_F(X,Y)$. For any $\alpha \in \overline{\mathbb{Q}} - \{0,1\}$ we have that $\phi_F(X,Y)$ is conjugate to a map of the form*
$$\phi_{F,\alpha}(X,Y) = (X^3 - 2(\alpha+1)X^2Y + 3\alpha XY^2, -3X^2Y + 2(\alpha+1)XY^2 - \alpha Y^3).$$

Proof. We can move three of the 4 fixed points to $\{0,1,\infty\}$ with an element of PGL_2 and label the fourth fixed point as α. Then we have
$$F(X,Y,\alpha) = (X)(Y)(X-Y)(X-\alpha Y) = X^3Y - (\alpha+1)X^2Y^2 + \alpha XY^3$$
and
$$\begin{aligned}\phi_{F,\alpha}(X,Y) &= (F_Y(X,Y,\alpha), -F_X(X,Y,\alpha)) \\ &= (X^3 - 2(\alpha+1)X^2Y + 3\alpha XY^2, -(3X^2Y - 2(\alpha+1)XY^2 + \alpha Y^3)).\end{aligned}$$
$\square$

Proposition 10. *Let $F(X,Y)$ be a degree 4 homogeneous polynomial with no multiple roots with associated morphism $\phi_F(X,Y)$. Assume that $\phi_F(X,Y)$ is in the form of Proposition 9. Then, the two periodic points are of the form*
$$\{\pm\sqrt{\alpha}, 1 \pm \sqrt{1-\alpha}, \alpha \pm \sqrt{\alpha^2-\alpha}\} \cup \{0,1,\infty,\alpha\}$$

Proof. Direct computation. $\square$

Proposition 11. *$\mathbb{Q}$-Rational affine two periodic points are parameterized by pythagorean triples.*

Proof. The values α and $1-\alpha$ are both squares and $0 < \alpha < 1$. Thus, there are relatively prime integers p and q so that $\alpha = \frac{p^2}{q^2}$ with $p < q$ and $1-\alpha = \frac{q^2-p^2}{q^2}$. Therefore, so $r^2 + p^2 = q^2$ a pythagorean triple, with $r^2 = (1-\alpha)q^2$. $\square$

Remark. The 2-periodic points are not the roots of $f(\tilde{\phi}(x))$, see Theorem 13 for the general relation.

For general $F(X,Y)$, $\phi_F^2(X,Y)$ does not come from a homogeneous polynomial G.

2.2. **Higher order periodic points.** We set the following notation

$$\begin{aligned} f(x) &= \frac{F(X,Y)}{Y^d} = \sum_{i=0}^{d-1} a_i x^i \\ \tilde{\phi}^n(x) &= \frac{A_n(x)}{B_n(x)} \\ c_n &= -\frac{B_{n+1}(x)}{F_X(A_n(x), B_n(x))} \end{aligned}$$

where $A_n(x)$ and $B_n(x)$ are polynomials and c_n is a constant.

Definition 12. Let $\Psi_n(x)$ be the polynomial whose zeros are affine n-periodic points.

The polynomial $\Psi_n(x)$ is the equivalent of the n^{th} division polynomial for elliptic curves, see [3, Chapter 2] for information on division polynomials.

While it is possible, to define $\Psi_n(x)$ recursively, the relation is not as simple as for elliptic curves. If we let $\Psi_{E,m}$ be the m-division polynomial for an elliptic curve E, then

$$\begin{aligned} \Psi_{E,2m+1} &= \Psi_{E,m+2}\Psi_{E,m}^3 - \Psi_{E,m-1}\Psi_{E,m+1}^3 \quad \text{for } m \geq 2 \\ \Psi_{E,2m} &= \left(\frac{\Psi_{E,m}}{2y}\right)(\Psi_{E,m+2}\Psi_{E,m-1}^2 - \Psi_{E,m-2}\Psi_{E,m+1}^2) \quad \text{for } m \geq 3. \end{aligned}$$

Notice that these relations depend only on $\Psi_{E,m}$ for various m, whereas the formula in the following theorem also involves iterates of the map.

Theorem 13. *We have the following formulas*

$$\tilde{\phi}^n(x) = x + d\frac{\Psi_n(x)}{B_n(x)}$$

and

$$\Psi_{n+1}(x) = \frac{F(A_n(x), B_n(x)) - \Psi_n(x) F_X(A_n(x), B_n(x))}{B_n(x) c_n}$$

with multipliers

$$\prod_{i=0}^{n-1} \left(1 - d + d\frac{f(\phi^i(x)) f''(\phi^i(x))}{f'(\phi^i(x))^2}\right).$$

Proof. We proceed inductively. For $n = 1$ we know that the fixed points are the zeros of $f(x)$.

$$\tilde{\phi}(x) = x - d\frac{f(x)}{f'(x)} = x - d\frac{f(x)}{F_X(A_0(x), B_0(x))} = x - d\frac{f(x)}{-B_1(x)} = x + d\frac{\Psi_1(x)}{B_1(x)}.$$

Now assume that

$$\tilde{\phi}^n(x) = x + d\frac{\Psi_n(x)}{B_n(x)}.$$

Computing

$$\begin{aligned}
\tilde{\phi}^{n+1}(x) &= x + d\frac{\Psi_n(x)}{B_n(x)} - d\frac{f(\tilde{\phi}^n(x))}{f'(\tilde{\phi}^n(x))} \\
&= x + d\frac{\Psi_n(x)}{B_n(x)} - d\frac{F(A_n(x), B_n(x))}{B_n(x)F_X(A_n(x), B_n(x))} \\
&= x - d\frac{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))}{B_n(x)F_X(A_n(x), B_n(x))} \\
&= x - d\frac{F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))}{c_n B_n(x)B_{n+1}(x)}.
\end{aligned}$$

So we have to show that $B_n(x)$ divides $F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x))$. Working modulo $B_n(x)$ we see that

$$F(A_n(x), B_n(x)) - \Psi_n(x)F_X(A_n(x), B_n(x)) \equiv A_n(x)^d - (A_n(x)/d)dA_n(x)^{d-1} \equiv 0 \pmod{B_n(x)}$$

where we used the induction assumption for $\Psi_n(x)$. Thus, the n-periodic points are among the roots of $\Psi_n(x)$.

For equivalence, we count degrees. Again, proceeding inductively it is clear for $n = 1$. For $n + 1$ we have that

$$\deg(F(A_n(x), B_n(x)) = d(d-1)^n = (d-1)^{n+1} + (d-1)^n$$

and

$$\deg(\Psi_n(x)F_X(A_n(x), B_n(x))) \leq (d-1)^n + 1 + (d-1)^{n+1}$$

depending on whether the point at infinity is periodic or not. Thus,

$$\deg(\Psi_{n+1}(x)) \leq (d-1)^n + 1 + (d-1)^{n+1} - (d-1)^n = (d-1)^{n+1} + 1.$$

Since the number of (projective) periodic points of ϕ^n is $(d-1)^n + 1$, every affine fixed point must be a zero of $\Psi_n(x)$.

We compute the multipliers as

$$\begin{aligned}
\tilde{\phi}'(x) &= 1 - d\frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2} = 1 - d + d\frac{f(x)f''(x)}{f'(x)^2} \\
(\tilde{\phi}^n(x))' &= \prod_{i=0}^{n-1} \tilde{\phi}'(\tilde{\phi}^i(x)) = \prod_{i=0}^{n-1}\left(1 - d + d\frac{f(\tilde{\phi}^i(x))f''(\tilde{\phi}^i(x))}{f'(\tilde{\phi}^i(x))^2}\right).
\end{aligned}$$

□

2.3. **Replace d with r: Modified Newton-Raphson Iteration.** We have considered maps of the form

$$\tilde{\phi}_F(x) = x - d\frac{f(x)}{f'(x)}$$

where $d = \deg(F(X,Y))$. However, we could also consider affine maps of the form

$$\tilde{\phi}(x) = x - r\frac{f(x)}{f'(x)} \tag{2}$$

for some $r \neq 0$ and polynomial $f(x)$. When used for iterated root finding, such maps are often called the modified Newton-Raphson method. The fixed points are again the zeros of $f(x)$ and are all distinct with multipliers $1-r$. Thus, if $\deg f \neq r$, then the point at infinity must also be a fixed point by (1) with multiplier

$$\sum_{i=1}^{d+1} \frac{1}{1-\lambda_i} = \frac{\deg f(x)}{r} + \frac{1}{1-\lambda_\infty} = 1$$
$$\lambda_\infty = \frac{\deg f(x)}{\deg f(x) - r}.$$

These maps also form a family in the moduli space of dynamical systems and are determined by their fixed points..

Theorem 14. *Let r be a non-zero integer. Every rational map $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d-1$ which has $d-1$ affine fixed points all with multiplier $(1-r)$ and fixes $(1,0)$ with multiplier $\frac{d-1}{d-r-1}$ is a map of the form (2).*

Proof. The method of proof is identical to the proof of Theorem 7, so is omitted. □

Remark. Note that if we choose $r = 1$, then all of the affine fixed points are also critical points ($\tilde{\phi}'(x) = 0$) as noted in [1, Corollary 1].

2.4. **Connection to Maps with Automorphisms.** Let $\Gamma \subset \mathrm{PGL}_2$ be a finite group.

Definition 15. We say that a homogeneous polynomial F is an *invariant of* Γ if $F \circ \gamma = \chi(\gamma)F$ for all $\gamma \in \Gamma$ and some character χ of Γ. The *invariant ring of* Γ denoted $K[X,Y]^\Gamma$ is the set of all invariants.

The following was known as early as [2, footnote p.345].

Theorem 16. *If $F(X,Y)$ is a homogeneous invariant of a finite group $\Gamma \subset \mathrm{PGL}_2$, then $\Gamma \subset \mathrm{Aut}(\phi_F)$.*

Proof. Easy application of the chain rule. □

3. Connection to Lattès Maps

Consider an elliptic curve with Weierstrass equation $E : y^2 = g(x)$ for $g(x) = x^3 + ax^2 + bx + c$. The solutions $g(x) = 0$ are the 2-torsion points. If we integrate $g(x)$ we get $G(x) = x^4/4 + a/3x^3 + b/2x^2 + cx + C$ for some constant C. If we let $C = -(b^2 - 4ac)/12$, then the solutions $G(x) = 0$ are the 3-torsion points. In general, there are polynomials $\Psi_{E,m}(x)$ called the *division polynomials for E* for which the solutions of $\Psi_{E,m}(x)$ are the m-torsion points. See [3, Chapter 2] for more information on division polynomials.

A Lattès map is a rational function on the first coordinate of the multiplication map $[m] \in \mathrm{End}(E)$ on the rational points of an elliptic curve E; $\phi_{E,m}(x(P)) = x([m])$. For integers $m \geq 3$ we have

$$[m](x,y) = \left(x - \frac{\Psi_{E,m-1}\Psi_{E,m+1}}{\Psi_{E,m}^2}, \frac{\Psi_{E,m+2}\Psi_{E,m-1}^2 - \Psi_{E,m-2}\Psi_{E,m+1}^2}{4y\Psi_{E,m}^3} \right).$$

In other words, the induced Lattès map is given by

$$\phi_{E,m}(x) = x - \frac{\Psi_{E,m-1}\Psi_{E,m+1}}{\psi_m^2}.$$

Hence the fixed points of the Lattès maps are the x-coordinates of the $m-1$ and $m+1$ torsion points. For $m = 2$, the fixed points are the 3 torsion points.

Example 17. Given an elliptic curve of the form $y^2 = g(x) = x^3 + ax^2 + bx + c$. The 2-torsion points satisfy $y^2 = 0$, so are fixed points of the map derived from homogenizing $g(x)$.

$$\begin{aligned} F(X,Y) &= X^3 + aX^2Y + bXY^2 + XY^3 \\ \phi_F(X,Y) &= (aX^2 + 2bXY + 3XY^2, -(2aX + bY^2)) \end{aligned}$$

The fixed points of the doubling map are the points where $x([2]P) = x(P)$, in other words, the points of order 3. They are the points which satisfy the equation

$$\Psi_{E,3}(x) = 3x^4 + 4ax^3 + 6bx^2 + 12cx + (4ac - b^2) = 2g(x)g''(x) - (g'(x))^2$$

So we have

$$\begin{aligned} F(X,Y) &= 3X^4 + 4aX^3Y + 6bX^2Y^2 + 12cXY^3 + (4ac - b^2)Y^4 \\ \phi_F(X,Y) &= (4aX^3 + 12bX^2Y + 36cXY^2 + 4(4ac - b^2)Y^3, \\ &\quad -(12X^3 + 12aX^2Y + 12bXY^2 + 12cY^3)). \end{aligned}$$

For $m = 2$ we get the following stronger connecting generalized ϕ_F and Lattès maps.

Theorem 18. *Maps of the form*

$$\tilde{\phi}(x) = x - 3\frac{f(x)}{f'(x)}$$

are the Lattès maps from multiplication by $[2]$ *and* $f(x) = \prod(x - x_i)$ *where* x_i *are the* x*-coordinates of the* 3*-torsion points.*

Proof. From [7, Proposition 6.52] we have the multiplies are all -2 except at ∞ where it is 4 and the fixed points are the 3 torsion points (plus ∞). Now apply Theorem 14. □

3.1. **Complex Multiplication and Automorphisms.** For an elliptic curve E, every automorphism is of the form $(x, y) \mapsto (u^2x, u^3y)$ for some $u \in \mathbb{C}^*$ [5, III.10]. In general, the only possibilities are $u = \pm 1$ and $\mathrm{Aut}(E) \cong \mathbb{Z}/2\mathbb{Z}$. However, in the case of complex multiplication $\mathrm{End}(E) \supsetneq \mathbb{Z}$ and it is possible to contain additional roots of unity, thus having $\mathrm{Aut}(E) \supsetneq \mathbb{Z}/2\mathbb{Z}$. The two cases are $j(E) = 0, 1728$ having $\mathrm{Aut}(E) \cong \mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$ respectively [5, III.10]. These additional automorphisms induce a linear action $x \mapsto u^2x$ which fixes a polynomial whose roots are torsion points. Thus, the corresponding map ϕ_F has a non-trivial automorphism of the form

$$\begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{PGL}_2 .$$

Thus we have shown the following theorem.

Theorem 19. *If* E *has* $\mathrm{Aut}(E) \supsetneq \mathbb{Z}/2\mathbb{Z}$ *and the zeros of* $F(X,Y)$ *are torsion points of* E*, then an induced map* ϕ_F *has a non-trivial automorphism group.*

Example 20. Let $E = y^2 = x^3 + ax$, for $a \in \mathbb{Z}$, then $j(E) = 1728$ and $\mathrm{End}(E)$ contains the map $(x, y) \mapsto (-x, iy)$. Thus, the automorphism group of every ϕ_F coming from torsion points satisfies

$$\langle \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rangle \subset \mathrm{Aut}(\phi_F).$$

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VERY AMPLE POLARIZED SELF MAPS EXTEND TO PROJECTIVE SPACE

ANUPAM BHATNAGAR AND LUCIEN SZPIRO

ABSTRACT. Let X be a projective variety defined over an infinite field, equipped with a line bundle L, giving an embedding of X into $\mathbb{P}^m$ and let $\phi : X \to X$ be a morphism such that $\phi^* L \cong L^{\otimes q}, q \geq 2$. Then there exists an integer $r > 0$ extending ϕ^r to $\mathbb{P}^m$.

1. Introduction

Let X be a projective variety defined over an infinite field k and ϕ a finite self morphim of X. We say ϕ is *polarized* by a line bundle L on X if $\phi^* L \cong L^{\otimes q}$ for $q > 1$. We say that the polarization is very ample if the line bundle L is very ample i.e. the morphism $X \to \mathbb{P}(H^0(X, L)) \cong \mathbb{P}_k^m$ obtained by evaluating the sections of L at points of X is a closed embedding ([4], pp. 151). In this paper we show that there exists an integer $r \geq 1$ such that ϕ^r extends to a finite self map of $\mathbb{P}_k^m$, where X is embedded. We give an example where $r > 1$ is required. Fakhruddin shows in ([3], Cor. 2.2) that ϕ itself can be extended provided one chooses carefully a different embedding of X in projective space. Our proof and Fakhruddin's proof are closely related. We explain the differences and similarities in the proofs among the two papers in the third remark at the end of this article.

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2. Main result

Theorem 1. *Let X be a projective variety defined over an infinite field k, L a very ample line bundle on X and $\phi : X \to X$ a polarized morphism. Then there exists a positive integer r and a finite morphism $\psi : \mathbb{P}_k^m \to \mathbb{P}_k^m$ extending ϕ^r, where $m + 1 = \dim_k H^0(X, L)$.*

Proof: Let $\dim(X) = g$ and let $s_0, \ldots, s_m$ be a basis of $H^0(X, L)$. Let $\mathcal{I}$ be the sheaf of ideals on $\mathbb{P}^m$ defining X. Then

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^m} \to \mathcal{O}_X \to 0 \tag{2}$$

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is a short exact sequence of sheaves on $\mathbb{P}^m$. Tensoring (2) with $\mathcal{O}_{\mathbb{P}^m}(n)$ and taking cohomology we get the long exact sequence

$$0 \to H^0(\mathbb{P}^m, \mathcal{I}(n)) \to H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(n)) \to$$
$$\to H^0(X, L^{\otimes n}) \to H^1(\mathbb{P}^m, \mathcal{I}(n)) \to \dots$$

By Serre's vanishing theorem there exists n_0 depending on $\mathcal{I}$ such that $H^1(\mathbb{P}^m, \mathcal{I}(n)) = 0$ for each $n \geq n_0$. Let $\{f_i\}$ be the set of homogeneous polynomials defining X. Choose an integer r such that $q^r > \max_i\{\deg(f_i), n_0\}$. Since $(\phi^r)^*L \cong L^{\otimes q^r}$, $(\phi^r)^*(s_i)$ can be lifted to a homogeneous polynomial h_i of degree q^r in the s_i's defined up to an element of $H^0(\mathbb{P}^m, \mathcal{I}(q^r))$. The polynomials $h_i, 0 \leq i \leq m$ define a rational map $\psi : \mathbb{P}^m \dashrightarrow \mathbb{P}^m$. We show using induction that if the h_i's are chosen appropriately, then ψ is a morphism.

Let W_i be the hypersurface defined by h_i. We can choose $s_0, \dots, s_g$ with no common zeros on X, then each component (say Z) of $\cap_{i=0}^g W_i$ has codimension at most $g+1$ since it is defined by $g+1$ equations. By ([4], Thm 7.2, pp. 48), it follows that $\operatorname{codim}(Z) \geq g+1$. Suppose we have $h_0, \dots, h_j, 0 \leq j \leq m$ such that each component of $\cap_{i=0}^j W_i$ has codimension $j+1$ and we want to choose h_{j+1}. Let α_1 be the lifting of $(\phi^r)^*(s_{j+1})$ to $H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(q^r))$. If $V(\alpha_1)$ does not contain any of the components of $\cap_{i=0}^j W_i$, then set $h_{j+1} = \alpha_1$. Otherwise we invoke the Prime Avoidance Lemma which states:

Lemma 3. *Let A be a ring and let $\mathfrak{p}_1, \dots, \mathfrak{p}_m, \mathfrak{q}$ be ideals of A. Suppose that all but possibly two of the $\mathfrak{p}_i$'s are prime ideals. If $\mathfrak{q} \not\subseteq \mathfrak{p}_i$ for each i, then $\mathfrak{q}$ is not contained in the set theoretical union $\cup \mathfrak{p}_i$.*

Proof: [5], pp. 2. □

Taking $A = k[s_0, \dots, s_n], \mathfrak{q} = \mathcal{I}(q^r)$, and $\mathfrak{p}_i$'s the ideals corresponding to the distinct components of $\cap_{i=0}^j W_i$ we can choose $\alpha_2 \in H^0(\mathbb{P}^m, \mathcal{I}(q^r))$ such that $V(\alpha_2)$ does not contain any of the components of $\cap_{i=0}^j W_i$. Consider the family of hypersurfaces $V(a\alpha_1 + b\alpha_2)$ with $[a : b] \in \mathbb{P}^1_k$. If $a = 0$, then the corresponding hypersurface does not contain any components of $\cap_{i=0}^j W_i$. Otherwise, since k is infinite there exists $c \in k$ such that $V(\alpha_1 + c\alpha_2)$ does not contain any component of $\cap_{i=0}^j W_i$. Let $h_{j+1} = \alpha_1 + c\alpha_2$. This concludes the induction and the proof of the theorem. □

We give an example of a self map of a rational quintic in $\mathbb{P}^3$ that does not extend to $\mathbb{P}^3$. This illustrates that the condition $r > 1$ in Theorem 1 is at times necessary.

Proposition 4. *Let u, v be the coordinates of $C \cong \mathbb{P}^1$ embedded in $\mathbb{P}^3$ with coordinates $(x_0 = u^5, x_1 = u^4v, x_2 = uv^4, x_3 = v^5)$. Then a self map ϕ of C of degree 2 defined by two homogeneous polynomials $P(u,v)$ and $Q(u,v)$ does not extend to $\mathbb{P}^3$ if $P(u,v) = au^2 + buv + cv^2$ with $abc \neq 0$.*

Proof: Considering the restriction map $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(10))$. The image of $x_0^2, x_1^2, x_2^2, x_3^2, x_0x_1, x_0x_2, x_0x_3, x_1x_2, x_1x_3, x_2x_3$ under this map is $u^{10}, u^8v^2, u^2v^8, v^{10}, u^9v, u^6v^4, u^5v^5, u^4v^6, uv^9$. Thus u^7v^3 and u^3v^7 are linearly independent. (Note that it is easy to find two quadratic equations for C). One has the possible commutative diagram:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\phi"] \arrow[d, "i"] & \mathbb{P}^1 \arrow[d, "i"] \\ \mathbb{P}^3 \arrow[r, dashed, "\psi"] & \mathbb{P}^3 \end{tikzcd}$$

The composition $(i \circ \phi)$ is given by four homogeneous polynomials of degree 10, namely $(P(u,v)^5, P(u,v)^4Q(u,v), P(u,v)Q(u,v)^4, Q(u,v)^5)$. If ϕ extended to a self map ψ of $\mathbb{P}^3$, some degree two homogeneous polynomial F_i in the x_i's will restrict to $(P(u,v)^5, P(u,v)^4Q(u,v), P(u,v)Q(u,v)^4, Q(u,v)^5)$ on C, by substituting the expressions of the x_i in (u,v). Since $abc \neq 0$ the coefficients of u^7v^3 and u^3v^7 in $P(u,v)^5$ are non-zero. So $P(u,v)^5$ is not in the image of $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(10))$.

3. Remarks

(1) If k is finite, ϕ^r extends to ψ if we allow ψ to be defined over a finite extension of k. Indeed, applying the theorem to $\bar{k}$(the algebraic closure of k), ψ is defined by $m+1$ polynomials in $m+1$ variables with coefficients in $\bar{k}$. Hence ψ is defined over the finite extension of k containing the finite set of coefficients of these polynomials.

(2) We say $P \in X$ is *preperiodic* for ϕ if $\phi^m(P) = \phi^n(P)$ for $m > n \geq 1$. Denote the set of preperiodic points of the dynamical system (X, L, ϕ) by $\mathrm{Prep}(\phi)$. It can be easily verified that $\mathrm{Prep}(\phi) = \mathrm{Prep}(\phi^r)$. Thus from an algebraic dynamics perspective, we do not lose any information by replacing ϕ by ϕ^r. The same holds true for points of canonical height [1] zero as well.

(3) One of the technical conditions required to extend ϕ from a self map of X to a self map of $\mathbb{P}^m_k$ is that $\phi^*L \cong L^s$ where s is larger than the degrees of equations defining X. We choose to replace ϕ by ϕ^r and fix L. The integer q being at least 2 gives the result immediately. On the other hand in ([3], Prop 2.1) Fakhruddin chooses to replace L by $L^{\otimes n}$ and keeps ϕ fixed. To finish the proof he uses a result of Castelnuevo-Mumford ([6], Theorem 1 and 3), stating that if n is large enough, X will be defined by equations of degree at most two in $\mathbb{P}(H^0(X, L^{\otimes n}))$.

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