Arithmetic differential invariants of dynamical systems

Alexandru Buium

Department of Mathematics and Statistics
University of New Mexico
buium@math.unm.edu

January 31, 2012

Dynamical systems

$X, \tilde{X}$ non-singular algebraic varieties over $\mathbb{C}$

Dynamical systems

$X, \tilde{X}$ non-singular algebraic varieties over $\mathbb{C}$

$\sigma_1, \sigma_2 : \tilde{X} \to X$ dominant morphisms, $\sigma = (\sigma_1, \sigma_2)$ correspondece.

Dynamical systems

$X, \tilde{X}$ non-singular algebraic varieties over $\mathbb{C}$

$\sigma_1, \sigma_2 : \tilde{X} \to X$ dominant morphisms, $\sigma = (\sigma_1, \sigma_2)$ correspondece.

Example: $\tilde{X} \subset X \times X$, in particular $\tilde{X}$ the graph of a map $X \to X$

Dynamical systems

$X, \tilde{X}$ non-singular algebraic varieties over $\mathbb{C}$

$\sigma_1, \sigma_2 : \tilde{X} \to X$ dominant morphisms, $\sigma = (\sigma_1, \sigma_2)$ correspondece.

Example: $\tilde{X} \subset X \times X$, in particular $\tilde{X}$ the graph of a map $X \to X$

View (X, σ) as a (generalized) dynamical system

Dynamical systems

$X, \tilde{X}$ non-singular algebraic varieties over $\mathbb{C}$

$\sigma_1, \sigma_2 : \tilde{X} \to X$ dominant morphisms, $\sigma = (\sigma_1, \sigma_2)$ correspondece.

Example: $\tilde{X} \subset X \times X$, in particular $\tilde{X}$ the graph of a map $X \to X$

View (X, σ) as a (generalized) dynamical system

Orbit of $x \in X$ under σ is the set

Dynamical systems

$X, \tilde{X}$ non-singular algebraic varieties over $\mathbb{C}$

$\sigma_1, \sigma_2 : \tilde{X} \to X$ dominant morphisms, $\sigma = (\sigma_1, \sigma_2)$ correspondece.

Example: $\tilde{X} \subset X \times X$, in particular $\tilde{X}$ the graph of a map $X \to X$

View (X, σ) as a (generalized) dynamical system

Orbit of $x \in X$ under σ is the set

$\{x' \in X; \text{there exist } x_1, ..., x_n \in X, x_1 = x, x_n = x', \sigma_1(x_i) = \sigma_2(x_{i-1})\}$

Dynamical systems

$X, \tilde{X}$ non-singular algebraic varieties over $\mathbb{C}$

$\sigma_1, \sigma_2 : \tilde{X} \to X$ dominant morphisms, $\sigma = (\sigma_1, \sigma_2)$ correspondece.

Example: $\tilde{X} \subset X \times X$, in particular $\tilde{X}$ the graph of a map $X \to X$

View (X, σ) as a (generalized) dynamical system

Orbit of $x \in X$ under σ is the set

$\{x' \in X; \text{there exist } x_1, \ldots, x_n \in X, x_1 = x, x_n = x', \sigma_1(x_i) = \sigma_2(x_{i-1})\}$

Most of the times there is a Zariski dense orbit

Basic pathology

Basic pathology

(X, σ) correspondence

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^\sigma = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^\sigma = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

If σ has a Zariski dense orbit then $\mathcal{O}(X)^\sigma = \mathbb{C}$

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^\sigma = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

If σ has a Zariski dense orbit then $\mathcal{O}(X)^\sigma = \mathbb{C}$

More generally: L line bundle on X, $\beta : \sigma_1^* L \simeq \sigma_2^* L$

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^\sigma = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

If σ has a Zariski dense orbit then $\mathcal{O}(X)^\sigma = \mathbb{C}$

More generally: L line bundle on X, $\beta : \sigma_1^* L \simeq \sigma_2^* L$

$R(X, L) = \bigoplus_{0 \neq m \in \mathbb{Z}_+} H^0(X, L^m)$

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^{\sigma} = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

If σ has a Zariski dense orbit then $\mathcal{O}(X)^{\sigma} = \mathbb{C}$

More generally: L line bundle on X, $\beta : \sigma_1^* L \simeq \sigma_2^* L$

$R(X, L) = \bigoplus_{0 \neq m \in \mathbb{Z}_+} H^0(X, L^m)$

$R(X, L)^{\sigma} = \{s \in R(X, L); \beta \sigma_1^* s = \sigma_2^* s\}$ graded ring of invariants

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^{\sigma} = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

If σ has a Zariski dense orbit then $\mathcal{O}(X)^{\sigma} = \mathbb{C}$

More generally: L line bundle on X, $\beta : \sigma_1^* L \simeq \sigma_2^* L$

$R(X, L) = \bigoplus_{0 \neq m \in \mathbb{Z}_+} H^0(X, L^m)$

$R(X, L)^{\sigma} = \{s \in R(X, L); \beta\sigma_1^* s = \sigma_2^* s\}$ graded ring of invariants

$R(X, L)^{\sigma}_{(0)} = \{f/g; f, g \in R(X, L)^h, deg(f) = deg(g)\}$, field of invariants

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^\sigma = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

If σ has a Zariski dense orbit then $\mathcal{O}(X)^\sigma = \mathbb{C}$

More generally: L line bundle on X, $\beta : \sigma_1^* L \simeq \sigma_2^* L$

$R(X, L) = \bigoplus_{0 \neq m \in \mathbb{Z}_+} H^0(X, L^m)$

$R(X, L)^\sigma = \{s \in R(X, L); \beta\sigma_1^* s = \sigma_2^* s\}$ graded ring of invariants

$R(X, L)^\sigma_{(0)} = \{f/g; f, g \in R(X, L)^h, deg(f) = deg(g)\}$, field of invariants

If σ has an infinite orbit then $R(X, L)^\sigma_{(0)} = \mathbb{C}$

Basic pathology

(X, σ) correspondence

Define $\mathcal{O}(X)^\sigma = \{f \in \mathcal{O}(X); f \circ \sigma_1 = f \circ \sigma_2\}$ ring of invariants

If σ has a Zariski dense orbit then $\mathcal{O}(X)^\sigma = \mathbb{C}$

More generally: L line bundle on X, $\beta : \sigma_1^* L \simeq \sigma_2^* L$

$R(X, L) = \bigoplus_{0 \neq m \in \mathbb{Z}_+} H^0(X, L^m)$

$R(X, L)^\sigma = \{s \in R(X, L); \beta\sigma_1^* s = \sigma_2^* s\}$ graded ring of invariants

$R(X, L)^\sigma_{(0)} = \{f/g; f, g \in R(X, L)^h, deg(f) = deg(g)\}$, field of invariants

If σ has an infinite orbit then $R(X, L)^\sigma_{(0)} = \mathbb{C}$

NO INVARIANTS IN ALGEBRAIC GEOMETRY

What to do ?

What to do ?

Nothing

What to do ?

Nothing

OR

What to do ?

Nothing

OR

Search for new geometries where we have invariants

Strategy

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Geometry with δ-functions called δ-geometry

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Geometry with δ-functions called δ-geometry

Passing to δ-functions is analogous to

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Geometry with δ-functions called δ-geometry

Passing to δ-functions is analogous to

passing from functions $\mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x)$

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Geometry with δ-functions called δ-geometry

Passing to δ-functions is analogous to

passing from functions $\mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x)$

to differential functions (Lagrangians)

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Geometry with δ-functions called δ-geometry

Passing to δ-functions is analogous to

passing from functions $\mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x)$

to differential functions (Lagrangians)

$$C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad x(t) \mapsto f(x(t), x'(t), ..., x^{(n)}(t))$$

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Geometry with δ-functions called δ-geometry

Passing to δ-functions is analogous to

passing from functions $\mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x)$

to differential functions (Lagrangians)

$$C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad x(t) \mapsto f(x(t), x'(t), ..., x^{(n)}(t))$$

In physics: no invariant functions on space-time under the symmetries of space-time but one has invariant Lagrangians; the same will happen in δ-geometry

Strategy

Pass from the polynomial functions of algebraic geometry to more general functions called δ-functions

Hope: more functions $\Rightarrow$ more invariants

Geometry with δ-functions called δ-geometry

Passing to δ-functions is analogous to

passing from functions $\mathbb{R} \to \mathbb{R}, \quad x \mapsto f(x)$

to differential functions (Lagrangians)

$$C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad x(t) \mapsto f(x(t), x'(t), ..., x^{(n)}(t))$$

In physics: no invariant functions on space-time under the symmetries of space-time but one has invariant Lagrangians; the same will happen in δ-geometry

Want to apply δ-geometry to arithmetic geometry; but there are no derivations on $\mathbb{Z}$; what to do?

The Fermat quotient δ

The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N, p) = 1]\hat{}$

The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N, p) = 1]\hat{}$

where upperscript ˆ means p-adic completion

The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N,p)=1]\hat{}$

where upperscript $\hat{}$ means p-adic completion

Recall $\phi : R \to R$ unique ring homorphism with $\phi(x) \equiv x^p \bmod p$

The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N,p)=1]\hat{}$

where upperscript $\hat{}$ means p-adic completion

Recall $\phi : R \to R$ unique ring homorphism with $\phi(x) \equiv x^p \bmod p$

Define $\delta : R \to R$, $\delta x = \frac{\phi(x) - x^p}{p}$ Fermat quotient operator

The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N,p)=1]\hat{}$

where upperscript $\hat{}$ means p-adic completion

Recall $\phi : R \to R$ unique ring homorphism with $\phi(x) \equiv x^p$ mod p

Define $\delta : R \to R$, $\delta x = \frac{\phi(x) - x^p}{p}$ Fermat quotient operator

Morally R is the analogue of $C^\infty(\mathbb{R}) = \{x = x(t) \quad \text{smooth}, x : \mathbb{R} \to \mathbb{R}\}$

The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N,p) = 1]\hat{}$

where upperscript ˆ means p-adic completion

Recall $\phi : R \to R$ unique ring homorphism with $\phi(x) \equiv x^p$ mod p

Define $\delta : R \to R$, $\delta x = \frac{\phi(x) - x^p}{p}$ Fermat quotient operator

Morally R is the analogue of $C^\infty(\mathbb{R}) = \{x = x(t) \quad \text{smooth}, x : \mathbb{R} \to \mathbb{R}\}$

Morally $\delta =$ "$\frac{d}{dp}$" is the analogue of $\frac{d}{dt}$

The Fermat quotient δ

Define $R = W(\overline{\mathbb{F}}_p) = \hat{\mathbb{Z}}_p^{ur} = \mathbb{Z}_p[\zeta_N; (N,p)=1]\hat{}$

where upperscript $\hat{}$ means p-adic completion

Recall $\phi : R \to R$ unique ring homorphism with $\phi(x) \equiv x^p \bmod p$

Define $\delta : R \to R$, $\delta x = \frac{\phi(x) - x^p}{p}$ Fermat quotient operator

Morally R is the analogue of $C^\infty(\mathbb{R}) = \{x = x(t) \quad \text{smooth}, x : \mathbb{R} \to \mathbb{R}\}$

Morally $\delta =$ "$\frac{d}{dp}$" is the analogue of $\frac{d}{dt}$

Example: $p = 7$; $\delta 5 =$ "$\frac{d5}{d7}$" $= \frac{5-5^7}{7}$

δ-functions

δ-functions

For X smooth scheme over R

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

2) an embedding $U \subset \mathbb{A}^n$

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

2) an embedding $U \subset \mathbb{A}^n$

3) a restricted power series $F \in R[T, T', ..., T^{(r)}]\hat{}$ such that

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

2) an embedding $U \subset \mathbb{A}^n$

3) a restricted power series $F \in R[T, T', ..., T^{(r)}]\hat{}$ such that

$f(x) = F(x, \delta x, ..., \delta^r x), \quad x \in U(R) \subset R^n$

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

2) an embedding $U \subset \mathbb{A}^n$

3) a restricted power series $F \in R[T, T', ..., T^{(r)}]\hat{}$ such that

$f(x) = F(x, \delta x, ..., \delta^r x), \quad x \in U(R) \subset R^n$

Denote $\mathcal{O}^r(X)$ ring of δ-functions of order r

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

2) an embedding $U \subset \mathbb{A}^n$

3) a restricted power series $F \in R[T, T', ..., T^{(r)}]\hat{}$ such that

$f(x) = F(x, \delta x, ..., \delta^r x), \quad x \in U(R) \subset R^n$

Denote $\mathcal{O}^r(X)$ ring of δ-functions of order r

Note: $U \mapsto \mathcal{O}^r(U)$ sheaf $\mathcal{O}^r$ on X for the Zariski topology

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

2) an embedding $U \subset \mathbb{A}^n$

3) a restricted power series $F \in R[T, T', \dots, T^{(r)}]\hat{}$ such that

$f(x) = F(x, \delta x, \dots, \delta^r x), \quad x \in U(R) \subset R^n$

Denote $\mathcal{O}^r(X)$ ring of δ-functions of order r

Note: $U \mapsto \mathcal{O}^r(U)$ sheaf $\mathcal{O}^r$ on X for the Zariski topology

δ-functions are arithmetic analogues of differential functions (Lagrangians)

δ-functions

For X smooth scheme over R

Say $f : X(R) \to R$ is a δ-function of order r if for any point in $X(R)$ there exist

1) an affine neighborhood $U \subset X$

2) an embedding $U \subset \mathbb{A}^n$

3) a restricted power series $F \in R[T, T', ..., T^{(r)}]\hat{}$ such that

$f(x) = F(x, \delta x, ..., \delta^r x), \quad x \in U(R) \subset R^n$

Denote $\mathcal{O}^r(X)$ ring of δ-functions of order r

Note: $U \mapsto \mathcal{O}^r(U)$ sheaf $\mathcal{O}^r$ on X for the Zariski topology

δ-functions are arithmetic analogues of differential functions (Lagrangians)

Example $f : \mathbb{A}^1(R) = R \to R$, $f(x) = \sum_{n \geq 1} p^n x^n (\delta x)^{n^3} (\delta^2 x)^{n^n}$, δ-function of order 2

δ-line bundles

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i \phi^i ; a_i \in \mathbb{Z}\}$

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i \phi^i; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i \phi^i; a_i \in \mathbb{Z}_+\}$

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i\phi^i; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i\phi^i; a_i \in \mathbb{Z}_+\}$

For $w = \sum a_i\phi^i \in W$, $f \in \mathcal{O}^r(X)^\times$ set $f^w = \prod(\phi^i(f))^{a_i}$.

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i \phi^i; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i \phi^i; a_i \in \mathbb{Z}_+\}$

For $w = \sum a_i \phi^i \in W$, $f \in \mathcal{O}^r(X)^\times$ set $f^w = \prod(\phi^i(f))^{a_i}$.

For L line bundle on X defined by cocycle (f_{ij}) define δ-line bundle L^w by cocycle (f_{ij}^w)

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i\phi^i; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i\phi^i; a_i \in \mathbb{Z}_+\}$

For $w = \sum a_i\phi^i \in W$, $f \in \mathcal{O}^r(X)^\times$ set $f^w = \prod(\phi^i(f))^{a_i}$.

For L line bundle on X defined by cocycle (f_{ij}) define δ-line bundle L^w by cocycle (f_{ij}^w)

For $\sigma_1, \sigma_2 : \tilde{X} \to X$ etale between smooth schemes over R

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i \phi^i; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i \phi^i; a_i \in \mathbb{Z}_+\}$

For $w = \sum a_i \phi^i \in W$, $f \in \mathcal{O}^r(X)^\times$ set $f^w = \prod (\phi^i(f))^{a_i}$.

For L line bundle on X defined by cocycle (f_{ij}) define δ-line bundle L^w by cocycle (f_{ij}^w)

For $\sigma_1, \sigma_2 : \tilde{X} \to X$ etale between smooth schemes over R

$R_\delta(X, L) = \bigoplus_{0 \neq m \in W_+} H^0(X, L^w)$

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i \phi^i; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i \phi^i; a_i \in \mathbb{Z}_+\}$

For $w = \sum a_i \phi^i \in W$, $f \in \mathcal{O}^r(X)^\times$ set $f^w = \prod(\phi^i(f))^{a_i}$.

For L line bundle on X defined by cocycle (f_{ij}) define δ-line bundle L^w by cocycle (f_{ij}^w)

For $\sigma_1, \sigma_2 : \tilde{X} \to X$ etale between smooth schemes over R

$R_\delta(X, L) = \bigoplus_{0 \neq m \in W_+} H^0(X, L^w)$

$R_\delta(X, L)^\sigma = \{s \in R_\delta(X, L); \beta\sigma_1^* s = \sigma_2^* s\}$ graded ring of δ-invariants

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i\phi^i; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i\phi^i; a_i \in \mathbb{Z}_+\}$

For $w = \sum a_i\phi^i \in W$, $f \in \mathcal{O}^r(X)^\times$ set $f^w = \prod(\phi^i(f))^{a_i}$.

For L line bundle on X defined by cocycle (f_{ij}) define δ-line bundle L^w by cocycle (f_{ij}^w)

For $\sigma_1, \sigma_2 : \tilde{X} \to X$ etale between smooth schemes over R

$R_\delta(X, L) = \bigoplus_{0 \neq m \in W_+} H^0(X, L^w)$

$R_\delta(X, L)^\sigma = \{s \in R_\delta(X, L); \beta\sigma_1^* s = \sigma_2^* s\}$ graded ring of δ-invariants

$R_\delta(X, L)^\sigma_{(0)} = \{f/g; f, g \in R_\delta(X, L)^h, p \not| g, deg(f) = deg(g)\}$, δ-DVR of δ-invariants (δ acts on this)

δ-line bundles

Define δ-line bundle on X as a locally free sheaf of $\mathcal{O}^r$-modules of rank 1

Define $W = \mathbb{Z}[\phi] = \{\sum a_i \phi^i ; a_i \in \mathbb{Z}\}$

Define $W_+ = \{\sum a_i \phi^i ; a_i \in \mathbb{Z}_+\}$

For $w = \sum a_i \phi^i \in W$, $f \in \mathcal{O}^r(X)^\times$ set $f^w = \prod(\phi^i(f))^{a_i}$.

For L line bundle on X defined by cocycle (f_{ij}) define δ-line bundle L^w by cocycle (f_{ij}^w)

For $\sigma_1, \sigma_2 : \tilde{X} \to X$ etale between smooth schemes over R

$R_\delta(X, L) = \bigoplus_{0 \neq m \in W_+} H^0(X, L^w)$

$R_\delta(X, L)^\sigma = \{s \in R_\delta(X, L); \beta \sigma_1^* s = \sigma_2^* s\}$ graded ring of δ-invariants

$R_\delta(X, L)^\sigma_{(0)} = \{f/g; f, g \in R_\delta(X, L)^h, p \not| g, deg(f) = deg(g)\}$, δ-DVR of δ-invariants (δ acts on this)

Basic Example $L = K^{-1}$, anticanonical bundle

Main result

Main result

Theorem

Main result

Theorem

The ring $R_\delta(X, K^{-1})^\sigma$ is "δ-birationally equivalent" to the ring $R_\delta(\mathbb{P}^1, \mathcal{O}(1))$ if the correspondence σ on X "comes from" one of the following cases:

Main result

Theorem

The ring $R_\delta(X, K^{-1})^\sigma$ is "δ-birationally equivalent" to the ring $R_\delta(\mathbb{P}^1, \mathcal{O}(1))$ if the correspondence σ on X "comes from" one of the following cases:

1) (spherical case) The standard action of $SL_2(\mathbb{Z}_p)$ on $\mathbb{P}^1$.

Main result

Theorem

The ring $R_\delta(X, K^{-1})^\sigma$ is "δ-birationally equivalent" to the ring $R_\delta(\mathbb{P}^1, \mathcal{O}(1))$ if the correspondence σ on X "comes from" one of the following cases:

1) (spherical case) The standard action of $SL_2(\mathbb{Z}_p)$ on $\mathbb{P}^1$.

2) (flat case) A dynamical system $f : \mathbb{P}^1 \to \mathbb{P}^1$ which is post-critically finite with (orbifold) Euler characteristic zero.

Main result

Theorem

The ring $R_\delta(X, K^{-1})^\sigma$ is "δ-birationally equivalent" to the ring $R_\delta(\mathbb{P}^1, \mathcal{O}(1))$ if the correspondence σ on X "comes from" one of the following cases:

1) (spherical case) The standard action of $SL_2(\mathbb{Z}_p)$ on $\mathbb{P}^1$.

2) (flat case) A dynamical system $f : \mathbb{P}^1 \to \mathbb{P}^1$ which is post-critically finite with (orbifold) Euler characteristic zero.

3) (hyperbolic case) The action of a Hecke correspondence on a modular (or Shimura) curve.

Morally

Morally

In all these cases the categorical quotient X/σ is a "rational variety in δ-geometry"

Explanations

Explanations

By σ "comes from" a group action on X we mean that ("up to some specific finite schemes") $\tilde{X}$ is the disjoint union of the graphs of finitely many automorphisms generating the action.

Explanations

By σ "comes from" a group action on X we mean that ("up to some specific finite schemes") $\tilde{X}$ is the disjoint union of the graphs of finitely many automorphisms generating the action.

By σ "comes from" an endomorphism of X we mean that ("up to some specific finite subschemes") $\tilde{X}$ is the graph of the endomorphism.

Explanations

By σ "comes from" a group action on X we mean that ("up to some specific finite schemes") $\tilde{X}$ is the disjoint union of the graphs of finitely many automorphisms generating the action.

By σ "comes from" an endomorphism of X we mean that ("up to some specific finite subschemes") $\tilde{X}$ is the graph of the endomorphism.

δ-birational equivalence means isomorphism (compatible with the actions of δ) between the p-adic completions of the rings $R_\delta(X, K^{-1})^\sigma_{(0)}$ and $R_\delta(\mathbb{P}^1, \mathcal{O}(1))_{(0)}$.

Explanations

By σ "comes from" a group action on X we mean that ("up to some specific finite schemes") $\tilde{X}$ is the disjoint union of the graphs of finitely many automorphisms generating the action.

By σ "comes from" an endomorphism of X we mean that ("up to some specific finite subschemes") $\tilde{X}$ is the graph of the endomorphism.

δ-birational equivalence means isomorphism (compatible with the actions of δ) between the p-adic completions of the rings $R_\delta(X, K^{-1})^\sigma_{(0)}$ and $R_\delta(\mathbb{P}^1, \mathcal{O}(1))_{(0)}$.

post-critically finite with (orbifold) Euler characteristic zero is essentially equivalent to f multiplicative ($f(x) = x^N$), Chebyshev, or Lattèes (i.e. induced from an endomorphism of an elliptic curve)

Converse result

Converse result

Theorem

Converse result

Theorem

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be defined over a number field and assume for $p >> 0$ the correspondence (X, σ) obtained from the graph of f satisfies $R_\delta(X, K^{-1})^{\sigma}_{(0)} \neq R$. Then f is post critically finite with (orb) Euler characteristic zero.

Proofs

Proofs

1) Construct δ-invariants

Proofs

1) Construct δ-invariants

Spherical case: elementary

Proofs

1) Construct δ-invariants

Spherical case: elementary

Flat case: δ-characters (B-, Inventiones 95, B-Zimmerman, Crelle 05)

Proofs

1) Construct δ-invariants

Spherical case: elementary

Flat case: δ-characters (B-, Inventiones 95, B-Zimmerman, Crelle 05)

Hyperbolic case: δ-modular forms (B-, Crelle 00, Barcau, Compositio 02, B-, Compositio 03, 04)

Proofs

1) Construct δ-invariants

Spherical case: elementary

Flat case: δ-characters (B-, Inventiones 95, B-Zimmerman, Crelle 05)

Hyperbolic case: δ-modular forms (B-, Crelle 00, Barcau, Compositio 02, B-, Compositio 03, 04)

2) Show that the constructed δ-invariants generate all δ-invariants

Proofs

1) Construct δ-invariants

Spherical case: elementary

Flat case: δ-characters (B-, Inventiones 95, B-Zimmerman, Crelle 05)

Hyperbolic case: δ-modular forms (B-, Crelle 00, Barcau, Compositio 02, B-, Compositio 03, 04)

2) Show that the constructed δ-invariants generate all δ-invariants

All cases: use arithmetic analogues of Lie-Cartan prolongations of vector fields (Barcau, Compositio 02, B-Zimmerman, Crelle 05)

Proofs

1) Construct δ-invariants

Spherical case: elementary

Flat case: δ-characters (B-, Inventiones 95, B-Zimmerman, Crelle 05)

Hyperbolic case: δ-modular forms (B-, Crelle 00, Barcau, Compositio 02, B-, Compositio 03, 04)

2) Show that the constructed δ-invariants generate all δ-invariants

All cases: use arithmetic analogues of Lie-Cartan prolongations of vector fields (Barcau, Compositio 02, B-Zimmerman, Crelle 05)

3) Converse results

Proofs

1) Construct δ-invariants

Spherical case: elementary

Flat case: δ-characters (B-, Inventiones 95, B-Zimmerman, Crelle 05)

Hyperbolic case: δ-modular forms (B-, Crelle 00, Barcau, Compositio 02, B-, Compositio 03, 04)

2) Show that the constructed δ-invariants generate all δ-invariants

All cases: use arithmetic analogues of Lie-Cartan prolongations of vector fields (Barcau, Compositio 02, B-Zimmerman, Crelle 05)

3) Converse results

Flat case: study dynamical systems with invariant tensor differential forms mod p (B-, IRMN 05)

δ-characters

δ-characters

Theorem

δ-characters

Theorem

E/R elliptic curve. There exists a δ-function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

δ-characters

Theorem

E/R elliptic curve. There exists a δ-function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

Remarks

δ-characters

Theorem

E/R elliptic curve. There exists a δ-function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

Remarks

1) ψ an arithmetic analogue of the Manin map $E(K) \to K$ for E/K elliptic curve over a function field K

δ-characters

Theorem

E/R elliptic curve. There exists a δ-function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

Remarks

1) ψ an arithmetic analogue of the Manin map $E(K) \to K$ for E/K elliptic curve over a function field K

2) ψ^2 descends to a δ-function on $(E \backslash E[2])/\langle[-1]\rangle = \mathbb{P}^1 \backslash \{4 \text{ points}\}$

δ-characters

Theorem

E/R elliptic curve. There exists a δ-function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

Remarks

1) ψ an arithmetic analogue of the Manin map $E(K) \to K$ for E/K elliptic curve over a function field K

2) ψ^2 descends to a δ-function on $(E \backslash E[2])/\langle[-1]\rangle = \mathbb{P}^1 \backslash \{4 \text{ points}\}$

which is an invariant for Lattès dynamical system $\mathbb{P}^1 \to \mathbb{P}^1$

δ-characters

Theorem

E/R elliptic curve. There exists a δ-function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

Remarks

1) ψ an arithmetic analogue of the Manin map $E(K) \to K$ for E/K elliptic curve over a function field K

2) ψ^2 descends to a δ-function on $(E \backslash E[2])/\langle[-1]\rangle = \mathbb{P}^1 \backslash \{4 \text{ points}\}$

which is an invariant for Lattès dynamical system $\mathbb{P}^1 \to \mathbb{P}^1$

induced by $[n] : E \to E$:

δ-characters

Theorem

E/R elliptic curve. There exists a δ-function $\psi : E(R) \to R$, $ord(\psi) = 2$, ψ group homomorphism.

Remarks

1) ψ an arithmetic analogue of the Manin map $E(K) \to K$ for E/K elliptic curve over a function field K

2) ψ^2 descends to a δ-function on $(E \backslash E[2])/\langle[-1]\rangle = \mathbb{P}^1 \backslash \{4 \text{ points}\}$

which is an invariant for Lattès dynamical system $\mathbb{P}^1 \to \mathbb{P}^1$

induced by $[n] : E \to E$:

$\psi^2(nP) = n^2\psi(P), \psi^2(-P) = \psi^2(P)$

δ-modular forms

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

$V^* = Spec(\bigoplus_{m \in \mathbb{Z}} L_X^m)$ physical line bundle on X minos zero section

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

$V^* = Spec(\bigoplus_{m\in\mathbb{Z}} L_X^m)$ physical line bundle on X minos zero section

$M^n = \mathcal{O}^n(V^*)$ ring of δ-modular functions of order n

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

$V^* = Spec(\bigoplus_{m \in \mathbb{Z}} L_X^m)$ physical line bundle on X minos zero section

$M^n = \mathcal{O}^n(V^*)$ ring of δ-modular functions of order n

$M^n(w)$ space of δ-modular forms of weight $w = \sum a_i \phi^i \in W$:

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

$V^* = Spec(\bigoplus_{m \in \mathbb{Z}} L_X^m)$ physical line bundle on X minos zero section

$M^n = \mathcal{O}^n(V^*)$ ring of δ-modular functions of order n

$M^n(w)$ space of δ-modular forms of weight $w = \sum a_i \phi^i \in W$:

$f(\lambda \cdot P) = \lambda^w f(P)$, $\lambda \in R^\times$.

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

$V^* = Spec(\bigoplus_{m \in \mathbb{Z}} L_X^m)$ physical line bundle on X minos zero section

$M^n = \mathcal{O}^n(V^*)$ ring of δ-modular functions of order n

$M^n(w)$ space of δ-modular forms of weight $w = \sum a_i \phi^i \in W$:

$f(\lambda \cdot P) = \lambda^w f(P)$, $\lambda \in R^\times$.

$M^n \to R((q))[q', ..., q^{(n)}]\hat{}$ δ-Fourier map

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

$V^* = Spec(\bigoplus_{m\in\mathbb{Z}} L_X^m)$ physical line bundle on X minos zero section

$M^n = \mathcal{O}^n(V^*)$ ring of δ-modular functions of order n

$M^n(w)$ space of δ-modular forms of weight $w = \sum a_i \phi^i \in W$:

$f(\lambda \cdot P) = \lambda^w f(P)$, $\lambda \in R^\times$.

$M^n \to R((q))[q', ..., q^{(n)}]\hat{}$ δ-Fourier map

(not injective but injective on each $M^n(w)$)

δ-modular forms

$X \subset X_1(N)$, $N > 4$, X affine, disjoint from cusps and supersingular locus

L line bundle on $X_1(N)$ whose m-power has sections modular forms of weight m

$V^* = Spec(\bigoplus_{m \in \mathbb{Z}} L_X^m)$ physical line bundle on X minos zero section

$M^n = \mathcal{O}^n(V^*)$ ring of δ-modular functions of order n

$M^n(w)$ space of δ-modular forms of weight $w = \sum a_i \phi^i \in W$:

$f(\lambda \cdot P) = \lambda^w f(P)$, $\lambda \in R^\times$.

$M^n \to R((q))[q', ..., q^{(n)}]\hat{}$ δ-Fourier map

(not injective but injective on each $M^n(w)$)

$M^\infty = \bigcup M^n$, $R((q))^\infty = \bigcup R((q))[q', ..., q^{(n)}]\hat{}$

The generators

The generators

Theorem

The generators

Theorem

1. There exists $f^1 \in M^1(-1-\phi)$ with δ-Fourier expansion $\frac{1}{p} \log \left(1 + p \frac{q'}{q^p}\right)$

The generators

Theorem

1. There exists $f^1 \in M^1(-1-\phi)$ with δ-Fourier expansion $\frac{1}{p}\log\left(1+p\frac{q'}{q^p}\right)$
2. There exists $f^\partial \in M^1(\phi-1)$ with δ-Fourier expansion 1

The generators

Theorem

1. There exists $f^1 \in M^1(-1-\phi)$ with δ-Fourier expansion $\frac{1}{p} \log \left(1 + p\frac{q'}{q^p}\right)$
2. There exists $f^\partial \in M^1(\phi - 1)$ with δ-Fourier expansion 1
3. f^1 and f^∂ "δ-generate" $R_\delta(X, K^{-1})^{\text{Hecke}}$.

The generators

Theorem

1. There exists $f^1 \in M^1(-1-\phi)$ with δ-Fourier expansion $\frac{1}{p} \log\left(1 + p\frac{q'}{q^p}\right)$
2. There exists $f^\partial \in M^1(\phi - 1)$ with δ-Fourier expansion 1
3. f^1 and f^∂ "δ-generate" $R_\delta(X, K^{-1})^{\text{Hecke}}$.
4. $f^\partial - 1$ "δ-generates" $Ker(M^\infty \to R((q))^\infty)$ (B-, Saha JNT 2012)

The generators

Theorem

1. There exists $f^1 \in M^1(-1-\phi)$ with δ-Fourier expansion $\frac{1}{p}\log\left(1+p\frac{q'}{q^p}\right)$
2. There exists $f^\partial \in M^1(\phi-1)$ with δ-Fourier expansion 1
3. f^1 and f^∂ "δ-generate" $R_\delta(X,K^{-1})^{\text{Hecke}}$.
4. $f^\partial - 1$ "δ-generates" $Ker(M^\infty \to R((q))^\infty)$ (B-, Saha JNT 2012)
4. f^1 and $f^\partial - 1$ "δ-generate" $Ker(M^\infty \to R((q))\hat{})$ (B-, Saha JNT 2012)

Back to motivation for finding invariant functions

Back to motivation for finding invariant functions

Rings of invariant functions for (X, σ) identified with rings of functions on X/σ

Back to motivation for finding invariant functions

Rings of invariant functions for (X, σ) identified with rings of functions on X/σ

A different approach to quotients: groupoid strategy. Comes in 2 flavors:
1) Grothendieck's descent and 2) Connes' NC-geometry.
Rings of functions on X/σ replaced by 1) descent data or 2) convolution rings

Back to motivation for finding invariant functions

Rings of invariant functions for (X, σ) identified with rings of functions on X/σ

A different approach to quotients: groupoid strategy. Comes in 2 flavors:
1) Grothendieck's descent and 2) Connes' NC-geometry.
Rings of functions on X/σ replaced by 1) descent data or 2) convolution rings

Grothendieck descent not strong enough to deal with correspondences that have dense orbits but Connes' NC-geometry strong enough to deal with some cases

Comparison between δ-geometry and NC-geometry

	δ-geometry	NC- geometry
spherical	$\frac{\mathbb{P}^1(R)}{SL_2(\mathbb{Z}_p)}$	$\frac{\mathbb{P}^1(\mathbb{R})}{SL_2(\mathbb{Z})}$ = NC-modular curve
flat	$\frac{E(R)}{\langle \gamma_i \rangle}, \frac{E(R)}{[n]}$	$\frac{S^1}{\langle e^{2\pi i \theta} \rangle}$ = NC-elliptic curve
hyperbolic	$\Gamma \backslash \mathbb{H} = Sh_\Gamma \to \frac{Sh_\Gamma}{\text{Hecke}}$	$\lim Sh_\Gamma = Sh^0 \subset Sh \subset Sh^{(nc)}$

Comparison between δ-geometry and NC-geometry

	δ-geometry	NC- geometry
spherical	$\frac{\mathbb{P}^1(R)}{SL_2(\mathbb{Z}_p)}$	$\frac{\mathbb{P}^1(\mathbb{R})}{SL_2(\mathbb{Z})}$ = NC-modular curve
flat	$\frac{E(R)}{\langle\gamma_i\rangle}, \frac{E(R)}{[n]}$	$\frac{S^1}{\langle e^{2\pi i\theta}\rangle}$ = NC-elliptic curve
hyperbolic	$\Gamma\backslash\mathbb{H} = Sh_\Gamma \to \frac{Sh_\Gamma}{\text{Hecke}}$	$\lim Sh_\Gamma = Sh^0 \subset Sh \subset Sh^{(nc)}$

The 2 geometries are very different but they apply to similar situations

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

$\Phi : X = X_1(N) \to A$ modular parametrization, A elliptic curve

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

$\Phi : X = X_1(N) \to A$ modular parametrization, A elliptic curve

$p >> 0$ "good" prime

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

$\Phi : X = X_1(N) \to A$ modular parametrization, A elliptic curve

$p >> 0$ "good" prime

$Q \in X(R)$ an ordinary point.

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

$\Phi : X = X_1(N) \to A$ modular parametrization, A elliptic curve

$p >> 0$ "good" prime

$Q \in X(R)$ an ordinary point.

S set of primes inert in imaginary quadratic field of Q

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

$\Phi : X = X_1(N) \to A$ modular parametrization, A elliptic curve

$p >> 0$ "good" prime

$Q \in X(R)$ an ordinary point.

S set of primes inert in imaginary quadratic field of Q

C the S-isogeny class of Q in $X(R)$

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

$\Phi : X = X_1(N) \to A$ modular parametrization, A elliptic curve

$p >> 0$ "good" prime

$Q \in X(R)$ an ordinary point.

S set of primes inert in imaginary quadratic field of Q

C the S-isogeny class of Q in $X(R)$

Then there exists a constant c such that for any subgroup $\Gamma \leq A(R)$ with $r := rank(\Gamma) < \infty$ the set $\Phi(C) \cap \Gamma$ is finite of cardinality at most cp^r.

Applications to arithmetic geometry

Theorem (Poonen+B-, Compositio 2009)

$\Phi : X = X_1(N) \to A$ modular parametrization, A elliptic curve

$p >> 0$ "good" prime

$Q \in X(R)$ an ordinary point.

S set of primes inert in imaginary quadratic field of Q

C the S-isogeny class of Q in $X(R)$

Then there exists a constant c such that for any subgroup $\Gamma \leq A(R)$ with $r := rank(\Gamma) < \infty$ the set $\Phi(C) \cap \Gamma$ is finite of cardinality at most cp^r.

Similar results for Heegner points (C replaced by CL)

Idea of proof

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^{\sharp} = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^\sharp = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

$f^\flat : X(R) \subset X_1(N)(R) \to R$, $f^1 | f^\flat$, $f^\flat(CL) = 0$, order 1

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^\sharp = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

$f^\flat : X(R) \subset X_1(N)(R) \to R$, $f^1 | f^\flat$, $f^\flat(CL) = 0$, order 1

Any $P \in X(R) \cap \Phi(CL) \cap \Gamma$ satisfies the system of "differential equations of order ≤ 2 in 1 unknown"

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^{\sharp} = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

$f^{\flat} : X(R) \subset X_1(N)(R) \to R$, $f^1 | f^{\flat}$, $f^{\flat}(CL) = 0$, order 1

Any $P \in X(R) \cap \Phi(CL) \cap \Gamma$ satisfies the system of "differential equations of order ≤ 2 in 1 unknown"

$$\begin{cases} f^{\sharp}(P) = 0 \\ f^{\flat}(P) = 0 \end{cases}$$

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^\sharp = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

$f^\flat : X(R) \subset X_1(N)(R) \to R$, $f^1 | f^\flat$, $f^\flat(CL) = 0$,   order 1

Any $P \in X(R) \cap \Phi(CL) \cap \Gamma$ satisfies the system of "differential equations of order ≤ 2 in 1 unknown"

$$\begin{cases} f^\sharp(P) = 0 \\ f^\flat(P) = 0 \end{cases}$$

Intuitively

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^\sharp = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

$f^\flat : X(R) \subset X_1(N)(R) \to R$, $f^1 | f^\flat$, $f^\flat(CL) = 0$, order 1

Any $P \in X(R) \cap \Phi(CL) \cap \Gamma$ satisfies the system of "differential equations of order ≤ 2 in 1 unknown"

$$\begin{cases} f^\sharp(P) = 0 \\ f^\flat(P) = 0 \end{cases}$$

Intuitively

$$\begin{cases} f^\sharp(x, x', x'') = 0 \\ f^\flat(x, x') = 0 \end{cases}$$

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^{\sharp} = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

$f^{\flat} : X(R) \subset X_1(N)(R) \to R$, $f^1 | f^{\flat}$, $f^{\flat}(CL) = 0$, order 1

Any $P \in X(R) \cap \Phi(CL) \cap \Gamma$ satisfies the system of "differential equations of order ≤ 2 in 1 unknown"

$$\begin{cases} f^{\sharp}(P) = 0 \\ f^{\flat}(P) = 0 \end{cases}$$

Intuitively

$$\begin{cases} f^{\sharp}(x, x', x'') = 0 \\ f^{\flat}(x, x') = 0 \end{cases}$$

"Eliminate" x', x'' and get $f^0(x) = 0$ of "order 0"

Idea of proof

Assume $\Gamma = A(R)_{tors}$ and C replaced by CL

Consider $f^\sharp = \psi \circ \Phi : X_1(N)(R) \to A(R) \to R$ order 2

$f^\flat : X(R) \subset X_1(N)(R) \to R$, $f^1 | f^\flat$, $f^\flat(CL) = 0$, order 1

Any $P \in X(R) \cap \Phi(CL) \cap \Gamma$ satisfies the system of "differential equations of order ≤ 2 in 1 unknown"

$$\begin{cases} f^\sharp(P) = 0 \\ f^\flat(P) = 0 \end{cases}$$

Intuitively

$$\begin{cases} f^\sharp(x, x', x'') = 0 \\ f^\flat(x, x') = 0 \end{cases}$$

"Eliminate" x', x'' and get $f^0(x) = 0$ of "order 0"

Finitely many solutions (by Krasner's theorem) plus bound