On approximating non-Archimedean Julia sets

joint work with Jean-Yves Briend

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Outline

1. Introduction
2. Non-Archimedean Julia set
3. Polynomials
4. Question

K : a field complete with respect to absolute value $|\cdot|$

$$\varphi : \mathbb{P}^1 \to \mathbb{P}^1 \quad \text{of degree } d \geq 2 \text{ over } K.$$

$\mathcal{J}_\varphi(K)$ = the K-rational Julia set of φ
= the subset of points in $\mathbb{P}^1(K)$ where $\{\varphi^n\}_{n\geq 1}$ is not equicontinuous.

Problem

To compute/visualize $\mathcal{J}_\varphi(K)$.

In general this is a difficult problem in the case where $K = \mathbb{C}$.

Theorem (Braverman-Yampolsky)

There exist a complex number c such that the (complex) Julia set of the quadratic polynomial $\varphi_c(z) = z^2 + c$ is not computable.

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Notation for non-Archimedean field

K a discretely valued field,

$v : K^* \twoheadrightarrow \mathbb{Z}$ valuation on K,

$|x| = a^{-v(x)}$ for some $a > 1$,

$\mathcal{O}_K$ the ring of integers of K,

π a uniformizer such that $\mathfrak{M}_K = \pi\mathcal{O}_K$,

$\widetilde{K} = \mathcal{O}_K/\mathfrak{M}_K$ assumed to be algebraically closed,

$p = \mathrm{Char}(\widetilde{K}) \geq 0$,

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Reduction

Write $\varphi(x,y) = [f(x,y), g(x,y)]$ with $f, g \in \mathcal{O}_K[x,y]$, homogeneous of degree d with at least one coefficient being a unit. Set $\widetilde{\varphi} = [\widetilde{f}, \widetilde{g}]$.

Good reduction: φ is said to have good reduction (over $\mathbb{C}_v$) if there exists a $\gamma \in \mathrm{PGL}(2, \mathbb{C}_v)$ such that

$$\varphi^\gamma(z) = \left(\gamma^{-1} \circ \varphi \circ \gamma\right)(z) = \frac{f(z)}{g(z)}, \; f, g \in \widehat{\mathcal{O}}_v[z]$$

satisfying

$$v(\mathrm{Res}(\varphi)) = 0.$$

$$\begin{array}{ccc} \mathbb{P}^1_K & \xrightarrow{\varphi} & \mathbb{P}^1_K \\ \downarrow \mathbf{r} & & \downarrow \mathbf{r} \\ \mathbb{P}^1_{\widetilde{K}} & \xrightarrow{\widetilde{\varphi}} & \mathbb{P}^1_{\widetilde{K}} \end{array}$$

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Reduction and Julia set

Although it's not explicitly stated, P. Morton and J. Silverman's work shows that:

Theorem (Morton-Silverman)

If φ has good reduction over $\mathbb{C}_v$ then $\mathcal{J}_\varphi = \mathcal{J}_\varphi(\mathbb{C}_v)$ is empty.

Remark (Properties of Julia set)

(1) $\mathcal{J}_\varphi \subset \overline{\cup_m \mathrm{Per}_m(\varphi)}$ (closure in $\mathbb{P}^1(\mathbb{C}_p)$).
(2) $\mathcal{J}_\varphi$ may not be compact in $\mathbb{P}^1(\mathbb{C}_p)$.
(3) A periodic point for φ is in the Julia set $\mathcal{J}_\varphi$ if and only if it is a repelling periodic point.

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Detecting the Julia set

- $\mathcal{J}_\varphi(K) \subset \mathbb{P}^1(K) \subset \mathbf{P}^1_{\mathrm{Berk}}$ (the Berkovich projective line).

We would like to compute $\mathcal{J}_\varphi(K)$ as a subtree of $\mathbf{P}^1_{\mathrm{Berk}}$.

Julia set and Indeterminacies:

X a smooth separated scheme of finite type over $\mathcal{O}_K$, satisfying

(i) the generic fiber $X_\eta \simeq \mathbb{P}^1_K$; and

(ii) $\mathbb{P}^1(K) \simeq X(\mathcal{O}_K)$.

In this talk, we call such an X a model of $\mathbb{P}^1_K$.

- Let ϕ denote the extension of φ on X. Then, in general we get a rational map

$$\phi : X \dashrightarrow X.$$

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$\overline{Q}$ = the closure of Q in X.
$\widetilde{Q}$ = the closed point where $\overline{Q}$ meets with $\widetilde{X}$.

Theorem

Assume that $\mathcal{J}_\varphi(K)$ is non-empty and let $Q \in \mathcal{J}_\varphi(K)$. Then $\{\widetilde{\varphi^n(Q)} \mid n \geq 1\}$ has non-empty intersection with the set of indeterminacies of ϕ.

Remark

(1) If there is a model X of $\mathbb{P}^1_K$ such that the extension ϕ is a morphism on X, then $\mathcal{J}_\varphi(K)$ is empty.

(2) Such a model is called a *weak Néron model* for the pair $(\mathbb{P}^1_K, \varphi)$ by Call and Silverman.

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(1) If there is a model X of $\mathbb{P}^1_K$ such that the extension ϕ is a morphism on X, then $\mathcal{J}_\varphi(K)$ is empty.

(2) Such a model is called a *weak Néron model* for the pair $(\mathbb{P}^1_K, \varphi)$ by Call and Silverman.

Detecting the Julia set

Let $\widetilde{X}$ denote the special fiber of X and let $Q \in \mathbb{P}^1(K)$.
$\overline{Q}$ = the closure of Q in X.
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An algorithm

Assume that we are given a model X of $\mathbb{P}^1$ and an extension of φ

$$\phi : X \dashrightarrow X.$$

By blowing up the indeterminacies of ϕ we can find a model X' and a birational morphism $\tau : X' \to X$ such that ϕ is lifted to a morphism

$$\widehat{\phi} : X' \to X$$

such that we have a (triangle) diagram ! Repeat the process, we obtain a sequence of models $X_0 \overset{\tau_0}{\leftarrow} X_1 \overset{\tau_1}{\leftarrow} X_2 \leftarrow \cdots X_n \leftarrow \cdots$
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p-adic Julia sets

Example

(1) $\varphi(z) = f(z)/p$ where $f(z) \in \mathbb{Z}_p[z]$ monic and

$$f(z) \equiv z^p - z \pmod{p}$$

Then, $\mathcal{J}_\varphi = \mathbb{Z}_p$.

(2) Let $p \neq 2$ and $\varphi(z) = pz^3 + az^2 + b \in \mathbb{Z}_p[z]$ with $a \in \mathbb{Z}_p^*$.

Then,

- $\mathcal{J}_\varphi(\mathbb{Q}_p^{\mathrm{nr}}) = \{\mathrm{pt}\}$.
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Density of repelling periodic points

Density question: Is $\mathcal{J}_\varphi$ the closure of repelling periodic points? Partial results under some conditions have been obtained.

Theorem (J.-P. Bézivin)

If φ has at least one repelling periodic point, then $\mathcal{J}_\varphi$ is the closure of repelling periodic points.

Theorem (Y. Okuyama)

If the Lyapunov exponent $L(\varphi)$ is positive, then $\mathcal{J}_\varphi$ is the closure of repelling periodic points.

Okuyama's theorem holds over the complex field as well.

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Question

(1). How to detect whether or not $\mathcal{J}_\varphi(K) = \emptyset$ effectively?

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The converse statement of Morton-Silverman Theorem does not hold in general.

Example

(1). *Lattès family* Let E be an elliptic curve over K and consider diagram:

$$\begin{array}{ccc} E & \xrightarrow{[m]} & E \\ \downarrow & & \downarrow \\ \mathbb{P}^1_K & \xrightarrow{\varphi} & \mathbb{P}^1_K \end{array} \Longrightarrow \mathcal{J}_\varphi = \emptyset.$$

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(2). (Favre and Rivera-Letelier) Let $k \geq 2$ and $d_1, \ldots, d_k > 1$ be integers. Let $a_2, \ldots, a_k \in \mathbb{C}_v^*$ such that $|a_k| > \cdots > |a_2| > 0$. Set $\delta_1 = d_1, \delta_j = d_j + d_{j-1}$ for $j = 2, \ldots, k$ and

$$\varphi(z) = z^{d_1} \prod_{j=2}^{k} \left(1 + (a_j z)^{\delta_j}\right)^{(-1)^j}.$$

If $\sum d_j^{-1} \leq 1$ then there exist $a_2, \ldots, a_k$ such that $\mathcal{J}_\varphi = \emptyset$ and φ does not have good reduction over $\mathbb{C}_v$.

Polynomial dynamics

We restrict to the case $\varphi(z) \in K[z]$.

- $\mathcal{J}_\varphi(K)$ is a compact subset of $\mathbb{P}^1(K)$.

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Quadratic family

Low degrees $d = 2, 3$:

For the quadratic family, the situation is much simpler than the classical case ($K = \mathbb{C}$). Let $\varphi_c(z) = z^2 + c$ and $\mathcal{J}_c(K) = \mathcal{J}_{\varphi_c}(K)$.

Theorem (Benedetto-Briend-Perdry)

$\mathcal{J}_c(K) \neq \emptyset$ if and only if one of the following conditions holds.

1. $p \neq 2$: $v(c) = -2k < 0$;
2. $p = 2$: $v(4c) < 0$ *and* $1 - 4c$ *is a square in* K.

In this case ($\mathcal{J}_c(K) \neq \emptyset$), we have $\mathcal{J}_c(K) = \mathcal{J}_c$ and the dynamics of φ on $\mathcal{J}_c(K)$ is topologically conjugated to the full (one-sided) 2-shift.

- We can rephrase the theorem as the following $\mathcal{J}_c(K) \neq \emptyset$ if and only if the two (finite) fixed points of φ_c are K-rational and repelling.

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For cubic polynomials criterion for the existence of K-rational Julia set is similar to the quadratic family.

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Let φ be a cubic polynomial. Then the K-rational Julia set $\mathcal{J}_\varphi(K) \neq \emptyset$ if and only if one of the fixed point of φ is K-rational and repelling.

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Remarks

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- Proofs show that $\mathcal{J}_\varphi = \emptyset \Longrightarrow \varphi$ has good reduction over $\mathbb{C}_v$.
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(1) Let the residue characteristic p be odd. Let $a \in \mathbb{C}_v$ satisfy $-p/(2p+2) \le v(a) < 0$. Let $\varphi(z) = z^2(z-a)^p$. Then, $\mathcal{J}_\varphi = \emptyset$ and φ does not have good reduction over $\mathbb{C}_v$.

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Higher degree polynomials

For polynomial with $\deg\varphi \geq 4$, the above criterion for $d = 2, 3$ does not hold in general.
Let $(p, d) \notin \{(2,4), (2,5), (2,7), (3,5)\}$.
Write $d = e_0 + e_1$ or $e_0 + e_1 + e_2$ such that $e_i \geq 2$ and $p \nmid e_i$ (if $p = 0$, this condition is empty).
Let $n = \mathrm{lcm}(\{e_i\})$.

Example ($d = e_0 + e_1$)

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$$\varphi(z) = \frac{1}{\pi^n} z^{e_0}(z-1)^{e_1} + z.$$
Then, $\mathrm{Fix}(\varphi) = \{0, 1, \infty\}$ and non-repelling.
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Integral points

Consider $\sigma : \mathcal{P}_d \to \mathbb{A}^N$ where $\mathcal{P}_d$ denotes the moduli space of polynomials of degree d. Let $[\varphi] \in \mathcal{P}_d(\mathbb{C}_v)$.
If $\mathcal{J}_\varphi = \emptyset$ then all periodic points are non-repelling.
Hence $\sigma([\varphi]) \in \mathbb{A}^N(\widehat{\mathcal{O}_K})$.

Question

Is it true that $\sigma^{-1}(\mathbb{A}^N(\widehat{\mathcal{O}_K}))$ consist of all polynomials φ with empty Julia set?

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We believe the following is true.

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