On local linearization in non-Archimedean dynamics

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Outline

Part I: Local linearization and small divisors

Let K be a complete valued non-Archimedean field. Given $\lambda \in K$ s.t. $\lambda \neq 0$ but not a root of unity, consider the germ of power series

$$\mathcal{F}_\lambda(K) = \{f(x) = \lambda x + a_2 x^2 + a_3 x^3 + \cdots \in K[[x]]\}$$

with multiplier $\lambda = f'(0)$ and f convergent on the open disc $D_{r_f}(0)$ of radius $r_f = 1/\limsup |a_i|^{1/i}$.

Definition (Linearizability)

A power series $f \in \mathcal{F}_\lambda(K)$ is said to be **linearizable** if there exist a convergent power series solution

$$g(x) = x + b_2 x^2 + b_3 x^3 + \cdots \in K[[x]]$$

to the Schröder functional equation

$$g(f(x)) = \lambda g(x). \tag{1}$$

Example (Lubin:1994, Arrowsmith&Vivaldi:1994)

Let $K = \mathbb{C}_p$, and $\lambda \in \mathbb{Z}_p \setminus \{0\}$ not a root of unity. Then

$$f_\lambda(x) = (1+x)^\lambda - 1 = \lambda x + \sum_{i=2}^{\infty} \binom{\lambda}{i} x^i,$$

is linearizable with conjugacy function

$$g(x) = \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}; \quad |x| < 1$$

with inverse

$$g^{-1}(x) = \exp(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}; \quad |x| < p^{-1/(p-1)}$$

Herman&Yoccoz 1981: there always exist a formal solution

Given $f(x) = \lambda x + a_2x^2 + a_3x^3 + \dots$, $\lambda \neq 0$ but not a root of unity, then the ansatz of

$$g(x) = x + b_2x^2 + b_3x^3 + \dots,$$

in the Schröder functional eq. $g(f(x)) = \lambda g(x)$ gives a recursive formula for coefficients of g:

$$b_k = \frac{1}{\lambda(1-\lambda^{k-1})} \sum_{l=1}^{k-1} b_l \Big(\sum \frac{l!}{\alpha_1! \cdot ... \cdot \alpha_k!} a_1^{\alpha_1} \cdot ... \cdot a_k^{\alpha_k}\Big) \qquad (2)$$

where $\alpha_1, \alpha_2, \dots, \alpha_k$ are nonnegative integer solutions of

$$\begin{cases} \alpha_1 + ... + \alpha_k = l, \\ \alpha_1 + 2\alpha_2 ... + k\alpha_k = k, \\ 1 \leq l \leq k-1. \end{cases} \qquad (3)$$

Small divisor problem: if λ is 'close to' a root of unity

Remark (Constructions from local arithmetic geometry)

1. *As described in Lubin:1994, in the p-adic case the conjugacy g can be obtained from an iterative logarithm*

$$g = L_f = \lim_{n\to\infty} f^{\circ n}/\lambda^n, \quad \textit{if } 0 < |\lambda| < 1,$$

$$f_* = \lim_{n\to\infty} \frac{f^{\circ p^n} - id}{p^n}, \quad \textit{if } |\lambda| = 1$$

respectively. In the latter case, formally $g = exp(\int \log(\lambda)/f_*)$.

2. *Rivera-Letelier:2000 use similar constructions of* L_f *and* f_* *in his work on the classification of Fatou components for rational maps over* $K = \mathbb{P}^1(\mathbb{C}_p)$.
3. *For* $K = \mathbb{F}_p((T))$ *there no such general constructions for all* $f \in \mathcal{F}_\lambda(K)$ *but for Drinfeld Modules (see e.g Goss:1996) of a subclass of such* f*, for which the multiplier* $|\lambda| \neq 0, 1$ *and for which all non-linear monomials are of degree divisible by p. The earliest examples being the so called Carlitz polynomials obtained by Carlitz in 1935.*

Complex field case

Cremer 1938: g **diverges** for every λ such that

$$\limsup \left(-\frac{1}{k} \log \left(\inf_{1 \le n \le k-1} |1 - \lambda^n| \right) \right) = +\infty. \tag{4}$$

Siegel 1942: g **converges** if

$$|1 - \lambda^n| \ge C n^{-\beta} \quad \text{for some real numbers } C, \beta > 0, \tag{5}$$

Brjuno 1971: g **converges** if

$$-\sum_{k=0}^{\infty} 2^{-k} \log \left(\inf_{1 \le n \le 2^{k+1}-1} |1 - \lambda^n| \right) < +\infty. \tag{6}$$

Yoccoz 1988: For quadratic polynomials, g **converges if and only if** λ satisfies the Brjuno condition (6).

See e.g. Milnor:2000 or Herman:1986 for a review.

Non-Archimedean Siegel Theorem - Herman&Yoccoz:1981

The conjugacy g **converges** if λ satisfies the Siegel condition

$$|1 - \lambda^n| \geq Cn^{-\beta} \quad \text{for some real numbers } C, \beta > 0. \tag{7}$$

1. If char $K = 0$
   - dim one every λ not a root of unity satisfy (7)
   - dim two there exist λ s.t. (7) is broken and g diverges.
   - multi-dim p-adic case Viegue:2007, Okuyama:2010.
2. If char $K = p > 0$, and $|1 - \lambda^m| < 1$ for some $m > 0$. Then λ does **not** satisfy (7) nor the Brjuno condition (6). In fact, if $|1 - \lambda^m| < 1$, then $|1 - \lambda^{mp^j}| = |1 - \lambda^m|^{p^j}$.

Small divisors in fields of prime characterstics and Herman's conjecture, int. congress of mathematical physics 1986

For a locally compact Ultrametric field K, the conjugacy 'usually' diverges even for polynomials of one variable.

Results from papers concerning Herman's conjecture L:2004,2010 and L&Rivera-Letelier:2011.

Let char $K = p > 0$ and $\lambda \in K$, not a root of unity be such that $|\lambda| = 1$ and $|1 - \lambda| < 1$. E.g. if $K = \mathbb{F}_p((T))$, then

$$\lambda = 1 + \mathcal{O}(T)$$

For such λ we study the germ of power series

$$f(x) = \lambda x + a_2 x^2 + a_3 x^4 + \cdots \in K[[x]].$$

E.g. $K = \mathbb{F}_p((T))$ and

$$f(x) = (1 + T)x + x^2.$$

Note $|1 - \lambda^{p^n}| = |T|^{p^n} \to 0$ 'fast' as $n \to \infty$.

Hence, for $f(x) = \lambda x + a_2 x^2$ it 'seems' from formal solution

$$b_k = \frac{1}{\lambda(1-\lambda^{k-1})} \sum_{l=1}^{k-1} b_l \Big(\sum \frac{l!}{\alpha_1! \cdot \alpha_2!} \lambda^{\alpha_1} \cdot a_2^{\alpha_k} \Big) \qquad (8)$$

where

$$\begin{cases} \alpha_1 + \alpha_2 = l, \\ \alpha_1 + 2\alpha_2 = k, \\ 1 \le l \le k-1. \end{cases} \qquad (9)$$

we could have small denominators in b_{p^N+1}

$$\prod_{i=1}^{p^{N-1}} |1 - \lambda^{ip}| = |1-\lambda|^{p^N(1+\frac{p-1}{p}(N-1))}$$

so that $\limsup |b_{p^N+1}|^{1/(p^N+1)} = \infty$ and $g(x) = \sum b_k x^k$ diverges.

Theorem (L)

Quadratic polynomials of the form $f(x) = \lambda x + a_2 x^2 \in K[x]$, *where* $|1-\lambda| < 1$ *are analytically* **linearizable** *at the origin if and only if char* $K = 2$.

In fact, for char$K = 2$ all bad terms with small denominators cancel and we found an explicit formula for the conjugacy

$$g(x) = x + \sum_{j=1}^{\infty} \frac{a_2^{\frac{2^j-1}{2-1}} x^{2^j}}{\lambda^j (1-\lambda^{2^j-1})(1-\lambda^{2^{j-1}-1})\dots(1-\lambda^{2-1})}$$

whereas for char$K > 2$ we proved

$$|b_{p^N+1}| = \frac{|a_2|^{p^N} |\lambda - 1|^{p^{N-1}}}{|\lambda - 1|^{p^N(N\frac{p-1}{p}+1)}}.$$

Positive characteritic case (L:2004,L:2010,L&Rivera-Letelier:2011)

Let char$K = p$, and $\lambda \in K$ not a root of unity, be s.t. $|1-\lambda| < 1$.

For future reference below, put $\gamma_i = |1-\lambda|^{\frac{p-i}{p-2}}$.

Convergence	Divergence
$\lambda x + a_2 x^2,\ p = 2$	$\lambda x + a_2 x^2,\ p > 3$
$\lambda x + (\lambda - \lambda^2) x^2 + \dots$	$\lambda x + a_2 x^2 + \dots,\ \lvert a_i \rvert < \lvert 1-\lambda \rvert \lvert a_2 \rvert$
$\lambda x + \sum_{p \mid i} a_i x^i$	$\lambda x + a_{p+1} x^{p+1}$
$\lambda x (1 + x + x^2 + \dots)$	6) $\lambda x + \sum a_i x^i,\ \lvert a_{p+1} \rvert = 1, \lvert a_i \rvert < \gamma_i,\ i \in [2, p]$

Remark

6) says that there is an open set of non-linearizable power series

Concluding remarks linearization in positive characteristic

Let K be locally compact of positive characteristic p and let $\lambda \in K$ with $|\lambda| = 1$. How 'likey' is it that

$$f(x) = \lambda x + a_2 x^2 + \cdots \in K[[x]]$$

is linearizable at the origin?

1. The fact that we have an open set of non-linearizable power series indicate that with high probability f is non-linearizable.
2. For polynomials, it seems they are only linearizable only if their non-linear monomials are of degree divisible by p.

Part II: Geometry of linearization discs

Definition (Semi-disc)

The **semi-disc** will be referred to as the maximal disc D about the fixed point at the origin such that the semi-conjugacy $g(f(x)) = \lambda g(x)$ holds for all $x \in D$.

Definition (Linearization disc)

The **linearization-disc** will be referred to as the maximal disc Δ such that the full conjugacy $g \circ f \circ g^{-1}(x) = \lambda x$, holds for all $x \in \Delta$.

Remark

1. $\Delta \setminus \{0\}$ ***cannot contain*** *any periodic point, nor any root of its iterates since* $g \circ f^n \circ g^{-1}(x) = \lambda^n x$.
2. *The semi-disc* D ***may contain*** *other periodic points or roots of iterates.*
3. *Ergodic behavior on* Δ *is discussed in (L:2009).*

Theorem (Weierstrass Preparation Theorem (WPT))

Let K be algebraically closed. Let $f(x) = \sum_{i=1}^{\infty} a_i x^i$ be a nonzero power series over K which converges on a rational closed disc $U = \overline{D}_R(0)$, and let $0 < r \leq R$. Then

$$\begin{aligned} s &= \max\{|a_i| r^i : i \geq 1\}, \\ d &= \max\{i \geq 1 : |a_i| r^i = s\}, \quad \textit{and} \\ d' &= \min\{i \geq 1 : |a_i| r^i = s\} \end{aligned}$$

are all attained and finite. Furthermore,

a. *f maps $\overline{D}_r(0)$ onto $\overline{D}_s(0)$ exactly d-to-1 (counting multiplicity).*

b. *f maps $D_r(0)$ onto $D_s(0)$ exactly d'-to-1 (counting multiplicity).*

This is a generalization (Benedetto:2003) of the WPT.
We will refer to $deg(f, \overline{D}_r(0)) = d$ and $deg(f, D_r(0)) = d'$ as the **Weierstrass degree** of f on the corresponding discs.

Example Lubin:1994, Arrowsmith&Vivaldi:1994

Let $K = \mathbb{C}_p$, and $\lambda \in \mathbb{N} \setminus \{0\}$. Then

$$f_\lambda(x) = (1+x)^\lambda - 1 = \lambda x + \sum_{i=2}^{\lambda} \binom{\lambda}{i} x^i,$$

and $g(x) = \log(1+x)$ and $g^{-1}(x) = \exp(x)$.

1. semi-disc $D = D_1(0)$ linearization disc $\Delta = D_{p^{-1/(p-1)}}(0)$
2. For $n \geq 1$, put
$$r_n = p^{-1/p^{n-1}(p-1)}.$$
3. (indifferent) e.g. $\lambda = p + 1$;
$f_\lambda : D_1(0) \to D_1(0)$ one-to-one and isometric.
f_λ has $p^n - 1$ **periodic points** on the sphere $S_{r_n}(0)$.
4. (attracting) e.g. $\lambda = p$;
f_λ has $p^n - 1$ **roots of iterates** on the sphere $S_{r_n}(0)$.

Comments on Fatou components at indifferent fixed points

Suppose a linearizable $f(x) = \lambda x + \ldots, |\lambda| = 1$ is the power series of some rational map R. Let S be the corresponding Siegel disc, that is the corresponding fixed analytic component of the Fatou set.

1. For $K = \mathbb{C}$, R is linear throughout S.
2. For $K = \mathbb{C}_p$, the linearization disc $\Delta \subset S$.
3. Indeed, For $K = \mathbb{C}_p$, the Siegel disc will contain infinitely many periodic points and the dynamics is quasi-periodic, as proven by Rivera-Letelier:2000.

Hyperbolic case - general K (joint work with Zieve)

Let K be a complete non-Archimedean field. For $\lambda \in K$ s.t. $0 < |\lambda| < 1$, we consider the two-parameter family

$$\mathcal{F}_{\lambda,a}(K) = \{\lambda x + a_2 x^2 + a_3 x^3 + \cdots \in K[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)}\}.$$

Theorem (L&Zieve:2010 Attracting fixed point)

If $f \in \mathcal{F}_{\lambda,a}(K)$, where $0 < |\lambda| < 1$. Then, the semi-disc $D \supseteq D_{1/a}(0)$ and the linearization disc $\Delta \supseteq D_{\lambda/a}(0)$.

Corollary

$f \in \mathcal{F}_{\lambda,a}(K)$ is attracting in $D_{1/a}(0)$ and strictly attracting (no preperiodic points except $x = 0$) on $D_{\lambda/a}(0)$.

Example

1. $f(x) = \lambda x + a_2 x^2$, then $a = |a_2|$, $D = D_{1/a}(0)$, $\Delta = D_{\lambda/a}(0)$.
2. same for $f(x) = \lambda x + x^2 + x^3 + \ldots$
3. $f(x) = \lambda x(1 + x + x^2 + \ldots)$, then $D = \Delta = D_{1/a}(0) = D_1(0)$.

Geometry of linearization discs at indifferent fixed points

Given $\lambda \in K$ s.t. $|\lambda| = 1$ but not a root of unity, and the two-parameter family

$$\mathcal{F}_{\lambda,a}(K) = \{\lambda x + a_2x^2 + a_3x^3 + \cdots \in K[[x]] : a = \sup_{i\geq 2} |a_i|^{1/(i-1)}\},$$

with multiplier $\lambda = f'(0)$.

Lemma

The radius of the linearization disc $rad(\Delta) \leq 1/a$. On Δ we have that $f \in \mathcal{F}_{\lambda,a}(K)$ is ergodic if and only if the multiplier map $T_\lambda : x \to \lambda x$ is. The same is true for transitivity and minimality.

Theorem (Periodic points on the boundary, L:2010)

Suppose that Δ is rational open, and that the radius of the corresponding semi-disc $rad(D) > rad(\Delta)$, then f has an indifferent periodic point on the boundary of Δ.

Theorem (L:2010, char K>0 explicit solution)

Let char $K = p > 0$ and $f \in \mathcal{F}_{\lambda,a}(K)$ be polynomial of the form $f(x) = \lambda x + a_p x^p$, $a_p \neq 0$. Then $g(x) = x + \sum_{j=1}^{\infty} b_{p^j} x^{p^j}$, where

$$b_{p^j} = \frac{a_p^{\frac{p^j-1}{p-1}}}{\lambda^j (1-\lambda^{p^j-1})(1-\lambda^{p^{j-1}-1})\dots(1-\lambda^{p-1})}. \qquad (10)$$

$rad(D) = \rho_p = 1/a$ where $a = |a_p|^{1/(p-1)}$.
$rad(\Delta) = \sigma_p$ where

$$\sigma_p = \frac{\sqrt[p^{m'}-1]{|1-\lambda^m|}}{a}, \quad \textit{where } m' = 1 \textit{ if } m = 1, \textit{ and otherwise}$$

$$m' = \min\{n \in \mathbb{Z} : n \geq 1, p^n \equiv 1 \mod m\}.$$

Moreover in the algebraic closure $\widehat{K}$ we have $\deg(g, \overline{D}_{\sigma_p}(0)) = p^{m'}$. Furthermore, f has an indifferent periodic point of period $\kappa \leq p^{m'}$ on the sphere $S_{\sigma_p}(0)$ in $\widehat{K}$, with multiplier λ^κ.

Example

$f(x) = \lambda x + x^p$ and $|1-\lambda| < 1$ so that $m = m' = 1$, then

$$g(x) = x + \sum_{j=1}^{\infty} \frac{x^{p^j}}{\lambda^j (1-\lambda^{p^j-1})(1-\lambda^{p^{j-1}-1}) \dots (1-\lambda^{p-1})}.$$

For $n \geq 1$, put

$$r_n = |1-\lambda|^{1/p^{n-1}(p-1)}$$

The semi-disc $D = D_1(0)$ and the linearization disc $\Delta = D_{r_1}(0)$.

The Weierstrass degree
$deg(g, \overline{D}_{r_n}(0)) = p^n$
$deg(g, D_{r_n}(0)) = p^{n-1}$.

f has a periodic point on each sphere $S_{r_n}(0)$ in the alg. closure $\hat{K}$.

Estimates of linearization discs in pos. characteristics

Let $\lambda \in K$, not a root of unity, be such that the integer $m = \min\{n \in \mathbb{Z} : n \geq 1, |1-\lambda^n| < 1\}$, exists and let $\mathcal{F}_{\lambda,a,p}(K) = \{\lambda x + \sum_{p|i} a_p x^i + \cdots \in K[[x]] : a = \sup_{i\geq 2} |a_i|^{1/(i-1)}\}$.

Theorem (L:2010 General estimate - sometimes optimal)

Given $f \in \mathcal{F}_{\lambda,a,p}(K)$, the semi-disc $D \supseteq D_\rho(0)$ and lin. disc $\Delta \supseteq D_\sigma(0)$ where

$$\rho = \frac{|1-\lambda^m|^{\frac{1}{mp}}}{a}, \quad \sigma = \frac{|1-\lambda^m|^{\frac{1}{p-1}}}{a}.$$

Suppose $a = |a_p|^{1/(p-1)}$. Then, $\Delta = D_\sigma(0)$ and $\deg(g, \overline{D}_\sigma(0)) = p$ and f has an indifferent periodic point in $\widehat{K}$ on the sphere $S_\sigma(0)$.

Example

$\lambda = 1 + T \Rightarrow m = 1$. Then, for

$$f(x) = (1+T)x + x^p + \sum_{n\geq 2} x^{np} \quad \Delta = D_{|T|^{1/(p-1)}}(0).$$

Estimates of indifferent linearization discs in characteristic zero

$P-$adic case

Ben-Menahem:1988
ThiranVerstegenWeyers:1989
Arrowsmith&Vivaldi:1994
Pettigrew&Roberts&Vivaldi:2001
Khrennikov:2001
Zieve:1996
Viegue:2007 (multi-dimensional case)
L:2009

Function field case

L:2009

Indifferent linearization discs in $\mathbb{C}_p$

Given λ, not a root of unity, and a real number a, we define the family

$$\mathcal{F}_{\lambda,a}(\mathbb{C}_p) = \{\lambda x + \sum a_i x^i \in \mathbb{C}_p[[x]] : a = \sup_{i \geq 2} |a_i|^{1/(i-1)}\}$$

Theorem (General estimate)

Let $f \in \mathcal{F}_{\lambda,a}(\mathbb{C}_p)$. Then, the linearization disc $\Delta_f(0) \supseteq D_\sigma(0)$ where

$$\sigma = \sigma(\lambda, a) := a^{-1} R(s+1)^{\frac{1}{m}} |1-\lambda^m|^{\frac{1}{m}(1+\frac{p-1}{p}s)} \left(\frac{|\alpha - \lambda^m|}{|1-\lambda^m|}\right)^{1/mp^s}. \tag{11}$$

Theorem (Exact disc quadratic case)

If f is a quadratic polynomial with $\lambda \in \{z : p^{-1} < |1-z| < 1\}$, then $\Delta_f = D_\tau(0)$, where $\tau = |1-\lambda|^{-1/p}\sigma(\lambda, a)$. The same is true for power series with a sufficiently large quadratic term.

Ergodicity

Lemma

The radius of the linearization disc $rad(\Delta) \leq 1/a$*. On* Δ *we have that* $f \in \mathcal{F}_{\lambda,a}(K)$ *is ergodic if and only if the multiplier map* $T_\lambda : x \to \lambda x$ *is. The same is true for transitivity and minimality.*

Theorem (Ergodicity on spheres about fixed points in non-Archimedean dynamics (L))

Let K *be a complete Ultrametric field and let* $f \in \mathcal{F}_{\lambda,a}(K)$ *be holomorphic on a disc* U *in* K*. Suppose that* f *has a linearization disc* $\Delta \subset U$ *and* $S \subset \Delta$ *is a sphere about the corresponding fixed point. Then*

$f : S \to S$ ***is ergodic if and only if*** K ***is isomorphic to*** $\mathbb{Q}_p$ ***and the multiplier is a generator of the group of units*** $(\mathbb{Z}/p^2\mathbb{Z})^*$.

Furthermore, if $K = \mathbb{Q}_p$ *and* λ *is a generator of the group of units* $(\mathbb{Z}/p^2\mathbb{Z})^*$*, then the radius of* Δ *is* $1/a$ *(considered as a disc in* $\mathbb{Q}_p$*).*

Concluding remarks concerning linearization discs

1. For linearizable power series at indifferent fixed points in positive characteristic, our examples indicate that it is common that the semi-disc D is strictly larger than the linearization disc Δ, forcing f to have a periodic point on the boundary of Δ.
2. How common is it that the semi-disc is strictly larger than the linearization disc at indifferent fixed points in the p-adic case?
3. So far we know it happens for the family $f(x) = (1 + x)^{\lambda} - 1$. Another candidate is the quadratic family $\lambda x + a_2 x^2 + (\text{'suff. small terms'})$ with multiplier $\lambda \in \{z : p^{-1} < |1 - z| < 1\}$ for which we found the exact size of Δ.
4. What can we say about the dynamics of non-linearizable series in positive characteristic? (recent/present joint work with Rivera-Letelier)
5. Are there normal forms of non-linearizable power series?

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