Preperiodic points for families of polynomials

Dragos Ghioca

A special case of the Manin-Mumford Conjecture

The Manin-Mumford Conjecture asks that only *special* subvarieties of semiabelian varieties S may contain a Zariski dense set of torsion points. In this context, **special** means that the subvariety is a translate of an algebraic subgroup of S by a torsion point.

A special case of the Manin-Mumford Conjecture

The Manin-Mumford Conjecture asks that only *special* subvarieties of semiabelian varieties S may contain a Zariski dense set of torsion points. In this context, **special** means that the subvariety is a translate of an algebraic subgroup of S by a torsion point.
In the case $S = \mathbb{G}_m^2$, the statement is much simpler.

Theorem

(Lang) If there exist infinitely many points (x, y) on a plane curve C, where both x and y are roots of unity, then the equation of C (embedded in $\mathbb{G}_m^2$) is of the form $X^m Y^n = \alpha$, where $m, n \in \mathbb{Z}$ and α is a root of unity.

A reformulation

Lang's Theorem yields the following result.

Theorem

Let $F_1, F_2 \in \mathbb{C}(\lambda)$. *If there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $F_1(\lambda)$ *and* $F_2(\lambda)$ *are roots of unity, then* F_1 *and* F_2 *are multiplicatively dependent, i.e., there exist* $m, n \in \mathbb{Z}$ *(not both equal to* 0*) such that* $F_1^m F_2^n = 1$.

A reformulation

Lang's Theorem yields the following result.

Theorem

Let $F_1, F_2 \in \mathbb{C}(\lambda)$. If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $F_1(\lambda)$ and $F_2(\lambda)$ are roots of unity, then F_1 and F_2 are multiplicatively dependent, i.e., there exist $m, n \in \mathbb{Z}$ (not both equal to 0) such that $F_1^m F_2^n = 1$.

Furthermore, under the above hypothesis, we conclude that for each $\lambda \in \mathbb{C}$, $F_1(\lambda)$ is a root of unity if and only if $F_2(\lambda)$ is a root of unity.

A reformulation

Lang's Theorem yields the following result.

Theorem

Let $F_1, F_2 \in \mathbb{C}(\lambda)$. If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $F_1(\lambda)$ and $F_2(\lambda)$ are roots of unity, then F_1 and F_2 are multiplicatively dependent, i.e., there exist $m, n \in \mathbb{Z}$ (not both equal to 0) such that $F_1^m F_2^n = 1$.

Furthermore, under the above hypothesis, we conclude that for each $\lambda \in \mathbb{C}$, $F_1(\lambda)$ is a root of unity if and only if $F_2(\lambda)$ is a root of unity. Versions of the above theorem hold in higher dimensions, where sets with "infinitely many points" are replaced by "Zariski dense subsets".

A family of elliptic curves

Consider the 1-parameter Legendre family of elliptic curves E_λ given by the equation

$$y^2 = x(x-1)(x-\lambda),$$

indexed by all $\lambda \in \mathbb{C}$.

A family of elliptic curves

Consider the 1-parameter Legendre family of elliptic curves E_λ given by the equation

$$y^2 = x(x-1)(x-\lambda),$$

indexed by all $\lambda \in \mathbb{C}$. Let $P_\lambda \in E_\lambda(\mathbb{C})$ be the point on E_λ with x-coordinate equal to 2, and let Q_λ be the point on E_λ with x-coordinate 3, i.e.,

$$P_\lambda = \left(2, \sqrt{2(2-\lambda)}\right)$$

and

$$Q_\lambda = \left(3, \sqrt{6(3-\lambda)}\right).$$

A family of elliptic curves

Consider the 1-parameter Legendre family of elliptic curves E_λ given by the equation

$$y^2 = x(x-1)(x-\lambda),$$

indexed by all $\lambda \in \mathbb{C}$. Let $P_\lambda \in E_\lambda(\mathbb{C})$ be the point on E_λ with x-coordinate equal to 2, and let Q_λ be the point on E_λ with x-coordinate 3, i.e.,

$$P_\lambda = \left(2, \sqrt{2(2-\lambda)}\right)$$

and

$$Q_\lambda = \left(3, \sqrt{6(3-\lambda)}\right).$$

Alternatively, we can view P_λ and Q_λ as sections on the above elliptic surface.

$$E_\lambda : \ y^2 = x(x-1)(x-\lambda)$$
$$P_\lambda = (2, \sqrt{2(2-\lambda)});\ Q_\lambda = (3, \sqrt{6(3-\lambda)})$$

Question: Are there infinitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion points on E_λ?

$$E_\lambda: \ y^2 = x(x-1)(x-\lambda)$$
$$P_\lambda = (2, \sqrt{2(2-\lambda)}); \ Q_\lambda = (3, \sqrt{6(3-\lambda)})$$

Question: Are there infinitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion points on E_λ?
The question in not trivial since one can easily check that for P_λ (and same for Q_λ) there exist infinitely many $\lambda \in \mathbb{C}$ such that P_λ (resp. Q_λ) is torsion for E_λ (simply solve the equation $[n]P_\lambda = 0$ for various $n \in \mathbb{N}$).

$$E_\lambda:\ y^2 = x(x-1)(x-\lambda)$$
$$P_\lambda = (2, \sqrt{2(2-\lambda)});\ Q_\lambda = (3, \sqrt{6(3-\lambda)})$$

Question: Are there infinitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion points on E_λ?
The question in not trivial since one can easily check that for P_λ (and same for Q_λ) there exist infinitely many $\lambda \in \mathbb{C}$ such that P_λ (resp. Q_λ) is torsion for E_λ (simply solve the equation $[n]P_\lambda = 0$ for various $n \in \mathbb{N}$).
On the other hand, neither P_λ nor Q_λ is a torsion section on the elliptic surface. One can see this by noting that $P_3 = (2, i\sqrt{2})$ is not torsion on E_3:
$$y^2 = x(x-1)(x-3)$$
and similarly $Q_2 = (3, \sqrt{6})$ is not torsion on E_2:
$$y^2 = x(x-1)(x-2).$$

Also, the two sections P_λ and Q_λ are linearly independent over $\mathbb{Z}$, i.e., there exist no nonzero $m, n \in \mathbb{Z}$ such that

$$mP_\lambda + nQ_\lambda = 0,$$

since otherwise we would get that P_λ is torsion for E_λ if and only if Q_λ is torsion for E_λ. That would be impossible since $P_2 = (2, 0)$ is torsion for E_2:

$$y^2 = x(x-1)(x-2)$$

while $Q_2 = (3, \sqrt{6})$ is not torsion for E_2.

Also, the two sections P_λ and Q_λ are linearly independent over $\mathbb{Z}$, i.e., there exist no nonzero $m, n \in \mathbb{Z}$ such that

$$mP_\lambda + nQ_\lambda = 0,$$

since otherwise we would get that P_λ is torsion for E_λ if and only if Q_λ is torsion for E_λ. That would be impossible since $P_2 = (2, 0)$ is torsion for E_2:

$$y^2 = x(x-1)(x-2)$$

while $Q_2 = (3, \sqrt{6})$ is not torsion for E_2.
So, there exists a countable set $T(P)$ of numbers $\lambda \in \mathbb{C}$ such that P_λ is torsion for E_λ, and another countable set $T(Q)$ containing all $\lambda \in \mathbb{C}$ such that Q_λ is torsion for E_λ. On the other hand, it *seems* that the two sets shouldn't have many elements in common. Is this enough evidence to convince us that $T(P) \cap T(Q)$ is a finite set?

Also, the two sections P_λ and Q_λ are linearly independent over $\mathbb{Z}$, i.e., there exist no nonzero $m, n \in \mathbb{Z}$ such that

$$mP_\lambda + nQ_\lambda = 0,$$

since otherwise we would get that P_λ is torsion for E_λ if and only if Q_λ is torsion for E_λ. That would be impossible since $P_2 = (2, 0)$ is torsion for E_2:

$$y^2 = x(x-1)(x-2)$$

while $Q_2 = (3, \sqrt{6})$ is not torsion for E_2.

So, there exists a countable set $T(P)$ of numbers $\lambda \in \mathbb{C}$ such that P_λ is torsion for E_λ, and another countable set $T(Q)$ containing all $\lambda \in \mathbb{C}$ such that Q_λ is torsion for E_λ. On the other hand, it *seems* that the two sets shouldn't have many elements in common. Is this enough evidence to convince us that $T(P) \cap T(Q)$ is a finite set?

Yes.

Theorem

(Masser, Zannier) There exist at most finitely many $\lambda \in \mathbb{C}$ such that both P_λ and Q_λ are torsion points on the elliptic curve E_λ.

Theorem

(Masser, Zannier) There exist at most finitely many $\lambda \in \mathbb{C}$ *such that both* P_λ *and* Q_λ *are torsion points on the elliptic curve* E_λ.

Masser and Zannier extended their original result to the case of arbitrary sections P_λ and Q_λ as long as they are linearly independent over $\mathbb{Z}$.

A dynamical reformulation

Consider the 1-parameter of rational maps

$$f_\lambda(x) = \frac{(x^2 - \lambda)^2}{4x(x-1)(x-\lambda)}.$$

A dynamical reformulation

Consider the 1-parameter of rational maps

$$f_\lambda(x) = \frac{(x^2-\lambda)^2}{4x(x-1)(x-\lambda)}.$$

Then for each $\lambda \in \mathbb{C}$, $f_\lambda(2)$ is the x-coordinate of the point $[2]P_\lambda$, where $P_\lambda \in E_\lambda(\mathbb{C})$ is the point on E_λ with x-coordinate equal to 2. Similarly, $f_\lambda(3)$ is the x-coordinate of the point $[2]Q_\lambda$, where $Q_\lambda \in E_\lambda(\mathbb{C})$ is the point on E_λ with x-coordinate equal to 3. The map f_λ is the Lattès map induced by the multiplication-by-2-map on E_λ.

A dynamical reformulation

Consider the 1-parameter of rational maps

$$f_\lambda(x) = \frac{(x^2-\lambda)^2}{4x(x-1)(x-\lambda)}.$$

Then for each $\lambda \in \mathbb{C}$, $f_\lambda(2)$ is the x-coordinate of the point $[2]P_\lambda$, where $P_\lambda \in E_\lambda(\mathbb{C})$ is the point on E_λ with x-coordinate equal to 2. Similarly, $f_\lambda(3)$ is the x-coordinate of the point $[2]Q_\lambda$, where $Q_\lambda \in E_\lambda(\mathbb{C})$ is the point on E_λ with x-coordinate equal to 3. The map f_λ is the Lattès map induced by the multiplication-by-2-map on E_λ.

Therefore, 2 is preperiodic for f_λ if and only if the point P_λ is a torsion point for the elliptic curve E_λ. Hence, Masser-Zannier result is equivalent with the fact that there are at most finitely many $\lambda \in \mathbb{C}$ such that both 2 and 3 are preperiodic under f_λ.

A dynamical reformulation

Consider the 1-parameter of rational maps

$$f_\lambda(x) = \frac{(x^2-\lambda)^2}{4x(x-1)(x-\lambda)}.$$

Then for each $\lambda \in \mathbb{C}$, $f_\lambda(2)$ is the x-coordinate of the point $[2]P_\lambda$, where $P_\lambda \in E_\lambda(\mathbb{C})$ is the point on E_λ with x-coordinate equal to 2. Similarly, $f_\lambda(3)$ is the x-coordinate of the point $[2]Q_\lambda$, where $Q_\lambda \in E_\lambda(\mathbb{C})$ is the point on E_λ with x-coordinate equal to 3. The map f_λ is the Lattès map induced by the multiplication-by-2-map on E_λ.

Therefore, 2 is preperiodic for f_λ if and only if the point P_λ is a torsion point for the elliptic curve E_λ. Hence, Masser-Zannier result is equivalent with the fact that there are at most finitely many $\lambda \in \mathbb{C}$ such that both 2 and 3 are preperiodic under f_λ. The most general theorem proved by Masser and Zannier in this direction is the following.

Theorem

(Masser-Zannier) With the above notation, let $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{C}(\lambda)$ *be rational functions with the property that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic under the action of* f_λ*. Then the points* P_λ *and* Q_λ *with x-coordinates* $\mathbf{a}(\lambda)$*, respectively* $\mathbf{b}(\lambda)$ *are linearly dependent over* $\mathbb{Z}$ *on the generic fiber of the elliptic surface.*

Theorem

(Masser-Zannier) With the above notation, let $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{C}(\lambda)$ *be rational functions with the property that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic under the action of* f_λ*. Then the points* P_λ *and* Q_λ *with x-coordinates* $\mathbf{a}(\lambda)$*, respectively* $\mathbf{b}(\lambda)$ *are linearly dependent over* $\mathbb{Z}$ *on the generic fiber of the elliptic surface.*

In particular, the conclusion may be reformulated as follows:

- the point (P_λ, Q_λ) lives in a 1-dimensional algebraic subgroup (given by the equation $[m]P + [n]Q = 0$) of the abelian surface $E_\lambda \times E_\lambda$ over $\mathbb{C}(\lambda)$; or

Theorem

(Masser-Zannier) With the above notation, let $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{C}(\lambda)$ *be rational functions with the property that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic under the action of* f_λ*. Then the points* P_λ *and* Q_λ *with x-coordinates* $\mathbf{a}(\lambda)$*, respectively* $\mathbf{b}(\lambda)$ *are linearly dependent over* $\mathbb{Z}$ *on the generic fiber of the elliptic surface.*

In particular, the conclusion may be reformulated as follows:

- the point (P_λ, Q_λ) lives in a 1-dimensional algebraic subgroup (given by the equation $[m]P + [n]Q = 0$) of the abelian surface $E_\lambda \times E_\lambda$ over $\mathbb{C}(\lambda)$; or
- the point $(\mathbf{a}, \mathbf{b}) \in (\mathbb{P}^1 \times \mathbb{P}^1)$ lives on a curve which is preperiodic under the action of $(\mathbf{f}, \mathbf{f})$, where $\mathbf{f}$ is the Lattés map induced by the multiplication-by-2-map on the generic fiber of E_λ.

Theorem

(Masser-Zannier) With the above notation, let $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{C}(\lambda)$ *be rational functions with the property that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic under the action of* f_λ. *Then the points* P_λ *and* Q_λ *with x-coordinates* $\mathbf{a}(\lambda)$, *respectively* $\mathbf{b}(\lambda)$ *are linearly dependent over* $\mathbb{Z}$ *on the generic fiber of the elliptic surface.*

In particular, the conclusion may be reformulated as follows:

- the point (P_λ, Q_λ) lives in a 1-dimensional algebraic subgroup (given by the equation $[m]P + [n]Q = 0$) of the abelian surface $E_\lambda \times E_\lambda$ over $\mathbb{C}(\lambda)$; or
- the point $(\mathbf{a}, \mathbf{b}) \in (\mathbb{P}^1 \times \mathbb{P}^1)$ lives on a curve which is preperiodic under the action of $(\mathbf{f}, \mathbf{f})$, where $\mathbf{f}$ is the Lattés map induced by the multiplication-by-2-map on the generic fiber of E_λ.

It is natural to ask the same question for an arbitrary family of rational maps f_λ.

Conjecture

(Ghioca, Hsia, Tucker) Let $f_\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ *be a* 1*-parameter family of rational maps defined over* $\mathbb{C}$ *of degree greater than* 1*. Let* $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{P}^1(\mathbb{C}(\lambda))$ *such that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic for* f_λ*. Then at least one of the following conditions holds:*

Conjecture

(Ghioca, Hsia, Tucker) Let $f_\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ *be a* 1*-parameter family of rational maps defined over* $\mathbb{C}$ *of degree greater than* 1*. Let* $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{P}^1(\mathbb{C}(\lambda))$ *such that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic for* f_λ*. Then at least one of the following conditions holds:*

(1) $\mathbf{a}(\lambda)$ *is preperiodic for* f_λ *for all* λ*;*

Conjecture

(Ghioca, Hsia, Tucker) Let $f_\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ *be a* 1*-parameter family of rational maps defined over* $\mathbb{C}$ *of degree greater than* 1*. Let* $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{P}^1(\mathbb{C}(\lambda))$ *such that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic for* f_λ*. Then at least one of the following conditions holds:*

(1) $\mathbf{a}(\lambda)$ *is preperiodic for* f_λ *for all* λ*;*

(2) $\mathbf{b}(\lambda)$ *is preperiodic for* f_λ *for all* λ*;*

Conjecture

(Ghioca, Hsia, Tucker) Let $f_\lambda : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ *be a* 1*-parameter family of rational maps defined over* $\mathbb{C}$ *of degree greater than* 1*. Let* $\mathbf{a}(\lambda), \mathbf{b}(\lambda) \in \mathbb{P}^1(\mathbb{C}(\lambda))$ *such that there exist infinitely many* $\lambda \in \mathbb{C}$ *such that both* $\mathbf{a}(\lambda)$ *and* $\mathbf{b}(\lambda)$ *are preperiodic for* f_λ*. Then at least one of the following conditions holds:*

(1) $\mathbf{a}(\lambda)$ *is preperiodic for* f_λ *for all* λ*;*

(2) $\mathbf{b}(\lambda)$ *is preperiodic for* f_λ *for all* λ*;*

(3) $\mathbf{a}(\lambda)$ *is preperiodic for* f_λ *if and only if* $\mathbf{b}(\lambda)$ *is preperiodic for* f_λ*.*

The above conditions (1)-(3) are the correct analogue of the Masser-Zannier conclusion that the points P_λ and Q_λ are linearly dependent over $\mathbb{Z}$.

A polynomial family and constant starting points

We could focus first on the case f_λ is totally ramified at infinity, i.e., we're dealing with a family of polynomials, and in addition $\mathbf{a}$ and $\mathbf{b}$ are constants. This is already a difficult question.

A polynomial family and constant starting points

We could focus first on the case f_λ is totally ramified at infinity, i.e., we're dealing with a family of polynomials, and in addition **a** and **b** are constants. This is already a difficult question. A very important special case was proved by Baker and DeMarco (their result also motivated our previous conjecture).

Theorem

(Baker, DeMarco) Let $a, b \in \mathbb{C}$, and let d be an integer greater than 1. If there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for $x^d + \lambda$, then $a^d = b^d$.

A polynomial family and constant starting points

We could focus first on the case f_λ is totally ramified at infinity, i.e., we're dealing with a family of polynomials, and in addition **a** and **b** are constants. This is already a difficult question. A very important special case was proved by Baker and DeMarco (their result also motivated our previous conjecture).

Theorem

(Baker, DeMarco) Let $a, b \in \mathbb{C}$, and let d be an integer greater than 1. If there exist infinitely many $\lambda \in \mathbb{C}$ such that both a and b are preperiodic for $x^d + \lambda$, then $a^d = b^d$.

It is easy to see that neither a nor b is preperiodic for all the maps $x^d + \lambda$. So, according to the previous conjecture, one expects that the conclusion be that a is preperiodic for $x^d + \lambda$ exactly when b is preperiodic for $x^d + \lambda$. Baker and DeMarco proved the more precise statement that after just one iteration under f_λ, both a and b are in the same point, and thus they are preperiodic for the same values of λ.

An example

Consider the family of polynomials $f_\lambda(x) = x^3 - \lambda x^2 + (\lambda^2 - 1)x + \lambda$ indexed by all $\lambda \in \mathbb{C}$. Let $\mathbf{a}(\lambda) = \lambda$ and $\mathbf{b}(\lambda) = \lambda^3 - 1$.

Question: Are there infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for the same f_λ?

An example

Consider the family of polynomials $f_\lambda(x) = x^3 - \lambda x^2 + (\lambda^2 - 1)x + \lambda$ indexed by all $\lambda \in \mathbb{C}$. Let $\mathbf{a}(\lambda) = \lambda$ and $\mathbf{b}(\lambda) = \lambda^3 - 1$.

Question: Are there infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for the same f_λ?

For example, $\lambda = 0$ satisfies the above conditions since then

- $f_0(x) = x^3 - x$;
- $\mathbf{a}(0) = 0$ and $\mathbf{b}(0) = -1$,

and $f_0(0) = 0$ while $f_0(-1) = 0$.

An example

Consider the family of polynomials $f_\lambda(x) = x^3 - \lambda x^2 + (\lambda^2 - 1)x + \lambda$ indexed by all $\lambda \in \mathbb{C}$. Let $\mathbf{a}(\lambda) = \lambda$ and $\mathbf{b}(\lambda) = \lambda^3 - 1$.

Question: Are there infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for the same f_λ?

For example, $\lambda = 0$ satisfies the above conditions since then

- $f_0(x) = x^3 - x$;
- $\mathbf{a}(0) = 0$ and $\mathbf{b}(0) = -1$,

and $f_0(0) = 0$ while $f_0(-1) = 0$.

Also $\lambda = 1$ works since then

- $f_1(x) = x^3 - x^2 + 1$;
- $\mathbf{a}(1) = 1$ and $\mathbf{b}(1) = 0$,

and $f_1(1) = 1$ while $f_1(0) = 1$.

An example

Consider the family of polynomials $f_\lambda(x) = x^3 - \lambda x^2 + (\lambda^2 - 1)x + \lambda$ indexed by all $\lambda \in \mathbb{C}$. Let $\mathbf{a}(\lambda) = \lambda$ and $\mathbf{b}(\lambda) = \lambda^3 - 1$.

Question: Are there infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for the same f_λ?

For example, $\lambda = 0$ satisfies the above conditions since then

- $f_0(x) = x^3 - x$;
- $\mathbf{a}(0) = 0$ and $\mathbf{b}(0) = -1$,

and $f_0(0) = 0$ while $f_0(-1) = 0$.

Also $\lambda = 1$ works since then

- $f_1(x) = x^3 - x^2 + 1$;
- $\mathbf{a}(1) = 1$ and $\mathbf{b}(1) = 0$,

and $f_1(1) = 1$ while $f_1(0) = 1$.

Are there infinitely many more such λ's? Note that *individually*, there exist infinitely many $\lambda \in \mathbb{C}$ such that either $\mathbf{a}(\lambda)$ or $\mathbf{b}(\lambda)$ are preperiodic for f_λ (simply solve the equation $f_\lambda^n(\mathbf{a}(\lambda)) = \mathbf{a}(\lambda)$ for varying $n \in \mathbb{N}$).

On the other hand, $\lambda = -1$ does not work since

- $f_{-1}(x) = x^3 + x^2 - 1$;
- $\mathbf{a}(-1) = -1$ and $\mathbf{b}(-1) = -2$,

and $f_{-1}(-1) = -1$, while

$$f_{-1}(-2) = -5;\ f_{-1}^2(-2) = -101;\ \ldots\ldots$$

On the other hand, $\lambda = -1$ does not work since

- $f_{-1}(x) = x^3 + x^2 - 1$;
- $\mathbf{a}(-1) = -1$ and $\mathbf{b}(-1) = -2$,

and $f_{-1}(-1) = -1$, while

$$f_{-1}(-2) = -5;\ f_{-1}^2(-2) = -101;\ \dots\dots$$

So, it's not true that $\mathbf{a}(\lambda)$ is preperiodic exactly when $\mathbf{b}(\lambda)$ is preperiodic, and it's not true that $\mathbf{b}(\lambda)$ is always preperiodic under f_λ.

On the other hand, $\lambda = -1$ does not work since

- $f_{-1}(x) = x^3 + x^2 - 1$;
- $\mathbf{a}(-1) = -1$ and $\mathbf{b}(-1) = -2$,

and $f_{-1}(-1) = -1$, while

$$f_{-1}(-2) = -5;\ f_{-1}^2(-2) = -101;\ \ldots\ldots$$

So, it's not true that $\mathbf{a}(\lambda)$ is preperiodic exactly when $\mathbf{b}(\lambda)$ is preperiodic, and it's not true that $\mathbf{b}(\lambda)$ is always preperiodic under f_λ. Nor it is true that $\mathbf{a}(\lambda)$ is always preperiodic, as it's shown by the case $\lambda = 2$.

On the other hand, $\lambda = -1$ does not work since

- $f_{-1}(x) = x^3 + x^2 - 1$;
- $\mathbf{a}(-1) = -1$ and $\mathbf{b}(-1) = -2$,

and $f_{-1}(-1) = -1$, while

$$f_{-1}(-2) = -5;\ f_{-1}^2(-2) = -101;\ \ldots\ldots$$

So, it's not true that $\mathbf{a}(\lambda)$ is preperiodic exactly when $\mathbf{b}(\lambda)$ is preperiodic, and it's not true that $\mathbf{b}(\lambda)$ is always preperiodic under f_λ. Nor it is true that $\mathbf{a}(\lambda)$ is always preperiodic, as it's shown by the case $\lambda = 2$. In that case,

- $f_2(x) = x^3 - 2x^2 + 3x + 2$ and $\mathbf{a}(2) = 2$, while
- $f_2(2) = 8$, $f_2^2(2) = 410$, ...

On the other hand, $\lambda = -1$ does not work since

- $f_{-1}(x) = x^3 + x^2 - 1$;
- $\mathbf{a}(-1) = -1$ and $\mathbf{b}(-1) = -2$,

and $f_{-1}(-1) = -1$, while

$$f_{-1}(-2) = -5;\ f_{-1}^2(-2) = -101;\ \ldots\ldots$$

So, it's not true that $\mathbf{a}(\lambda)$ is preperiodic exactly when $\mathbf{b}(\lambda)$ is preperiodic, and it's not true that $\mathbf{b}(\lambda)$ is always preperiodic under f_λ. Nor it is true that $\mathbf{a}(\lambda)$ is always preperiodic, as it's shown by the case $\lambda = 2$. In that case,

- $f_2(x) = x^3 - 2x^2 + 3x + 2$ and $\mathbf{a}(2) = 2$, while
- $f_2(2) = 8,\ f_2^2(2) = 410,\ \ldots\ldots$

The above two examples coupled with our conjecture suggest that there should only be finitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ since all three conditions (1)-(3) from our conjecture fail in this example.

On the other hand, $\lambda = -1$ does not work since

- $f_{-1}(x) = x^3 + x^2 - 1$;
- $\mathbf{a}(-1) = -1$ and $\mathbf{b}(-1) = -2$,

and $f_{-1}(-1) = -1$, while

$$f_{-1}(-2) = -5;\ f_{-1}^2(-2) = -101;\ \ldots\ldots$$

So, it's not true that $\mathbf{a}(\lambda)$ is preperiodic exactly when $\mathbf{b}(\lambda)$ is preperiodic, and it's not true that $\mathbf{b}(\lambda)$ is always preperiodic under f_λ. Nor it is true that $\mathbf{a}(\lambda)$ is always preperiodic, as it's shown by the case $\lambda = 2$. In that case,

- $f_2(x) = x^3 - 2x^2 + 3x + 2$ and $\mathbf{a}(2) = 2$, while
- $f_2(2) = 8,\ f_2^2(2) = 410,\ \ldots\ldots$

The above two examples coupled with our conjecture suggest that there should only be finitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ since all three conditions (1)-(3) from our conjecture fail in this example. This follows from the next result.

Theorem

(Ghioca, Hsia, Tucker) Let d be an integer greater than 1, let $c_d \in \mathbb{C}^*$, let $c_{d-1}, \ldots, c_0 \in \mathbb{C}[\lambda]$, and let

$$f_\lambda(x) = c_d x^d + c_{d-1}(\lambda) x^{d-1} + \cdots + c_1(\lambda) x + c_0(\lambda).$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ such that

- $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$;
- $\mathbf{a}$ *and* $\mathbf{b}$ *have the same leading coefficient.*

If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then $\mathbf{a} = \mathbf{b}$.

Theorem

(Ghioca, Hsia, Tucker) Let d be an integer greater than 1, let $c_d \in \mathbb{C}^*$, let $c_{d-1}, \ldots, c_0 \in \mathbb{C}[\lambda]$, and let

$$f_\lambda(x) = c_d x^d + c_{d-1}(\lambda)x^{d-1} + \cdots + c_1(\lambda)x + c_0(\lambda).$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ such that

- $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$;
- $\mathbf{a}$ *and* $\mathbf{b}$ *have the same leading coefficient.*

If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then $\mathbf{a} = \mathbf{b}$.

In particular, we get that $\mathbf{a}(\lambda)$ is preperiodic if and only if $\mathbf{b}(\lambda)$ is preperiodic.

Previous example:

$$f_\lambda(x) = x^3 - \lambda x^2 + (\lambda^2 - 1)x + \lambda$$

$$\mathbf{a}(\lambda) := f_\lambda^2(\lambda) = f_\lambda(\lambda^3) = \lambda^9 - \lambda^7 + \lambda^5 - \lambda^3 + \lambda$$

$$\mathbf{b}(\lambda) := f_\lambda(\lambda^3 - 1) = \lambda^9 - \lambda^7 - 3\lambda^6 + \lambda^5 + 2\lambda^4 + 2\lambda^3 - \lambda^2$$

satisfy the hypotheses of our theorem. So, there are at most finitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ (and thus there are finitely many $\lambda \in \mathbb{C}$ such that both λ and $\lambda^3 - 1$ are preperiodic under the action of f_λ).

Baker-DeMarco's theorem

Similarly, Baker-Demarco's result is a corollary of the above theorem. Indeed, if $a, b \in \mathbb{C}$, d is an integer greater than 1, and

$$f_\lambda(x) := x^d + \lambda$$

and

$$\mathbf{a}(\lambda) := f_\lambda^2(a) = (\lambda + a^d)^d + \lambda$$

and

$$\mathbf{b}(\lambda) := f_\lambda^2(b) = (\lambda + b^d)^d + \lambda,$$

then f_λ, $\mathbf{a}$ and $\mathbf{b}$ satisfy the hypotheses of the above theorem.

Baker-DeMarco's theorem

Similarly, Baker-Demarco's result is a corollary of the above theorem. Indeed, if $a, b \in \mathbb{C}$, d is an integer greater than 1, and

$$f_\lambda(x) := x^d + \lambda$$

and

$$\mathbf{a}(\lambda) := f_\lambda^2(a) = (\lambda + a^d)^d + \lambda$$

and

$$\mathbf{b}(\lambda) := f_\lambda^2(b) = (\lambda + b^d)^d + \lambda,$$

then f_λ, $\mathbf{a}$ and $\mathbf{b}$ satisfy the hypotheses of the above theorem. So, if there exist infinitely many $\lambda \in \mathbb{C}$ such that $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ (or equivalently, a and b) are preperiodic for f_λ, then $\mathbf{a} = \mathbf{b}$, i.e., $a^d = b^d$, as desired.

Another application

In the previous theorem we may consider the case that each c_i is constant, i.e., the family of polynomials f_λ is constant (equal to f, say). In this case we have the following interesting consequence.

Another application

In the previous theorem we may consider the case that each c_i is constant, i.e., the family of polynomials f_λ is constant (equal to f, say). In this case we have the following interesting consequence.

Corollary

Let $f \in \mathbb{C}[x]$ be a polynomial of degree larger than 1. Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ be two polynomials of same degree and same leading coefficient. If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f, then $\mathbf{a} = \mathbf{b}$.

A geometric reformulation of the previous statement

Corollary

Let f be a polynomial of degree larger than 1. Let $V \subset \mathbb{A}^2$ be a curve parametrized by $(\mathbf{a}(\lambda), \mathbf{b}(\lambda))$ for $\lambda \in \mathbb{C}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ are two polynomials of same degree and same leading coefficient. If there exist infinitely many points on $V(\mathbb{C})$ which are preperiodic under the map $(x, y) \mapsto (f(x), f(y))$ on $\mathbb{A}^2$, then V is the diagonal line in $\mathbb{A}^2$ (and thus it is itself preperiodic).

A geometric reformulation of the previous statement

Corollary

Let f be a polynomial of degree larger than 1. Let $V \subset \mathbb{A}^2$ be a curve parametrized by $(\mathbf{a}(\lambda), \mathbf{b}(\lambda))$ for $\lambda \in \mathbb{C}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ are two polynomials of same degree and same leading coefficient. If there exist infinitely many points on $V(\mathbb{C})$ which are preperiodic under the map $(x, y) \mapsto (f(x), f(y))$ on $\mathbb{A}^2$, then V is the diagonal line in $\mathbb{A}^2$ (and thus it is itself preperiodic).

This last result is a special case of the Dynamical Manin-Mumford Conjecture made by Zhang.

Observations

If the conditions

- $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$;
- $\mathbf{a}$ and $\mathbf{b}$ have the same leading coefficient.

are not met, then we cannot expect that $\mathbf{a} = \mathbf{b}$.

Observations

If the conditions

- $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \dots, \deg(c_{d-1})\}$;
- $\mathbf{a}$ and $\mathbf{b}$ have the same leading coefficient.

are not met, then we cannot expect that $\mathbf{a} = \mathbf{b}$. For example, if

$$f_\lambda \text{ is odd, and } \mathbf{b} = -\mathbf{a},$$

then $\mathbf{a}(\lambda)$ is preperiodic if and only if $\mathbf{b}(\lambda)$ is preperiodic.

Observations

If the conditions

- $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$;
- $\mathbf{a}$ and $\mathbf{b}$ have the same leading coefficient.

are not met, then we cannot expect that $\mathbf{a} = \mathbf{b}$. For example, if

$$f_\lambda \text{ is odd, and } \mathbf{b} = -\mathbf{a},$$

then $\mathbf{a}(\lambda)$ is preperiodic if and only if $\mathbf{b}(\lambda)$ is preperiodic.
On the other hand, if $\mathbf{b}(\lambda) = f_\lambda(\mathbf{a}(\lambda))$, then again $\mathbf{a}(\lambda)$ is preperiodic if and only if $\mathbf{b}(\lambda)$ is preperiodic.

Observations

If the conditions

- $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$;
- $\mathbf{a}$ and $\mathbf{b}$ have the same leading coefficient.

are not met, then we cannot expect that $\mathbf{a} = \mathbf{b}$. For example, if

$$f_\lambda \text{ is odd, and } \mathbf{b} = -\mathbf{a},$$

then $\mathbf{a}(\lambda)$ is preperiodic if and only if $\mathbf{b}(\lambda)$ is preperiodic.
On the other hand, if $\mathbf{b}(\lambda) = f_\lambda(\mathbf{a}(\lambda))$, then again $\mathbf{a}(\lambda)$ is preperiodic if and only if $\mathbf{b}(\lambda)$ is preperiodic.
So, without extra assumptions on $\mathbf{a}$ and $\mathbf{b}$ it is difficult to prove what are the precise relations between $\mathbf{a}$ and $\mathbf{b}$.

Theorem

Let d be an integer greater than 1, let $c_d \in \mathbb{C}^*$, let $c_{d-1}, \ldots, c_0 \in \mathbb{C}[\lambda]$, and let

$$f_\lambda(x) = c_d x^d + c_{d-1}(\lambda)x^{d-1} + \cdots + c_1(\lambda)x + c_0(\lambda).$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ such that

1. $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$*;*
2. *$\mathbf{a}$ and $\mathbf{b}$ have the same leading coefficient.*

If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then $\mathbf{a} = \mathbf{b}$.

In order to prove the result, first we focus on the algebraic case: $\mathbf{a}, \mathbf{b} \in \bar{\mathbb{Q}}[\lambda]$ and $c_i \in \bar{\mathbb{Q}}[\lambda]$. Using the technique of specializations, we can infer the general result from the algebraic case.

Theorem

Let d be an integer greater than 1, let $c_d \in \mathbb{C}^*$, let $c_{d-1}, \ldots, c_0 \in \mathbb{C}[\lambda]$, and let

$$f_\lambda(x) = c_d x^d + c_{d-1}(\lambda)x^{d-1} + \cdots + c_1(\lambda)x + c_0(\lambda).$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ such that

1. $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$;
2. $\mathbf{a}$ *and* $\mathbf{b}$ *have the same leading coefficient.*

If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then $\mathbf{a} = \mathbf{b}$.

In order to prove the result, first we focus on the algebraic case: $\mathbf{a}, \mathbf{b} \in \bar{\mathbb{Q}}[\lambda]$ and $c_i \in \bar{\mathbb{Q}}[\lambda]$. Using the technique of specializations, we can infer the general result from the algebraic case. Also, we may assume f_λ is monic (i.e., $c_d = 1$), at the expense of replacing the entire family by a suitable conjugate: $\mu^{-1} \circ f_\lambda \circ \mu$, where $\mu(z) = Az$ for a suitable number A.

Theorem

Let d be an integer greater than 1, let $c_d \in \mathbb{C}^*$, let $c_{d-1}, \ldots, c_0 \in \mathbb{C}[\lambda]$, and let

$$f_\lambda(x) = c_d x^d + c_{d-1}(\lambda)x^{d-1} + \cdots + c_1(\lambda)x + c_0(\lambda).$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ such that

1. $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$*;*
2. $\mathbf{a}$ *and* $\mathbf{b}$ *have the same leading coefficient.*

If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then $\mathbf{a} = \mathbf{b}$.

In order to prove the result, first we focus on the algebraic case: $\mathbf{a}, \mathbf{b} \in \bar{\mathbb{Q}}[\lambda]$ and $c_i \in \bar{\mathbb{Q}}[\lambda]$. Using the technique of specializations, we can infer the general result from the algebraic case. Also, we may assume f_λ is monic (i.e., $c_d = 1$), at the expense of replacing the entire family by a suitable conjugate: $\mu^{-1} \circ f_\lambda \circ \mu$, where $\mu(z) = Az$ for a suitable number A. Secondly, if the family f_λ is constant, then we may assume $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq 1$ since otherwise the conclusion is vacuously true.

Ideea for our proof

Now, we go back to the Masser-Zannier problem for the Legendre family of elliptic curves E_λ. They proved that for two sections P_λ and Q_λ, if there exist infinitely many λ such that both P_λ and Q_λ are torsion points for E_λ, then there exist (nonzero) $m, n \in \mathbb{Z}$ such that $[m]P_\lambda = [n]Q_\lambda$.

Ideea for our proof

Now, we go back to the Masser-Zannier problem for the Legendre family of elliptic curves E_λ. They proved that for two sections P_λ and Q_λ, if there exist infinitely many λ such that both P_λ and Q_λ are torsion points for E_λ, then there exist (nonzero) $m, n \in \mathbb{Z}$ such that $[m]P_\lambda = [n]Q_\lambda$. Letting $\widehat{h}_\lambda$ be the canonical height for the elliptic curve E_λ, we would then have

$$\widehat{h}_\lambda(P_\lambda)/\widehat{h}_\lambda(Q_\lambda) = n^2/m^2$$

is *constant* on all elliptic fibers. Furthermore, even the local canonical heights of the two points have constant quotient on all ellliptic fibers.

Ideea for our proof

Now, we go back to the Masser-Zannier problem for the Legendre family of elliptic curves E_λ. They proved that for two sections P_λ and Q_λ, if there exist infinitely many λ such that both P_λ and Q_λ are torsion points for E_λ, then there exist (nonzero) $m, n \in \mathbb{Z}$ such that $[m]P_\lambda = [n]Q_\lambda$. Letting $\widehat{h}_\lambda$ be the canonical height for the elliptic curve E_λ, we would then have

$$\widehat{h}_\lambda(P_\lambda)/\widehat{h}_\lambda(Q_\lambda) = n^2/m^2$$

is *constant* on all elliptic fibers. Furthermore, even the local canonical heights of the two points have constant quotient on all ellliptic fibers.
In order to achieve our goal we use the method introduced by Baker and DeMarco.

Idea of proof (continued)

We can define the canonical height for $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ under the action of f_λ for any $\lambda \in \bar{\mathbb{Q}}$ as

$$\widehat{h}_\lambda(\mathbf{a}(\lambda)) = \lim_{n\to\infty} \frac{h(f_\lambda^n(\mathbf{a}(\lambda)))}{d^n},$$

where $d = \deg(f_\lambda)$ and $h(\cdot)$ is the naive Weil height. So, we may wonder if we could prove that $\widehat{h}_\lambda(\mathbf{a}(\lambda))/\widehat{h}_\lambda(\mathbf{b}(\lambda))$ is constant for all $\lambda \in \bar{\mathbb{Q}}$.

Idea of proof (continued)

We can define the canonical height for $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ under the action of f_λ for any $\lambda \in \bar{\mathbb{Q}}$ as

$$\widehat{h}_\lambda(\mathbf{a}(\lambda)) = \lim_{n\to\infty} \frac{h(f_\lambda^n(\mathbf{a}(\lambda)))}{d^n},$$

where $d = \deg(f_\lambda)$ and $h(\cdot)$ is the naive Weil height. So, we may wonder if we could prove that $\widehat{h}_\lambda(\mathbf{a}(\lambda))/\widehat{h}_\lambda(\mathbf{b}(\lambda))$ is constant for all $\lambda \in \bar{\mathbb{Q}}$.

Imagine we can prove the (seemingly) weaker statement that the local canonical heights of $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ with respect to the archimedean valuation given by a fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ have constant quotient for all $\lambda \in \bar{\mathbb{Q}}$.

Idea of proof (continued)

We can define the canonical height for $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ under the action of f_λ for any $\lambda \in \bar{\mathbb{Q}}$ as

$$\widehat{h}_\lambda(\mathbf{a}(\lambda)) = \lim_{n\to\infty} \frac{h(f_\lambda^n(\mathbf{a}(\lambda)))}{d^n},$$

where $d = \deg(f_\lambda)$ and $h(\cdot)$ is the naive Weil height. So, we may wonder if we could prove that $\widehat{h}_\lambda(\mathbf{a}(\lambda))/\widehat{h}_\lambda(\mathbf{b}(\lambda))$ is constant for all $\lambda \in \bar{\mathbb{Q}}$.

Imagine we can prove the (seemingly) weaker statement that the local canonical heights of $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ with respect to the archimedean valuation given by a fixed embedding of $\bar{\mathbb{Q}}$ into $\mathbb{C}$ have constant quotient for all $\lambda \in \bar{\mathbb{Q}}$. This fact follows from the equidistribution theorem proved by Baker and Rumely on Berkovich spaces.

More precisely, for each $\mathbf{c} \in \bar{\mathbb{Q}}[\lambda]$ of degree

$$m \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$$

we let

$$G_\lambda(\mathbf{c}(\lambda)) = \lim_{n \to \infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|}{md^n},$$

where $\log^+(z) := \log \max\{1, z\}$ for any positive real number z.

More precisely, for each $\mathbf{c} \in \bar{\mathbb{Q}}[\lambda]$ of degree

$$m \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$$

we let

$$G_\lambda(\mathbf{c}(\lambda)) = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|}{md^n},$$

where $\log^+(z) := \log\max\{1, z\}$ for any positive real number z. Baker-Rumely equidistribution theorem yields that

$$G_\lambda(\mathbf{a}(\lambda)) = G_\lambda(\mathbf{b}(\lambda)) \text{ for all } \lambda \in \bar{\mathbb{Q}}.$$

More precisely, for each $\mathbf{c} \in \bar{\mathbb{Q}}[\lambda]$ of degree

$$m \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$$

we let

$$G_\lambda(\mathbf{c}(\lambda)) = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|}{md^n},$$

where $\log^+(z) := \log\max\{1, z\}$ for any positive real number z. Baker-Rumely equidistribution theorem yields that

$$G_\lambda(\mathbf{a}(\lambda)) = G_\lambda(\mathbf{b}(\lambda)) \text{ for all } \lambda \in \bar{\mathbb{Q}}.$$

This last equality will be sufficient for us to conclude that $\mathbf{a} = \mathbf{b}$. But first we need to understand better the (Green) function $\mathbf{G}_c : \mathbb{C} \longrightarrow \mathbb{R}_{\geq 0}$ given by $\mathbf{G}_\mathbf{c}(\lambda) = G_\lambda(\mathbf{c}(\lambda))$ for any given $\mathbf{c} \in \bar{\mathbb{Q}}[\lambda]$.

Bötcher's Uniformization Theorem

For any (monic) polynomial $g \in \mathbb{C}[x]$ of degree $d \geq 2$, there exists a real number $R \geq 1$ and an analytic map $\Phi : U_R \longrightarrow U_R$, where

$$U_R = \{z \in \mathbb{C} : |z| > R\}$$

satisfying the following two conditions:

(i) Φ is univalent on U_R and at ∞,

$$\Phi(z) = z + O\left(\frac{1}{z}\right);$$

(ii) for all $z \in U_R$ we have

$$\Phi(g(z)) = \Phi(z)^d.$$

Bötcher's Uniformization Theorem

For any (monic) polynomial $g \in \mathbb{C}[x]$ of degree $d \geq 2$, there exists a real number $R \geq 1$ and an analytic map $\Phi : U_R \longrightarrow U_R$, where

$$U_R = \{z \in \mathbb{C} : |z| > R\}$$

satisfying the following two conditions:

(i) Φ is univalent on U_R and at ∞,

$$\Phi(z) = z + O\left(\frac{1}{z}\right);$$

(ii) for all $z \in U_R$ we have

$$\Phi(g(z)) = \Phi(z)^d.$$

More precisely,

$$\Phi(z) = z \cdot \prod_{n=0}^{\infty} \left(\frac{g^{n+1}(z)}{g^n(z)^d}\right)^{\frac{1}{d^{n+1}}}$$

The Green's Function

Then for $z \in U_R$, we know that $g(z) \in U_R$ and thus

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log |g^n(z)|}{d^n} \\
&= \lim_{n\to\infty} \frac{\log |\Phi(g^n(z))|}{d^n} \\
&= \lim_{n\to\infty} \frac{\log \left|\Phi(z)^{d^n}\right|}{d^n} \\
&= \log |\Phi(z)|.
\end{aligned}$$

The function $\mathbf{G}_c$

We recall that

$$\mathbf{G}_{\mathbf{c}}(\lambda) = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|}{md^n}$$

where $m = \deg(\mathbf{c}) \geq d \cdot \max\{\deg(c_0), \dots, \deg(c_{d-1})\}$.

The function $\mathbf{G}_c$

We recall that

$$\mathbf{G}_\mathbf{c}(\lambda) = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|}{md^n}$$

where $m = \deg(\mathbf{c}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$. We denote by Φ_λ the corresponding uniformizing map at ∞ for each f_λ; also we let R_λ be the radius of convergence for each Φ_λ.

The function $\mathbf{G}_c$

We recall that

$$\mathbf{G_c}(\lambda) = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|}{md^n}$$

where $m = \deg(\mathbf{c}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$. We denote by Φ_λ the corresponding uniformizing map at ∞ for each f_λ; also we let R_λ be the radius of convergence for each Φ_λ. We can prove that there exists a positive real number M such that for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| > M$,

$$\mathbf{c}(\lambda) \in U_{R_\lambda}.$$

This allows us to conclude that, if $|\lambda| > M$ then

$$\begin{aligned} &\mathbf{G_c}(\lambda) \\ &= \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|}{md^n} \\ &= \frac{\log |\Phi_\lambda(\mathbf{c}(\lambda))|}{m}. \end{aligned}$$

The function $\mathbf{G}$ (continued)

We note that

$$\Phi_\lambda(\mathbf{c}(\lambda)) = \mathbf{c}(\lambda) \cdot \prod_{n=0}^{\infty} \left(\frac{f_\lambda^{n+1}(\mathbf{c}(\lambda))}{f_\lambda^n(\mathbf{c}(\lambda))^d} \right)^{\frac{1}{d^{n+1}}}$$

So, using that the degree m of $\mathbf{c}$ is larger than the degrees of the c_i's, and letting q be the leading coefficient of $\mathbf{c}$, we conclude that $\lambda \mapsto \Phi_\lambda(f_\lambda(c))$ has the following properties:

(i) it's an analytic function on $U_M = \{\lambda \in \mathbb{C} : |\lambda| > M\}$.

(ii) at infinity, $\Phi_\lambda(\mathbf{c}(\lambda)) = q\lambda^m + O(\lambda^{m-1})$.

(iii) $\mathbf{G}_\mathbf{c}(\lambda) = \frac{\log|\Phi_\lambda(f_\lambda(\mathbf{c}))|}{m}$.

Conclusion of our proof

Using the existence of infinitely many λ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, Baker-Rumely equidistribution theorem yields

$$\mathbf{G_a}(\lambda) = \mathbf{G_b}(\lambda) \text{ for all } \lambda \in \bar{\mathbb{Q}}.$$

Conclusion of our proof

Using the existence of infinitely many λ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, Baker-Rumely equidistribution theorem yields

$$\mathbf{G_a}(\lambda) = \mathbf{G_b}(\lambda) \text{ for all } \lambda \in \bar{\mathbb{Q}}.$$

So, for $\lambda \in \bar{\mathbb{Q}}$ satfisfying $|\lambda| > M$ we conclude that

$$\mathbf{G_a}(\lambda) = \frac{\log|\Phi_\lambda(\mathbf{a}(\lambda))|}{\deg(\mathbf{a})} = \frac{\log|\Phi_\lambda(\mathbf{b}(\lambda))|}{\deg(\mathbf{b})} = \mathbf{G_b}(\lambda).$$

and thus, using that $\deg(\mathbf{a}) = \deg(\mathbf{b})$ we have

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \bar{\mathbb{Q}} \text{ s.t. } |\lambda| > M.$$

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \bar{\mathbb{Q}} \text{ s.t. } |\lambda| > M.$$

By continuity we obtain that

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \mathbb{C} \text{ s.t. } |\lambda| > M,$$

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \bar{\mathbb{Q}} \text{ s.t. } |\lambda| > M.$$

By continuity we obtain that

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \mathbb{C} \text{ s.t. } |\lambda| > M,$$

and by the Open Mapping Theorem we conclude that there exists $u \in \mathbb{C}$ of absolute value equal to 1 such that

$$\Phi_\lambda(\mathbf{a}(\lambda)) = u \cdot \Phi_\lambda(\mathbf{b}(\lambda)) \text{ if } |\lambda| > M.$$

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \bar{\mathbb{Q}} \text{ s.t. } |\lambda| > M.$$

By continuity we obtain that

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \mathbb{C} \text{ s.t. } |\lambda| > M,$$

and by the Open Mapping Theorem we conclude that there exists $u \in \mathbb{C}$ of absolute value equal to 1 such that

$$\Phi_\lambda(\mathbf{a}(\lambda)) = u \cdot \Phi_\lambda(\mathbf{b}(\lambda)) \text{ if } |\lambda| > M.$$

Since both $\Phi_\lambda(\mathbf{a}(\lambda))$ and $\Phi_\lambda(\mathbf{b}(\lambda))$ have the expansion $q\lambda^m + O(\lambda^{m-1})$ at infinity, we get that $u = 1$; therefore

$$\Phi_\lambda(\mathbf{a}(\lambda)) = \Phi_\lambda(\mathbf{b}(\lambda)) \text{ if } |\lambda| > M.$$

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \bar{\mathbb{Q}} \text{ s.t. } |\lambda| > M.$$

By continuity we obtain that

$$|\Phi_\lambda(\mathbf{a}(\lambda))| = |\Phi_\lambda(\mathbf{b}(\lambda))| \text{ for } \lambda \in \mathbb{C} \text{ s.t. } |\lambda| > M,$$

and by the Open Mapping Theorem we conclude that there exists $u \in \mathbb{C}$ of absolute value equal to 1 such that

$$\Phi_\lambda(\mathbf{a}(\lambda)) = u \cdot \Phi_\lambda(\mathbf{b}(\lambda)) \text{ if } |\lambda| > M.$$

Since both $\Phi_\lambda(\mathbf{a}(\lambda))$ and $\Phi_\lambda(\mathbf{b}(\lambda))$ have the expansion $q\lambda^m + O(\lambda^{m-1})$ at infinity, we get that $u = 1$; therefore

$$\Phi_\lambda(\mathbf{a}(\lambda)) = \Phi_\lambda(\mathbf{b}(\lambda)) \text{ if } |\lambda| > M.$$

Finally, using the fact that Φ_λ is univalent on U_{R_λ} and both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are in U_{R_λ} if $|\lambda| > M$, we obtain that

$$\mathbf{a}(\lambda) = \mathbf{b}(\lambda).$$

Remarks

Assume now that conditions (1)-(2) in our theorem are not met.

Theorem

Let d be an integer greater than 1, let $c_d \in \mathbb{C}^*$, let $c_{d-1}, \ldots, c_0 \in \mathbb{C}[\lambda]$, and let

$$f_\lambda(x) = c_d x^d + c_{d-1}(\lambda) x^{d-1} + \cdots + c_1(\lambda) x + c_0(\lambda).$$

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ such that

1. $\deg(\mathbf{a}) = \deg(\mathbf{b}) \geq d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$*;*
2. $\mathbf{a}$ *and* $\mathbf{b}$ *have the same leading coefficient.*

If there exist infinitely many $\lambda \in \mathbb{C}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then $\mathbf{a} = \mathbf{b}$.

Furthermore, assume f_λ is not a constant family. Then because f_λ is a polynomial family and $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ then $\mathbf{a}$ (or $\mathbf{b}$) is preperiodic if and only if

$$\deg_\lambda(f_\lambda^n(\mathbf{a}(\lambda))) \text{ is unbounded as } n \to \infty.$$

Furthermore, assume f_λ is not a constant family. Then because f_λ is a polynomial family and $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ then $\mathbf{a}$ (or $\mathbf{b}$) is preperiodic if and only if

$$\deg_\lambda(f_\lambda^n(\mathbf{a}(\lambda))) \text{ is unbounded as } n \to \infty.$$

The reason for this is that on the generic fiber, $\mathbf{a}$ (or $\mathbf{b}$) is preperiodic if and only if its height with respect to $\mathbf{f} = f_\lambda$ is 0 (by a theorem of Benedetto for non-isotrivial polynomial actions). Moreover, the only place of $\mathbb{C}(\lambda)$ for which the local height of $\mathbf{a}$ (of $\mathbf{b}$) might be nonzero is the place at infinity, since the coefficients c_i of $\mathbf{f}$ and also $\mathbf{a}$ (and $\mathbf{b}$) are integral everywhere else. And at the infinity place, the local height of $\mathbf{a}$ (or $\mathbf{b}$) with respect to $\mathbf{f}$ is nonzero if and only if the degrees in λ of the iterates of $\mathbf{a}$ (resp. $\mathbf{b}$) under $\mathbf{f}$ grow unbounded.

Furthermore, assume f_λ is not a constant family. Then because f_λ is a polynomial family and $\mathbf{a}, \mathbf{b} \in \mathbb{C}[\lambda]$ then $\mathbf{a}$ (or $\mathbf{b}$) is preperiodic if and only if

$$\deg_\lambda(f_\lambda^n(\mathbf{a}(\lambda))) \text{ is unbounded as } n \to \infty.$$

The reason for this is that on the generic fiber, $\mathbf{a}$ (or $\mathbf{b}$) is preperiodic if and only if its height with respect to $\mathbf{f} = f_\lambda$ is 0 (by a theorem of Benedetto for non-isotrivial polynomial actions). Moreover, the only place of $\mathbb{C}(\lambda)$ for which the local height of $\mathbf{a}$ (of $\mathbf{b}$) might be nonzero is the place at infinity, since the coefficients c_i of $\mathbf{f}$ and also $\mathbf{a}$ (and $\mathbf{b}$) are integral everywhere else. And at the infinity place, the local height of $\mathbf{a}$ (or $\mathbf{b}$) with respect to $\mathbf{f}$ is nonzero if and only if the degrees in λ of the iterates of $\mathbf{a}$ (resp. $\mathbf{b}$) under $\mathbf{f}$ grow unbounded.
Assume neither $\mathbf{a}$ nor $\mathbf{b}$ is identically preperiodic for our family of polynomials. Then the degrees in λ of the iterates of $\mathbf{a}$ and $\mathbf{b}$ under $\mathbf{f}$ are unbounded.

Thus we may assume there exists $k \in \mathbb{N}$ such that

$$m_{\mathbf{a}} := \deg_{\lambda}(f_{\lambda}^{k}(\mathbf{a}(\lambda))) > d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$$

and

$$m_{\mathbf{b}} := \deg_{\lambda}(f_{\lambda}^{k}(\mathbf{b}(\lambda))) > d \cdot \max\{\deg(c_0), \ldots, \deg(c_{d-1})\}$$

So, without loss of generality, we may replace **a** and **b** by their k-th iterate under f_{λ}.

Thus we may assume there exists $k \in \mathbb{N}$ such that

$$m_{\mathbf{a}} := \deg_\lambda(f_\lambda^k(\mathbf{a}(\lambda))) > d \cdot \max\{\deg(c_0), \dots, \deg(c_{d-1})\}$$

and

$$m_{\mathbf{b}} := \deg_\lambda(f_\lambda^k(\mathbf{b}(\lambda))) > d \cdot \max\{\deg(c_0), \dots, \deg(c_{d-1})\}$$

So, without loss of generality, we may replace **a** and **b** by their k-th iterate under f_λ. Then the exact same reasoning as above would still yield that if there exist infinitely many λ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic under f_λ, then the two functions

$$\mathbf{G}_{\mathbf{a}}(\lambda) := \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{a}(\lambda))|}{m_{\mathbf{a}} d^n} = \frac{\log |\Phi_\lambda(\mathbf{a}(\lambda))|}{m_{\mathbf{a}}}$$

and

$$\mathbf{G}_{\mathbf{b}}(\lambda) := \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{b}(\lambda))|}{m_{\mathbf{b}} d^n} = \frac{\log |\Phi_\lambda(\mathbf{b}(\lambda))|}{m_{\mathbf{b}}}$$

are equal.

So, again we can find a complex number u of absolute value equal to 1 such that

$$\Phi_\lambda(\mathbf{a}(\lambda))^{m_\mathbf{b}} = u \cdot \Phi_\lambda(\mathbf{b}(\lambda))^{m_\mathbf{a}}.$$

So, again we can find a complex number u of absolute value equal to 1 such that

$$\Phi_\lambda(\mathbf{a}(\lambda))^{m_\mathbf{b}} = u \cdot \Phi_\lambda(\mathbf{b}(\lambda))^{m_\mathbf{a}}.$$

Just as before we get that

$$\Phi_\lambda(\mathbf{a}(\lambda)) = q_\mathbf{a}\lambda^{m_\mathbf{a}} + O\left(q^{m_\mathbf{a}-1}\right)$$

and

$$\Phi_\lambda(\mathbf{b}(\lambda)) = q_\mathbf{b}\lambda^{m_\mathbf{b}} + O\left(q^{m_\mathbf{b}-1}\right).$$

So, again we can find a complex number u of absolute value equal to 1 such that

$$\Phi_\lambda(\mathbf{a}(\lambda))^{m_\mathbf{b}} = u \cdot \Phi_\lambda(\mathbf{b}(\lambda))^{m_\mathbf{a}}.$$

Just as before we get that

$$\Phi_\lambda(\mathbf{a}(\lambda)) = q_\mathbf{a}\lambda^{m_\mathbf{a}} + O\left(q^{m_\mathbf{a}-1}\right)$$

and

$$\Phi_\lambda(\mathbf{b}(\lambda)) = q_\mathbf{b}\lambda^{m_\mathbf{b}} + O\left(q^{m_\mathbf{b}-1}\right).$$

However this is not enough information to derive an exact relation between **a** and **b**.
It seems that even knowing that $m_\mathbf{a} = m_\mathbf{b}$ would not be enough (unless we also know that $q_\mathbf{a} = q_\mathbf{b}$).

Concluding remarks

Assume now in addition that f_λ, **a** and **b** are all defined over $\bar{\mathbb{Q}}$. Then the equidistribution theorem of Baker and Rumely still yields that

$$\frac{\widehat{h}_\lambda(\mathbf{a}(\lambda))}{\deg(\mathbf{a})} = \frac{\widehat{h}_\lambda(\mathbf{b}(\lambda))}{\deg(\mathbf{b})}$$

Concluding remarks

Assume now in addition that f_λ, $\mathbf{a}$ and $\mathbf{b}$ are all defined over $\bar{\mathbb{Q}}$. Then the equidistribution theorem of Baker and Rumely still yields that

$$\frac{\widehat{h}_\lambda(\mathbf{a}(\lambda))}{\deg(\mathbf{a})} = \frac{\widehat{h}_\lambda(\mathbf{b}(\lambda))}{\deg(\mathbf{b})}$$

Therefore for each $\lambda \in \bar{\mathbb{Q}}$, we obtain that

$$\widehat{h}_\lambda(\mathbf{a}(\lambda)) = 0 \text{ if and only if } \widehat{h}_\lambda(\mathbf{b}(\lambda)) = 0.$$

Concluding remarks

Assume now in addition that f_λ, $\mathbf{a}$ and $\mathbf{b}$ are all defined over $\bar{\mathbb{Q}}$. Then the equidistribution theorem of Baker and Rumely still yields that

$$\frac{\widehat{h}_\lambda(\mathbf{a}(\lambda))}{\deg(\mathbf{a})} = \frac{\widehat{h}_\lambda(\mathbf{b}(\lambda))}{\deg(\mathbf{b})}$$

Therefore for each $\lambda \in \bar{\mathbb{Q}}$, we obtain that

$$\widehat{h}_\lambda(\mathbf{a}(\lambda)) = 0 \text{ if and only if } \widehat{h}_\lambda(\mathbf{b}(\lambda)) = 0.$$

Over a number field, a point is preperiodic if and only if its canonical height equals 0; so

$$\mathbf{a}(\lambda) \text{ if preperiodic if and only if } \mathbf{b}(\lambda) \text{ is preperiodic.}$$

Conclusion

Therefore, for non-constant families $\mathbf{f} = f_\lambda$ of polynomials defined over $\bar{\mathbb{Q}}$, and for *any* $\mathbf{a}, \mathbf{b} \in \bar{\mathbb{Q}}[\lambda]$ we proved that if there exist infinitely many $\lambda \in \bar{\mathbb{Q}}$ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ, then

- either $\mathbf{a}$ or $\mathbf{b}$ is preperiodic for $\mathbf{f}$; or
- $\mathbf{a}(\lambda)$ is preperiodic for f_λ if and only if $\mathbf{b}(\lambda)$ is preperiodic for f_λ.

The **hard** part

The above argument was all based on the strong assumption that the local canonical heights of the two starting points under the maps f_λ are proportional. This assumption *happens* to be true, but it is very difficult to prove it. Below we will only sketch our proof.

The **hard** part

The above argument was all based on the strong assumption that the local canonical heights of the two starting points under the maps f_λ are proportional. This assumption *happens* to be true, but it is very difficult to prove it. Below we will only sketch our proof. We let K be a number field containing all coefficients of $\mathbf{a}$, $\mathbf{b}$ and of f_λ. (It is easy to see that if $\mathbf{a}$ or $\mathbf{b}$ is preperiodic under f_λ, then $\lambda \in \overline{K} = \bar{\mathbb{Q}}$.) For each place v of K (both archimedean and nonarchimedean) we let $\mathbb{C}_v$ be the completion of the algebraic closure of the completion of K at the place v (strictly speaking for nonarchimedean places v, we need to replace $\mathbb{C}_v$ with the corresponding Berkovich space since the former is not locally compact).

The **hard** part

The above argument was all based on the strong assumption that the local canonical heights of the two starting points under the maps f_λ are proportional. This assumption *happens* to be true, but it is very difficult to prove it. Below we will only sketch our proof. We let K be a number field containing all coefficients of $\mathbf{a}$, $\mathbf{b}$ and of f_λ. (It is easy to see that if $\mathbf{a}$ or $\mathbf{b}$ is preperiodic under f_λ, then $\lambda \in \overline{K} = \bar{\mathbb{Q}}$.) For each place v of K (both archimedean and nonarchimedean) we let $\mathbb{C}_v$ be the completion of the algebraic closure of the completion of K at the place v (strictly speaking for nonarchimedean places v, we need to replace $\mathbb{C}_v$ with the corresponding Berkovich space since the former is not locally compact).
Next we construct the generalized Mandelbrot sets $\mathbf{M}_{\mathbf{a},v}$ and $\mathbf{M}_{\mathbf{b},v}$.

The Generalized Mandelbrot sets

With the above notation, and for any $\mathbf{c} \in K[\lambda]$ of sufficiently high degree, we define $\mathbf{M}_{\mathbf{c},v}$ to be the set of all $\lambda \in \mathbb{C}_v$ such that the sequence $\{|f_\lambda^n(\mathbf{c}(\lambda))|_v\}_{n\in\mathbb{N}}$ is bounded. Alternatively, this is equivalent with asking that the local canonical height

$$\lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|_v}{d^n}$$

equals 0.

The Generalized Mandelbrot sets

With the above notation, and for any $\mathbf{c} \in K[\lambda]$ of sufficiently high degree, we define $\mathbf{M}_{\mathbf{c},v}$ to be the set of all $\lambda \in \mathbb{C}_v$ such that the sequence $\{|f_\lambda^n(\mathbf{c}(\lambda))|_v\}_{n\in\mathbb{N}}$ is bounded. Alternatively, this is equivalent with asking that the local canonical height

$$\lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|_v}{d^n}$$

equals 0.

Clearly, if $\mathbf{c}(\lambda)$ is preperiodic under f_λ, then $\lambda \in \mathbf{M}_{\mathbf{c},v}$ for *all* places v.

The Generalized Mandelbrot sets

With the above notation, and for any $\mathbf{c} \in K[\lambda]$ of sufficiently high degree, we define $\mathbf{M}_{\mathbf{c},v}$ to be the set of all $\lambda \in \mathbb{C}_v$ such that the sequence $\{|f_\lambda^n(\mathbf{c}(\lambda))|_v\}_{n\in\mathbb{N}}$ is bounded. Alternatively, this is equivalent with asking that the local canonical height

$$\lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{c}(\lambda))|_v}{d^n}$$

equals 0.
Clearly, if $\mathbf{c}(\lambda)$ is preperiodic under f_λ, then $\lambda \in \mathbf{M}_{\mathbf{c},v}$ for *all* places v.
The first important property of these generalized Mandelbrot sets is that they are compact.

The Green function of a compact subset of $\mathbb{C}_v$

Let E be a compact subset of $\mathbb{C}_v$. The logarithmic capacity $\gamma(E) = e^{-V(E)}$ and the Green's function $\mathbf{G}_E$ of E (relative to ∞) can be defined where $V(E)$ is the infimum of the *energy integral* with respect to all possible probability measures supported on E.

The Green function of a compact subset of $\mathbb{C}_v$

Let E be a compact subset of $\mathbb{C}_v$. The logarithmic capacity $\gamma(E) = e^{-V(E)}$ and the Green's function $\mathbf{G}_E$ of E (relative to ∞) can be defined where $V(E)$ is the infimum of the *energy integral* with respect to all possible probability measures supported on E. More precisely,

$$V(E) = \inf_{\mu} \int \int_{E \times E} -\log |x - y|_v d\mu(x) d\mu(y),$$

where the infimum is computed with respect to all probability measures μ supported on E.

The Green function of a compact subset of $\mathbb{C}_v$

Let E be a compact subset of $\mathbb{C}_v$. The logarithmic capacity $\gamma(E) = e^{-V(E)}$ and the Green's function $\mathbf{G}_E$ of E (relative to ∞) can be defined where $V(E)$ is the infimum of the *energy integral* with respect to all possible probability measures supported on E. More precisely,

$$V(E) = \inf_{\mu} \int\int_{E\times E} -\log|x-y|_v d\mu(x)d\mu(y),$$

where the infimum is computed with respect to all probability measures μ supported on E.
If $\gamma(E) > 0$ (i.e., if $V(E) \neq +\infty$), then the exists a unique probability measure μ_E attaining the infimum of the energy integral. Furthermore, the support of μ_E is contained in the boundary of the unbounded component of $\mathbb{C}_v \setminus E$.

The Green function of a compact subset of $\mathbb{C}_v$ (continued)

The Green's function $\mathbf{G}_E(z)$ of E relative to infinity is a well-defined nonnegative real-valued subharmonic function on $\mathbb{C}_v$ which is harmonic on $\mathbb{C}_v \setminus E$. Furthermore,

$$\mathbf{G}_E(z) = \log |z|_v + V(E) + o(1),$$

as $|z|_v \to \infty$.

The Green function of a compact subset of $\mathbb{C}_v$ (continued)

The Green's function $\mathbf{G}_E(z)$ of E relative to infinity is a well-defined nonnegative real-valued subharmonic function on $\mathbb{C}_v$ which is harmonic on $\mathbb{C}_v \setminus E$. Furthermore,

$$\mathbf{G}_E(z) = \log |z|_v + V(E) + o(1),$$

as $|z|_v \to \infty$.
If E is the closed unit disk, then $\gamma(E) = 1$ and $\mathbf{G}_E(z) = \log^+ |z|_v$.

The Green function of a compact subset of $\mathbb{C}_v$ (continued)

The Green's function $\mathbf{G}_E(z)$ of E relative to infinity is a well-defined nonnegative real-valued subharmonic function on $\mathbb{C}_v$ which is harmonic on $\mathbb{C}_v \setminus E$. Furthermore,

$$\mathbf{G}_E(z) = \log|z|_v + V(E) + o(1),$$

as $|z|_v \to \infty$.
If E is the closed unit disk, then $\gamma(E) = 1$ and $\mathbf{G}_E(z) = \log^+|z|_v$.
More importantly, for our generalized Mandelbrot set $\mathbf{M}_{\mathbf{c},v}$, we have

$$\mathbf{G}_{\mathbf{M}_{\mathbf{c},v}}(z) = \lim_{n\to\infty} \frac{\log^+|f_\lambda^n(\mathbf{c}(\lambda))|_v}{\deg(\mathbf{c})\cdot d^n}.$$

Berkovich adèlic sets

Assume now that for each place v of K, we have a compact subset E_v of $\mathbb{C}_v$ with the property that for all but finitely many places v, E_v is the closed unit disk in $\mathbb{C}_v$.

Berkovich adèlic sets

Assume now that for each place v of K, we have a compact subset E_v of $\mathbb{C}_v$ with the property that for all but finitely many places v, E_v is the closed unit disk in $\mathbb{C}_v$. We call

$$\mathbf{E} := \prod_v E_v$$

a Berkovich adèlic set, and define its capacity to be

$$\gamma(\mathbf{E}) := \prod_v \gamma(E_v)^{N_v},$$

where the positive integers N_v are the ones defined as in the product formula on the global field K, i.e., such that for each nonzero $x \in K$, we would have $\prod_v |x|_v^{N_v} = 1$.

Berkovich adèlic sets (continued)

Let $\mathbf{G}_v = \mathbf{G}_{E_v}$ be the Green's function of E_v relative for each place v. For every v we fix an embedding $\overline{K}$ into $\mathbb{C}_v$. Let $S \subset \overline{K}$ be any finite subset that is invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$.

Berkovich adèlic sets (continued)

Let $\mathbf{G}_v = \mathbf{G}_{E_v}$ be the Green's function of E_v relative for each place v. For every v we fix an embedding $\overline{K}$ into $\mathbb{C}_v$. Let $S \subset \overline{K}$ be any finite subset that is invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. We define the height $h_{\mathbf{E}}(S)$ of S relative to $\mathbf{E}$ by

$$h_{\mathbf{E}}(S) = \sum_{v} N_v \left(\frac{1}{|S|} \sum_{z \in S} \mathbf{G}_v(z) \right).$$

Berkovich adèlic sets (continued)

Let $\mathbf{G}_v = \mathbf{G}_{E_v}$ be the Green's function of E_v relative for each place v. For every v we fix an embedding $\overline{K}$ into $\mathbb{C}_v$. Let $S \subset \overline{K}$ be any finite subset that is invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. We define the height $h_{\mathbf{E}}(S)$ of S relative to $\mathbf{E}$ by

$$h_{\mathbf{E}}(S) = \sum_v N_v \left(\frac{1}{|S|} \sum_{z \in S} \mathbf{G}_v(z) \right).$$

If each E_v is the closed unit disk in $\mathbb{C}_v$, then the above definition reduces to the usual notion of the Weil height.

Berkovich adèlic sets (continued)

Let $\mathbf{G}_v = \mathbf{G}_{E_v}$ be the Green's function of E_v relative for each place v. For every v we fix an embedding $\overline{K}$ into $\mathbb{C}_v$. Let $S \subset \overline{K}$ be any finite subset that is invariant under the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$. We define the height $h_{\mathbf{E}}(S)$ of S relative to $\mathbf{E}$ by

$$h_{\mathbf{E}}(S) = \sum_v N_v \left(\frac{1}{|S|} \sum_{z \in S} \mathbf{G}_v(z) \right).$$

If each E_v is the closed unit disk in $\mathbb{C}_v$, then the above definition reduces to the usual notion of the Weil height.
Also, one can prove that the Berkovich adèlic set constructed with respect to all v-adic generalized Mandelbrot sets has capacity equal to 1.

The equidistribution statement

Theorem

(Baker, Rumely) Let $\mathbf{E}$ *be a Berkovich adelic set with* $\gamma(\mathbf{E}) = 1$*. Suppose that* S_n *is a sequence of* $\mathrm{Gal}(\overline{K}/K)$*-invariant finite subsets of* $\overline{K}$ *with* $|S_n| \to \infty$ *and* $h_{\mathbf{E}}(S_n) \to 0$ *as* $n \to \infty$*. For each place* v *and for each* n *let* δ_n *be the discrete probability measure supported equally on the elements of* S_n*. Then the sequence of measures* $\{\delta_n\}$ *converges weakly to* μ_v *the equilibrium measure on* E_v*.*

The equidistribution statement

Theorem

(Baker, Rumely) Let $\mathbf{E}$ *be a Berkovich adelic set with* $\gamma(\mathbf{E}) = 1$*. Suppose that* S_n *is a sequence of* $\mathrm{Gal}(\overline{K}/K)$*-invariant finite subsets of* $\overline{K}$ *with* $|S_n| \to \infty$ *and* $h_{\mathbf{E}}(S_n) \to 0$ *as* $n \to \infty$*. For each place* v *and for each* n *let* δ_n *be the discrete probability measure supported equally on the elements of* S_n*. Then the sequence of measures* $\{\delta_n\}$ *converges weakly to* μ_v *the equilibrium measure on* E_v*.*

The above equidistribution theorem allows us to finish the proof of our result.

Indeed, we construct the Berkovich adèlic sets $\mathbf{M_a} := \prod_v \mathbf{M}_{\mathbf{a},v}$ and $\mathbf{M_b} := \prod_v \mathbf{M}_{\mathbf{b},v}$. Then, assuming that there exist infinitely many λ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ we obtain $\mathrm{Gal}(\overline{K}/K)$-invariant finite subsets S_n of $\overline{K}$ with $|S_n| \to \infty$ for which both

$$h_{\mathbf{M_a}}(S_n) \to 0 \text{ and } h_{\mathbf{M_b}}(S_n) \to 0.$$

Indeed, we construct the Berkovich adèlic sets $\mathbf{M_a} := \prod_v \mathbf{M}_{\mathbf{a},v}$ and $\mathbf{M_b} := \prod_v \mathbf{M}_{\mathbf{b},v}$. Then, assuming that there exist infinitely many λ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ we obtain $\mathrm{Gal}(\overline{K}/K)$-invariant finite subsets S_n of $\overline{K}$ with $|S_n| \to \infty$ for which both

$$h_{\mathbf{M_a}}(S_n) \to 0 \text{ and } h_{\mathbf{M_b}}(S_n) \to 0.$$

Therefore, by the Baker-Rumely equidistribution theorem, $\mathbf{M}_{\mathbf{a},v} = \mathbf{M}_{\mathbf{b},v}$ for *each* place v.

Indeed, we construct the Berkovich adèlic sets $\mathbf{M_a} := \prod_v \mathbf{M}_{\mathbf{a},v}$ and $\mathbf{M_b} := \prod_v \mathbf{M}_{\mathbf{b},v}$. Then, assuming that there exist infinitely many λ such that both $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ are preperiodic for f_λ we obtain $\mathrm{Gal}(\overline{K}/K)$-invariant finite subsets S_n of $\overline{K}$ with $|S_n| \to \infty$ for which both

$$h_{\mathbf{M_a}}(S_n) \to 0 \text{ and } h_{\mathbf{M_b}}(S_n) \to 0.$$

Therefore, by the Baker-Rumely equidistribution theorem, $\mathbf{M}_{\mathbf{a},v} = \mathbf{M}_{\mathbf{b},v}$ for *each* place v. Then for each place v, using the fact that $\mathbf{M}_{\mathbf{a},v}$ and $\mathbf{M}_{\mathbf{b},v}$ share the same Green's function, we conclude that

$$\frac{\widehat{h}_\lambda(\mathbf{a}(\lambda))}{\deg(\mathbf{a})} = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{a}(\lambda))|_v}{\deg(\mathbf{a})d^n} = \lim_{n\to\infty} \frac{\log^+ |f_\lambda^n(\mathbf{b}(\lambda))|_v}{\deg(\mathbf{b})d^n} = \frac{\widehat{h}_\lambda(\mathbf{b}(\lambda))}{\deg(\mathbf{b})}.$$