Quantitative equidistribution in non-archimedean and complex dynamics

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§Berkovich projective line: Notation

K: *algebraically closed* field, complete WRT a non-trivial absolute value $|\cdot|$.

(either non-archimedean or archimedean. e.g. $\mathbb{C}_p, \mathbb{C}_v, \mathbb{C}$)

$\mathbb{P}^1 = \mathbb{P}^1(K)$: (classical) projective line

$[\cdot, \cdot]$: the normalized chordal distance on $\mathbb{P}^1$

$\mathsf{P}^1 = \mathsf{P}^1(K)$: Berkovich projective line, compactifying $\mathbb{P}^1$

(Fact: For archimedean K, $\mathsf{P}^1 \cong \mathbb{P}^1$)

$\mathsf{H}^1 := \mathsf{P}^1 \setminus \mathbb{P}^1$: endowed with the hyperbolic metric ρ

$\delta(\cdot, \cdot)_{\mathrm{can}}$: the generalized Hsia kernel on P^1 WRT $\mathcal{S}_{\mathrm{can}} \in \mathsf{H}^1$.

§Gauss variational approach to dynamics

A rational function of degree $d > 1$

$$f : \mathbb{P}^1 \to \mathbb{P}^1.$$

(Fact: this extends to $\mathsf{P}^1 \to \mathsf{P}^1$, $f(\mathsf{H}^1) = \mathsf{H}^1$, conti, surj, open, discrete)

$\exists 1$(non-degenerate homogeneous polynomial) lift of f

$$F : K^2 \to K^2$$

(upto $\times c \in K^*$), i.e. for canonical projection $\pi : K^2 \to \mathbb{P}^1(K)$,

$$\pi \circ F = f \circ \pi$$

and the homogeneous resultant $\mathrm{Res}\, F$ does not vanish.

Def . The dynamical Green function on $\mathbf{P}^1$

$$g_F := \sum_{j=0}^{\infty} \frac{1}{d^j} (f^j)^* \left(\frac{1}{d} \log |F| - \log | \cdot | \right)$$

(for $\forall c \in K^*$, $g_{cF} = g_F + (\log |c|)/(d-1)$).

An upper semicontinuous F-**kernel** on $\mathbf{P}^1$

$$\Phi_F(z, w) := \log \delta(z, w)_{\mathrm{can}} - g_F(z) - g_F(w).$$

The F-energy of a Radon measure μ on $\mathbf{P}^1$ (if exists)

$$I_F(\mu) := \int_{\mathbf{P}^1 \times \mathbf{P}^1} \Phi_F(z, w) \mathrm{d}(\mu \times \mu)(z, w).$$

The F-**equilibrium energy** of (the whole) $\mathbf{P}^1$

$$V_F := \sup\{I_F(\mu); \mu \text{ is a prob. Radon measure on } \mathbf{P}^1\} > -\infty.$$

A possible definition of the **canonical measure** μ_f is

Thm . *There is the unique solution of Gauss variational problem WRT external field* g_F.

Concretely, $\exists 1$ *probability Radon measure* μ_f *on* $\mathbf{P}^1$ *s.t.*

$$I_F(\mu_f) = V_F.$$

(Rem: μ_f is independent of choices of F)

§Fekete configuration in dynamics

Now we can be more canonical: the f-kernel on $\mathbf{P}^1$

$$\Phi_f(\cdot,\cdot) := \Phi_F(\cdot,\cdot) - V_F,$$

independent of choices of F. (Rem: $-\Phi_f$ is called the Arakelov Green (kernel) function of f on $\mathbf{P}^1$)

Def. A sequence (ν_n) of positive discrete measures on $\mathbf{P}^1$ is **f-asymptotically Fekete on $\mathbf{P}^1$** if as $n \to \infty$,

$$\nu_n(\mathbf{P}^1) \nearrow \infty, \quad (\nu_n \times \nu_n)(\mathrm{diag}_{\mathbb{P}^1}) = o(\nu_n(\mathbf{P}^1)^2),$$
$$\frac{1}{\nu_n(\mathbf{P}^1)^2} \int_{\mathbf{P}^1 \times \mathbf{P}^1 \setminus \mathrm{diag}_{\mathbb{P}^1}} \Phi_f \mathrm{d}(\nu_n \times \nu_n) \to 0.$$

(Rem: this is an analogue of Gauss variational problem for positive discrete measures. $\Phi_f(\mathcal{S}, \mathcal{S}) > 0$ if $\mathcal{S} \in \mathsf{H}^1$.)

Def. The averaged pullback of $a \in \mathsf{P}^1$

$$(f^n)^*(a) := \sum_{w \in f^{-n}(a)} \deg_w(f^n) \cdot (a)$$

((a): the Dirac measure at a on P^1).

The algebraic exceptional set of f (Rem: this is in $\mathbb{P}^1$)

$$E(f) := \{a \in \mathbb{P}^1; \# \bigcup_{n \in \mathbb{N}} f^{-n}(a) < \infty\}.$$

$SAT(f)$: superattracting periodic points of f

Def (main quantity). For each $a \in \mathbf{P}^1$ and each $n \in \mathbb{N}$,

$$\mathcal{E}_f(n,a) := \frac{1}{d^{2n}} \int_{\mathbf{P}^1 \times \mathbf{P}^1 \setminus \mathrm{diag}_{\mathbb{P}}^1} \Phi_f \, d((f^n)^*(a) \times (f^n)^*(a))$$

$$= -\left(\frac{(f^n)^*(a)}{d^n} - \mu_f, \frac{(f^n)^*(a)}{d^n} - \mu_f \right)_f$$

(: the dyn version of Favre and Rivera-Letelier's energy).

(Fact) Then

- For $\forall a \in \mathbf{P}^1 \setminus E(f)$,

  $((f^n)^*(a))$ is f-asymp Fekete on $\mathbf{P}^1 \Leftrightarrow \lim_{n\to\infty} \mathcal{E}_f(n,a) = 0$.

- For $\forall a \in E(f)$, $((f^n)^*(a))$ is NEVER f-asymp Fekete on $\mathbf{P}^1$.

Another fundamental quantity

Def. For $\forall a \in \mathbf{P}^1 \setminus E(f)$,

$$\eta_{a,n} = \eta_{a,n}(f) := \max_{w \in f^{-n}(a)} \deg_w(f) \in \mathbb{N}.$$

Rem: if K has characteristic 0, then

$$\limsup_{j\to\infty} \eta_{a,j}^{1/j} \begin{cases} \le (d^3 - 1)^{1/3} & (a \in \mathbb{P}^1 \setminus E(f)), \\ = d & (a \in E(f)), \end{cases} \tag{1}$$

$$\sup_{j \in \mathbb{N}} \eta_{a,j} \begin{cases} \le d^{2d-2} & (a \in \mathbb{P}^1 \setminus SAT(f)), \\ = \infty & (a \in SAT(f)). \end{cases} \tag{2}$$

§Main results: error estimates on Fekete

Let f be a rat function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$. Put

$$C(f) := \{c \in \mathbb{P}^1; f'(c) = 0\},$$

$$C(f)_{\text{wan}} := \{c \in C(f); (f^n(c)) \text{ is } \textbf{wandering} \text{ under } f\},$$

$$\mathrm{CO}(f)_{\text{wan}} := \{f^n(c); c \in C(f)_{\text{wan}}, n \in \mathbb{N}\}.$$

(Rem: if f has char 0, then $\sum_{c \in C(f)}(\deg_c f - 1) = 2d - 2$.)

Thm 1 (principal estimates). *For $\forall a \in \mathsf{H}^1$ and $\forall n \in \mathbb{N}$,*

$$|\mathcal{E}_f(n, a)| \leq C d^{-n} \tag{3}$$

for some $C > 0$ indep of n and loc bounded on a under ρ.

(cont.)

If in addition K *has char* 0, *then there is* $C' > 0$ *s.t. for* $\forall a \in \mathbb{P}^1$ *and* $\forall n \in \mathbb{N}$,

$$-\frac{1}{d^n}\sum_{j=1}^{n}\eta_{a,j}\sum_{c\in C(f)\setminus f^{-j}(a)}\frac{1}{d^j}\log\frac{1}{[f^j(c),a]}-\frac{C'}{d^n}\sum_{j=1}^{n}\eta_{a,j}-\frac{C_a}{d^n}$$

$$\leq \mathcal{E}_f(n,a) \tag{4}$$

$$\leq -\frac{1}{d^n}\sum_{j=1}^{n}\sum_{c\in C(f)\setminus f^{-j}(a)}\frac{1}{d^j}\log\frac{1}{[f^j(c),a]}+\frac{C'}{d^n}\sum_{j=1}^{n}\eta_{a,j}+\frac{C_a}{d^n}.$$

Here the constant $C_a \geq 0$, *which is independent of* n, *vanishes if* $a \in \mathbb{P}^1 \setminus \mathrm{CO}(f)_{\mathrm{wan}}$.

Def. The classical omega limit set of each $z_0 \in \mathbb{P}^1$

$$\omega(z_0) = \omega(z_0) := \bigcap_{N \in \mathbb{N}} \overline{\{f^n(z_0); n \geq N\}}^{\text{chordal}}.$$

A point $z_0 \in \mathbb{P}^1$ is **pre-recurrent** if $\exists n_0 \in \mathbb{N}$, $f^{n_0}(z_0) \in \omega(z_0)$.

(The chordal open ball with center $w \in \mathbb{P}^1$ and radius $r > 0$

$$B[w, r] := \{z \in \mathbb{P}^1; [z, w] < r\})$$

Thm 1 estimates **the non-Fekete locus**

$E_{\text{Fekete}}(f) := \{a \in$ **P**^{1}; $((f^n)^*(a))$ is *not* f-asymp Fekete on **P**$^1\}$

from above using

$$E_{\mathrm{wan}}(f) := \bigcap_{N\in\mathbb{N}} \bigcup_{j\geq N} \bigcup_{c\in C(f)_{\mathrm{wan}}} B[f^j(c), \exp(-d^j)].$$

Thm **2.** *Suppose K has characteristic 0. Then*

$$E(f) \subset E_{\mathrm{Fekete}}(f) \subset \mathbb{P}^1,$$

$$E_{\mathrm{Fekete}}(f) \setminus E(f) \subset E_{\mathrm{wan}}(f) \setminus E(f),$$

and $E_{\mathrm{wan}}(f)$ is of capacity 0. (finite Hyllengren meas for (d^j)).

Moreover, $E_{\mathrm{Fekete}}(f)$ is G_δ-dense in $\omega(c)$ for every pre-recurrent $c \in C(f)_{\mathrm{wan}}$.

(so, possibly $E(f) \subsetneq E_{\mathrm{Fekete}}(f)$)

§Application: quantitative equidistribution/K

Let f be a rational function on $\mathbb{P}^1 = \mathbb{P}^1(K)$ of degree $d > 1$.

Favre and Rivera-Letelier's **Cauchy-Schwarz** inequality is

Prop. For $\forall a \in \mathsf{P}^1$, C^1-test function $\forall \phi$ on P^1 and $\forall n \in \mathbb{N}$,

$$\left|\left\langle \phi, \frac{(f^n)^*(a)}{d^n} - \mu_f \right\rangle\right|$$
$$\leq \begin{cases} \langle \phi, \phi \rangle^{1/2} \sqrt{|\mathcal{E}_f(n,a)|} & (a \in \mathsf{H}^1), \\ C \max\{\mathrm{Lip}(\phi), \langle \phi, \phi \rangle^{1/2}\} \sqrt{|\mathcal{E}_f(n,a)| + nd^{-n}\eta_{a,n}} & (a \in \mathbb{P}^1). \end{cases}$$

Here $C > 0$ is independent of $a \in \mathbb{P}^1$, ϕ and n.

Theorem 1 establishes **a quantitative equidistribution in terms of the *proximity* of wandering crit orbits to $a \in \mathbb{P}^1$.**

Thm 3 (Special case)**.** *Suppose that K has characteristic 0. Then there is $C > 0$ s.t. for $\forall a \in \mathbb{P}^1$ excluding $E_{\mathrm{wan}}(f)$ of capacity 0 and $\forall n \in \mathbb{N}$ large enough,*

$$|\mathcal{E}_f(n,a)| \le Cnd^{-n}\eta_{a,n}, \tag{5}$$

and there is $C' > 0$ s.t. for C^1-test function $\forall \phi$ on P^1*, $\forall a \in \mathbb{P}^1 \setminus E_{\mathrm{wan}}(f)$ and $\forall n \in \mathbb{N}$ large enough,*

$$\left|\left\langle \phi, \frac{(f^n)^*(a)}{d^n} - \mu_f \right\rangle\right| \le C' \max\{\mathrm{Lip}(\phi), \langle \phi, \phi \rangle^{1/2}\} \sqrt{nd^{-n}\eta_{a,n}}$$

(*Recall that $\sup_{n\in\mathbb{N}} \eta_{a,n} \le d^{2d-2}$ if in addition $a \notin SAT(f)$*).

(cont.)

On the other hand, for $\forall a_0 \in \mathbb{P}^1$ *excluding*

$$\mathbf{PC}(f) := \overline{\{f^n(c); c \in C(f), n \in \mathbb{N}\}}^{\text{chordal}},$$

there are $r_0 > 0$ *and* $N = N(a_0)$ *s.t. for* $\forall a \in B[a_0, r_0]$, C^1*-test function* $\forall \phi$ *on* $\mathbf{P}^1$, $\forall k > N$, *the same (but locally uniform) estimate*

$$\left|\left\langle \phi, \frac{(f^n)^*(a)}{d^n} - \mu_f \right\rangle\right| \leq C' \max\{\mathrm{Lip}(\phi), \langle \phi, \phi \rangle^{1/2}\} \sqrt{nd^{-n}}.$$

Rem . For $K \cong \mathbb{C}$, the better $O(\sqrt{d^{-n}})$ estimate holds for $\forall a \in \mathbb{P}^1$ at which f is semihyp. (cf. D. Drasin and Ok, *BLMS 2007*).

§Arithmetic application/global fields

For a number field or a function field k, when f has its coefficients in k, the dynamics on **algebraic** points

$$f : \mathbb{P}^1(\overline{k}) \to \mathbb{P}^1(\overline{k}),$$

is also interesting.

Fix a non-trivial absolute value v on k, and set $K = \mathbb{C}_v$.

The **dynamical Diophantine approximation** (Silverman 1993, Szpiro and Tucker 2005):

For $\forall a \in \mathbb{P}^1(\overline{k}) \setminus E(f)$ *and wandering* $\forall z \in \mathbb{P}^1(\overline{k})$,

$$\lim_{n\to\infty} \frac{1}{d^n} \log[f^n(z), a]_v = 0.$$

Since $(E(f) \subset SAT(f) \subset) C(f) \subset \mathbb{P}^1(\overline{k})$, consequently

$$E_{\mathrm{wan}}(f)_v \cap \mathbb{P}^1(\overline{k}) \subset E(f), \tag{6}$$

and **Theorem 3 recovers (in a purely local manner)** Favre and Rivera-Letelier's arithmetic quantitative equidistribution:

Under the above arithmetic setting, *there is* $C > 0$ *s.t. for* $\forall a \in \mathbb{P}^1(\overline{k}) \setminus E(f)$*,* C^1*-test function* $\forall \phi$ *on* $\mathsf{P}^1(\mathbb{C}_v)$ *and* $\forall n \in \mathbb{N}$ *large enough,*

$$\left|\left\langle \phi, \frac{(f^n)^*(a)}{d^n} - \mu_{f,v} \right\rangle\right| \le C' \max\{\mathrm{Lip}(\phi), \langle \phi, \phi \rangle^{1/2}\} \sqrt{n d^{-n} \eta_{a,n}}.$$

Rem. By Thm 2 with (6), also $E_{\mathrm{Fekete}}(f)_v \cap \mathbb{P}^1(\overline{k}) = E(f)$.

§In complex dynamics/$\mathbb{C}$

Let f be a rational function on $\mathbb{P}^1(\mathbb{C})$ of degree > 1.

Q. When $E(f) = E_{\mathrm{Fekete}}(f)$?

Cor **1.** *If ∃ Cremer periodic points, Siegel disks or Herman rings of f, then $E(f) \subsetneq E_{\mathrm{Fekete}}(f)$.*

If f is geometrically finite, then $E_{\mathrm{Fekete}}(f) = E(f)$.

Rem. ∃ semihyperbolic real cubic polynomial f such that $E_{\mathrm{Fekete}}(f) \cap \mathcal{J}(f) \neq \emptyset$, so $E(f) \subsetneq E_{\mathrm{Fekete}}(f)$.