Equidistribution and the Dynamical Uniform Boundedness Conjecture

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ICERM Workshop on Global Arithmetic Dynamics

Monday, March 19, 2012

Dynamics on $\mathbb{P}^1$

Let K be a field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

[$\deg \phi := \max\{\deg f_1, \deg f_2\}$, where $\phi = f_1/f_2$ in lowest terms.]

Write $\phi^n := \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{n \text{ times}}$

Definition

A point $z \in \mathbb{P}^1(\overline{K}) = \overline{K} \cup \{\infty\}$ is called **preperiodic** if

$$\phi^n(z) = \phi^m(z) \quad \text{for some } n > m \geq 0.$$

Write $\text{Preper}(\phi, K) := \{z \in \mathbb{P}^1(K) : z \text{ is preperiodic under } \phi\}$.

The Dynamical Uniform Boundedness Conjecture

Let K be a global field.

Theorem (Northcott, 1950)

Let $\phi \in K(z)$ of degree $d \geq 2$. Then

$$\#Preper(\phi, K) < \infty.$$

Conjecture (Morton & Silverman, 1994)

For any integer $d \geq 2$, there is a constant $C = C(d, K)$ such that for any $\phi \in K(z)$ of degree d,

$$\#Preper(\phi, K) \leq C(d, K).$$

Quadratic Polynomial Records, $K = \mathbb{Q}$ (Morton, Poonen)

$$\phi(z) = z^2 - \frac{651}{100}. \qquad \infty \to \infty$$

$$\frac{21}{10} \to -\frac{21}{10} \to -\frac{21}{10} \qquad\qquad -\frac{31}{10} \to \frac{31}{10} \to \frac{31}{10}$$

$$-\frac{19}{10} \to -\frac{29}{10} \leftrightarrows \frac{19}{10} \leftarrow \frac{29}{10}$$

$$\phi(z) = z^2 - \frac{29}{16}. \qquad \infty \to \infty$$

$$\begin{array}{ccccccc} & & -\frac{1}{4} & \longrightarrow & -\frac{7}{4} & \longrightarrow & \frac{5}{4} & \longrightarrow & -\frac{1}{4} \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \pm\frac{3}{4} & \longrightarrow & -\frac{5}{4} & & \frac{1}{4} & & \frac{7}{4} & & \end{array}$$

Cubic Polynomial Records, $K = \mathbb{Q}$

$$\phi(z) = -\frac{3}{2}z^3 + \frac{19}{6}z. \qquad \infty \to \infty \qquad 0 \to 0$$

$$\begin{array}{ccccccccccc}
\frac{1}{3} & \to & 1 & \to & \frac{5}{3} & \leftrightarrows & -\frac{5}{3} & \leftarrow & -1 & \leftarrow & -\frac{1}{3} \\
 & & & & \uparrow & & \uparrow & & & & \\
 & & \frac{4}{3} & \to & \frac{2}{3} & & -\frac{2}{3} & \leftarrow & -\frac{4}{3} & &
\end{array}$$

$$\phi(z) = -\frac{3}{2}z^3 + \frac{73}{24}z. \qquad \infty \to \infty \qquad 0 \to 0$$

$$\frac{7}{6} \to \frac{7}{6} \qquad -\frac{7}{6} \to -\frac{7}{6}$$

$$\begin{array}{ccccccccccc}
-\frac{3}{2} & \to & \frac{1}{2} & \leftrightarrows & \frac{4}{3} & & \frac{3}{2} & \to & -\frac{1}{2} & \leftrightarrows & -\frac{4}{3} \\
 & & \uparrow & & & & & & \uparrow & & \\
 & & \frac{1}{6} & & & & & & -\frac{1}{6} & &
\end{array}$$

Lower Bounds for Canonical Heights

The canonical height of $P \in \mathbb{P}^1(K)$ is

$$\hat{h}_\phi(P) = \lim_{n\to\infty} \frac{1}{d^n} h(\phi^n(P)),$$

where

$$h(a/b) = \log\max\{|a|, |b|\}.$$

Conjecture (Silverman)

Let K be a number field and $d \geq 2$.
There is a constant $C = C(K, d)$ such that for any $\phi \in K(z)$ with $\deg\phi = d$, and for any non-preperiodic point $P \in \mathbb{P}^1(K)$,

$$\hat{h}_\phi(P) \geq Ch(\phi).$$

A Quadratic Polynomial Example of a Small Point

$$\phi(z) = z^2 - \frac{181}{144}$$

Not small height: $0 \mapsto \frac{-181}{144} \mapsto \frac{6697}{20736} \mapsto -\frac{495613295}{429981696} \mapsto \cdots$

Small height:

$$\frac{7}{12} \mapsto -\frac{11}{12} \quad \mapsto -\frac{5}{12} \quad \mapsto -\frac{13}{12} \quad \mapsto -\frac{1}{12}$$

$$\mapsto -\frac{5}{4} \quad \mapsto \frac{11}{36} \quad \mapsto -\frac{377}{324} \quad \mapsto \cdots$$

$\hat{h}_\phi(7/12) = 2^{-5}\log 3 = 0.03433\ldots$, vs.
$h(\phi) = h(181/144) = \log 181 = 5.198\ldots$
Ratio is $\hat{h}_\phi(7/12)/h(\phi) = 0.00660\ldots$

Another Example

$$\phi(z) = z^2 - \frac{931161001}{476985600} \qquad [476985600 = (2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13)^2]$$

Small height:

$$\frac{30379}{21840} \mapsto -\frac{379}{21840} \mapsto -\frac{42629}{21840} \mapsto \frac{40571}{21840} \mapsto \frac{32731}{21840}$$

$$\mapsto \frac{27809}{94640} \mapsto -\frac{76737829}{41127840} \mapsto -\frac{2534854375585937}{1657669238604288 0} \mapsto \cdots$$

$\hat{h}_\phi(30379/21840) = 0.28548\ldots$,
Ratio is $\hat{h}_\phi(30379/21840)/h(\phi) = 0.013824\ldots$

Yet Another Example

$$\phi(z) = z^2 - \frac{930065581}{509495184} \qquad [509495184 = (2^2 \cdot 3^3 \cdot 11 \cdot 19)^2]$$

Small height:

$$\frac{24281}{22572} \mapsto -\frac{15085}{22572} \mapsto -\frac{31123}{22572} \mapsto \frac{1709}{22572} \mapsto -\frac{41075}{22572} \mapsto \frac{7010093}{4717548}$$

$$\mapsto \frac{78844529861}{206067214188} \mapsto -\frac{660180820067424604775}{393182377437065449068} \mapsto \cdots$$

$\hat{h}_\phi(24281/22572) = 0.34463\ldots$,
Ratio is $\hat{h}_\phi(24281/22572)/h(\phi) = 0.016688\ldots$

The Cubic Polynomial Record Holder over $\mathbb{Q}$: Small Height

$$\phi(z) = -\frac{25}{24}z^3 + \frac{97}{24}z + 1$$

$$-\frac{7}{5} \mapsto -\frac{9}{5} \mapsto -\frac{1}{5} \mapsto \frac{1}{5} \mapsto \frac{9}{5}$$

$$\mapsto \frac{11}{5} \mapsto -\frac{6}{5} \mapsto -\frac{41}{20} \mapsto \frac{4323}{2560} \mapsto \cdots$$

$\hat{h}_\phi(-7) = 0.0011\ldots$, vs.

$h(\phi) = \log(97) = 4.57\ldots$

Ratio is $\hat{h}_\phi(-7)/h(\phi) = 0.00025\ldots$

Strong Non-Uniform Bounds for Polynomials

Theorem (RB, 2004)

Let $\phi(z) \in K[z]$ be a polynomial of degree $d \geq 2$. Let s be the number of bad places of ϕ (including archimedean places). Then

$$\#Preper(\phi, K) \leq O_K\left(\frac{d^2}{\log d} \cdot s \log s\right).$$

Can we generalize this result to rational functions?

And to points of small canonical height?

The Berkovich Projective Line

Let $\mathbb{C}_v$ be a complete and algebraically closed non-archimedean field (like $\mathbb{C}_p$).

The classical projective line $\mathbb{P}^1(\mathbb{C}_v) = \mathbb{C}_v \cup \{\infty\}$ is non-compact and totally disconnected.

The **Berkovich projective line** $\mathbb{P}^1_{\mathrm{Berk}}$ is a path-connected compactification of $\mathbb{P}^1(\mathbb{C}_p)$.

The points in $\mathbb{P}^1_{\mathrm{Berk}}$ come in four flavors:

- Type 1: points of $\mathbb{P}^1(\mathbb{C}_v)$
- Type 2: one point $\zeta(a, r)$ for each closed disk $\overline{D}(a, r)$ with $r \in |\mathbb{C}_v^\times|$
- Type 3: one point $\zeta(a, r)$ for each closed disk $\overline{D}(a, r)$ with $r \notin |\mathbb{C}_v^\times|$
- Type 4: point corresponding to decreasing chains of disks $D_1 \supset D_2 \supset \cdots$ with empty intersection

Pullback Measures

The action of a (nonconstant) rational function $\phi \in \mathbb{C}_v(z)$ on $\mathbb{P}^1(\mathbb{C}_v)$ extends continuously to $\mathbb{P}^1_{\mathsf{Berk}}$.

Let $\mathcal{M}$ be the space of (finite real) signed Borel measures on $\mathbb{P}^1_{\mathsf{Berk}}$. Given $\mu \in \mathcal{M}$, the **pullback measure** $\phi^*(\mu)$ is the signed Borel measure such that

$$\int_{\mathbb{P}^1_{\mathsf{Berk}}} f(\zeta)\, d\big(\phi^*(\mu)\big)(\zeta) = \int_{\mathbb{P}^1_{\mathsf{Berk}}} \sum_{\xi \in \phi^{-1}(\zeta)} (\deg_\xi \phi) \cdot f(\xi)\, d\mu(\zeta).$$

- If μ is a probability measure, then so is $\dfrac{1}{\deg \phi}\phi^*(\mu)$.
- $(\phi \circ \psi)^*(\mu) = \psi^*(\phi^*(\mu))$.

The Laplacian

Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the set of signed Borel measures $\mu \in \mathcal{M}$ such that $\mu(\mathbb{P}^1_{\text{Berk}}) = 0$.

There is a certain space $\mathcal{P}$ of "nice enough" functions $f : \mathbb{P}^1_{\text{Berk}} \to [-\infty, \infty]$ and a linear **Laplacian** operator $\Delta : \mathcal{P} \to \mathcal{M}_0$, satisfying

- $\Delta f = 0$ iff f is constant.
- $\displaystyle\int_{\mathbb{P}^1_{\text{Berk}}} f \Delta g = \int_{\mathbb{P}^1_{\text{Berk}}} g \Delta f.$
- For each $\mu \in \mathcal{M}_0$ and each $\zeta \in \mathbb{P}^1_{\text{Berk}}$, there is an associated potential function $g_\mu \in \mathcal{P}$ such that $\Delta(g_\mu) = \mu$.
- $\phi^*(\Delta f) = \Delta(f \circ \phi)$.
- If $f_n \to f$ uniformly on $\mathbb{P}^1_{\text{Berk}}$ (and $|\Delta f_n|$ is uniformly bounded on $\mathbb{P}^1_{\text{Berk}}$), then $\Delta f_n \to \Delta f$ weakly.

The Invariant/Canonical/Equilibrium Measure

Theorem (Favre & Rivera-Letelier; Baker & Rumely; Autissier, Chambert-Loir, & Thuillier, mid 2000s)

Let $\phi \in \mathbb{C}_v(z)$ be a rational function of degree $d \geq 2$.
Then there is a unique probability measure $\mu = \mu_\phi$ on $\mathbb{P}^1_{\text{Berk}}$ such that:

- $\phi^*(\mu) = d \cdot \mu$, *and*
- $\mu(E_\phi) = 0$,

where $E_\phi \subseteq \mathbb{P}^1(\mathbb{C}_v)$ is the (type 1) exceptional set of ϕ.

FYI:
There is a point $\zeta \in \mathbb{P}^1_{\text{Berk}}$ for which $\mu(\{\zeta\}) > 0$ **if and only if** ζ is type 2, $\mu = \delta_\zeta$, and ϕ has potentially good reduction, attained by moving ζ to $\zeta(0, 1)$.

Definition of the Equilibrium Measure

Given $\phi \in \mathbb{C}_v(z)$ of degree $d \geq 2$, choose any $\zeta \in \mathbb{P}^1_{\text{Berk}}$ that is **not** type 1.

It's easy to write down an explicit bounded function $g \in \mathcal{P}$ such that $\Delta g = d^{-1}\phi^*\delta_\zeta - \delta_\zeta$. For each $n \geq 1$, set

$$\mu_n = d^{-n}(\phi^n)^*\delta_\zeta \qquad \text{and} \qquad f_n = \sum_{i=0}^{n-1} d^{-i} \cdot g \circ \phi^i.$$

Then
$$\begin{aligned}\Delta f_n &= \sum_{i=0}^{n-1} d^{-i}\Delta\left(g \circ \phi^i\right) = \sum_{i=0}^{n-1} d^{-i}(\phi^i)^*(\Delta g)\\ &= \sum_{i=0}^{n-1} \left[d^{-i-1}(\phi^{i+1})^*(\delta_\zeta) - d^{-i}(\phi^i)^*(\delta_\zeta)\right]\\ &= d^{-n}(\phi^n)^*(\delta_\zeta) - \delta_\zeta = \mu_n - \delta_\zeta\end{aligned}$$

On the other hand, $\{f_n\}$ converges uniformly to some function f. Thus, setting $\mu_\phi = \Delta f + \delta_\zeta$, we immediately obtain $\mu_n \to \mu_\phi$ weakly, and $\phi^*\mu_\phi = d \cdot \mu_\phi$.

The Energy Pairing

If $\mu, \nu \in \mathcal{M}_0$ are nice enough measures, with $\mu = \Delta f$ and $\nu = \Delta g$, then their **energy pairing** is

$$(\mu, \nu) = -\int_{\mathbb{P}^1_{\mathrm{Berk}}} f \, d\Delta(\nu) = -\int_{\mathbb{P}^1_{\mathrm{Berk}}} g \, d\Delta(\mu).$$

If furthermore $\mu(\mathbb{P}^1_{\mathrm{Berk}}) = 0$, then $(\mu, \mu) \geq 0$, with equality iff $\mu = 0$.

Idea: If $\mu, \nu \in \mathcal{M}$ are **probability** measures, then

$$(\mu - \nu, \mu - \nu) \geq 0$$

quantifies how different μ and ν are.

Warning: Those facts may fail if μ has delta-masses at type 1 points.

Energy, Canonical Heights, and Equidistribution

Let K be a **number** field, with set of places M_K.
Let $\phi \in K(z)$ with $\deg \phi \geq 2$.

For each $v \in M_K$, let $\mu_{\phi,v}$ be the v-adic invariant measure on $\mathbb{P}^1_{\mathrm{Ber},v}$.

Then for any $x \in \mathbb{P}^1(K)$, the canonical height of x satisfies

$$\hat{h}_\phi(x) = [K:\mathbb{Q}]^{-1} \sum_{v \in M_K} \frac{n_v}{2} (\delta_x - \mu_{\phi,v}, \delta_x - \mu_{\phi,v}).$$

More generally, for any nonempty finite $\mathrm{Gal}(\overline{K}/K)$-invariant set $X \subseteq \mathbb{P}^1(\overline{K})$, we have

$$\frac{1}{\#X} \sum_{x \in X} \hat{h}_\phi(x) = [K:\mathbb{Q}]^{-1} \sum_{v \in M_K} \frac{n_v}{2} (\nu_X - \mu_{\phi,v}, \nu_X - \mu_{\phi,v}),$$

where $\quad \nu_X = \dfrac{1}{\#X} \sum_{x \in X} \delta_x.$

Applying Equidistribution to Uniform Boundedness

If ϕ has a large set X of K-rational preperiodic points, or even of small height points, then $\hat{h}_\phi(X) := \frac{1}{\#X}\sum_{x\in X}\hat{h}_\phi(x)$ is zero, or at least small.

So let's aim for a contradiction when $N = \#X$ is large, by bounding each local term $(\nu_X - \mu_{\phi,v}, \nu_X - \mu_{\phi,v})$ from below.

That is, find a good lower bound for $(\nu_X - \mu_{\phi,v}, \nu_X - \mu_{\phi,v})$, where $\nu_X = \frac{1}{\#X}\sum_{x\in X}\delta_x$.

When ϕ is a monic polynomial and x and y are both preperiodic, that simplifies to finding a good lower bound for

$$\sum_{x\neq y\in X} -\log|x-y|_v.$$

Sketch of Proof of RB's 2004 Theorem

(For ease, we assume ϕ is monic and all places are non-archimedean.)

At each $v \in M_K$, given $z_1, \ldots, z_N$ preperiodic, prove **Lemma 1**:

$$-\sum_{i \neq j} \log |z_i - z_j|_v \geq (d-1)(N \log_d N) r_{\phi,v},$$

where $r_{\phi,v} = \inf\{(\delta_x - \mu_{\phi,v}, \delta_y - \mu_{\phi,v}) : x \neq y\} \leq 0.$

At the place $w \in M_K$ maximizing $r_{\phi,w}$, prove **Lemma 2**: we can partition $\mathbb{P}^1(K)$ into two pieces so that given $z_1, \ldots, z_N$ preperiodic and all in a single piece,

$$-\sum_{i \neq j} \log |z_i - z_j|_v \geq \left(-\frac{N^2}{d-1} + (d-1)(N \log_d N) \right) r_{\phi,w}$$

Finishing the Proof

Set $R_v := n_v r_{\phi,v}$.

Let $w \in M_K$ be the place at which $R_w < 0$ is most negative, and partition $\mathcal{K}_{\phi,w} = U \sqcup V$.

Given $z_1, \dots, z_N \in K$ preperiodic and lying in U (resp., V) at w,

$$0 = \sum_{v \in M_K} n_v \sum_{i \neq j} -\log |z_i - z_j|_v \geq \sum_{v \text{ bad}} n_v \sum_{i \neq j} -\log |z_i - z_j|_v$$

$$\geq -\frac{N^2}{d-1} R_w + \sum_{v \text{ bad}} (d-1)(N \log_d N) \log R_v$$

$$\geq \left[-\frac{N^2}{d-1} + s(d-1)(N \log_d N) \right] \log R_w.$$

If $N \geq O_d(s \log s)$, we get $0 > 0$. Contradiction!

Generalizing to Rational Functions

The same proof works, if we:

- Remove the restriction that $z_1, \ldots z_N$ are preperiodic,
- Replace $-\log|z_i - z_j|_v$ by $(\delta_{z_i} - \mu_{\phi,v}, \delta_{z_j} - \mu_{\phi,v})$,
- Somehow prove analogues of Lemmas 1 and 2.

For Lemma 1:

- I proved Lemma 1 for polynomials by rewriting the sum as $-\log|\det A|_v$, where A was an $N \times N$ Vandermonde matrix, and using the dynamics of ϕ to do some appropriate row reduction.

- Matt Baker [2005] proved an analogue for rational functions using a similar Vandermonde strategy.

- Juan Rivera-Letelier [2011] proved an analogue for rational functions by a completely different strategy, involving smoothing certain discontinuous potential functions.

Lemma 2 for Rational Functions

A point $\zeta \in \mathbb{P}^1_{\text{Berk}}$ is a **barycenter** if $(\delta_\zeta - \mu_\phi, \delta_\zeta - \mu_\phi) \geq 0$ is minimized.

Equivalently, no connected component U of $\mathbb{P}^1_{\text{Berk}} \setminus \{\zeta\}$ has $\mu_\phi(U) > 1/2$.

In ongoing joint work with Juan Rivera-Letelier, we are proving that Lemma 2 will work with U being such a component of maximal mass $\mu_\phi(U)$, and with V being $\mathbb{P}^1_{\text{Berk}} \setminus U$.