Equidistribution and the Dynamical Uniform Boundedness Conjecture

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Dynamics on $\mathbb{P}^1$

Let $K$ be a field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$.

$$\deg \phi := \max\{\deg f_1, \deg f_2\}, \text{ where } \phi = f_1/f_2 \text{ in lowest terms}.$$ 

Write $\phi^n := \phi \circ \phi \circ \cdots \circ \phi$ \hspace{1cm} $n$ times

Definition

A point $z \in \mathbb{P}^1(K) = \overline{K} \cup \{\infty\}$ is called preperiodic if

$$\phi^n(z) = \phi^m(z) \text{ for some } n > m \geq 0.$$ 

Write $\text{Preper}(\phi, K) := \{z \in \mathbb{P}^1(K) : z \text{ is preperiodic under } \phi\}$. 
The Dynamical Uniform Boundedness Conjecture

Let $K$ be a global field.

Theorem (Northcott, 1950)
Let $\phi \in K(z)$ of degree $d \geq 2$. Then

$$\# \text{Preper}(\phi, K) < \infty.$$  

Conjecture (Morton & Silverman, 1994)
For any integer $d \geq 2$, there is a constant $C = C(d, K)$ such that for any $\phi \in K(z)$ of degree $d$,

$$\# \text{Preper}(\phi, K) \leq C(d, K).$$
Quadratic Polynomial Records, \( K = \mathbb{Q} \) (Morton, Poonen)

\[ \phi(z) = z^2 - \frac{651}{100}. \]
\[ \infty \rightarrow \infty \]

\[ \frac{21}{10} \rightarrow -\frac{21}{10} \rightarrow -\frac{21}{10} \rightarrow -\frac{31}{10} \rightarrow \frac{31}{10} \rightarrow \frac{31}{10} \]

\[ -\frac{19}{10} \rightarrow -\frac{29}{10} \leftrightarrow \frac{19}{10} \leftrightarrow \frac{29}{10} \]

\[ \phi(z) = z^2 - \frac{29}{16}. \]
\[ \infty \rightarrow \infty \]

\[ -\frac{1}{4} \rightarrow -\frac{7}{4} \rightarrow \frac{5}{4} \rightarrow -\frac{1}{4} \]

\[ \pm\frac{3}{4} \rightarrow -\frac{5}{4} \rightarrow \frac{1}{4} \rightarrow \frac{7}{4} \]
Cubic Polynomial Records, $K = \mathbb{Q}$

\[ \phi(z) = -\frac{3}{2}z^3 + \frac{19}{6}z. \quad \infty \rightarrow \infty \quad 0 \rightarrow 0 \]

\[ \frac{1}{3} \rightarrow 1 \rightarrow \frac{5}{3} \quad \leftrightarrow \quad -\frac{5}{3} \quad \leftrightarrow \quad -1 \leftrightarrow \quad -\frac{1}{3} \]

\[ \frac{4}{3} \rightarrow \frac{2}{3} \quad \leftrightarrow \quad -\frac{2}{3} \quad \leftrightarrow \quad -\frac{4}{3} \]

\[ \phi(z) = -\frac{3}{2}z^3 + \frac{73}{24}z. \quad \infty \rightarrow \infty \quad 0 \rightarrow 0 \]

\[ \frac{7}{6} \rightarrow \frac{7}{6} \quad \leftrightarrow \quad -\frac{7}{6} \rightarrow -\frac{7}{6} \]

\[ -\frac{3}{2} \rightarrow \frac{1}{2} \leftrightarrow \frac{4}{3} \quad \frac{3}{2} \rightarrow -\frac{1}{2} \leftrightarrow -\frac{4}{3} \]

\[ \frac{1}{6} \quad \frac{1}{6} \]
Lower Bounds for Canonical Heights

The canonical height of $P \in \mathbb{P}^1(K)$ is

$$\hat{h}_\phi(P) = \lim_{n \to \infty} \frac{1}{d^n} h(\phi^n(P)),$$

where

$$h(a/b) = \log \max\{|a|, |b|\}.$$

Conjecture (Silverman)

Let $K$ be a number field and $d \geq 2$.
There is a constant $C = C(K, d)$ such that for any $\phi \in K(z)$ with $\deg \phi = d$, and for any non-preperiodic point $P \in \mathbb{P}^1(K)$,

$$\hat{h}_\phi(P) \geq Ch(\phi).$$
A Quadratic Polynomial Example of a Small Point

\[ \phi(z) = z^2 - \frac{181}{144} \]

Not small height: 0 \[\mapsto\] \(-\frac{181}{144}\) \[\mapsto\] \(\frac{6697}{20736}\) \[\mapsto\] \(-\frac{495613295}{429981696}\) \[\mapsto\] \[
\]
Small height:

\[ \frac{7}{12} \mapsto -\frac{11}{12} \mapsto -\frac{5}{12} \mapsto -\frac{13}{12} \mapsto -\frac{1}{12} \]

\[ \mapsto -\frac{5}{4} \mapsto \frac{11}{36} \mapsto -\frac{377}{324} \mapsto \cdots \]

\[ \hat{h}_\phi(7/12) = 2^{-5} \log 3 = 0.03433 \ldots, \text{ vs.} \]

\[ h(\phi) = h(181/144) = \log 181 = 5.198 \ldots \]

Ratio is \[\hat{h}_\phi(7/12)/h(\phi) = 0.00660 \ldots \]
Another Example

\[ \phi(z) = z^2 - \frac{931161001}{476985600} \quad [476985600 = (2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 13)^2] \]

Small height:

\[
\begin{align*}
\frac{30379}{21840} & \mapsto -\frac{379}{21840} \mapsto -\frac{42629}{21840} \mapsto \frac{40571}{21840} \mapsto \frac{32731}{21840} \\
\frac{27809}{94640} & \mapsto -\frac{76737829}{41127840} \mapsto -\frac{25348543755859937}{16576692386042880} \mapsto \ldots
\end{align*}
\]

\[ \hat{h}_\phi(30379/21840) = 0.28548 \ldots, \]

Ratio is \( \hat{h}_\phi(30379/21840)/h(\phi) = 0.013824 \ldots \)
Yet Another Example

\[ \phi(z) = z^2 - \frac{930065581}{509495184} \]

\[ [509495184 = (2^2 \cdot 3^3 \cdot 11 \cdot 19)^2] \]

Small height:

\[
\begin{align*}
\frac{24281}{22572} & \mapsto -\frac{15085}{22572} \mapsto -\frac{31123}{22572} \mapsto -\frac{1709}{22572} \mapsto -\frac{41075}{22572} \mapsto \frac{7010093}{4717548} \\
\mapsto -\frac{78844529861}{206067214188} & \mapsto -\frac{6601808200674264604775}{393182377437065449068} \mapsto \ldots
\end{align*}
\]

\[ \hat{h}_\phi(24281/22572) = 0.34463 \ldots, \]

Ratio is \( \hat{h}_\phi(24281/22572)/h(\phi) = 0.016688 \ldots \)
The Cubic Polynomial Record Holder over $\mathbb{Q}$: Small Height

$$\phi(z) = -\frac{25}{24}z^3 + \frac{97}{24}z + 1$$

$$\begin{align*}
-\frac{7}{5} & \mapsto -\frac{9}{5} & \mapsto -\frac{1}{5} & \mapsto \frac{1}{5} & \mapsto \frac{9}{5} \\
\mapsto \frac{11}{5} & \mapsto -\frac{6}{5} & \mapsto -\frac{41}{20} & \mapsto \frac{4323}{2560} & \mapsto \ldots
\end{align*}$$

$$\hat{h}_\phi(-7) = 0.0011\ldots, \text{ vs.}$$

$$h(\phi) = \log(97) = 4.57\ldots$$

Ratio is $$\hat{h}_\phi(-7)/h(\phi) = 0.00025\ldots$$
Theorem (RB, 2004)

Let \( \phi(z) \in K[z] \) be a polynomial of degree \( d \geq 2 \). Let \( s \) be the number of bad places of \( \phi \) (including archimedean places). Then

\[
\#\text{Preper}(\phi, K) \leq O_K \left( \frac{d^2}{\log d} \cdot s \log s \right).
\]

Can we generalize this result to rational functions?

And to points of small canonical height?
The Berkovich Projective Line

Let $\mathbb{C}_v$ be a complete and algebraically closed non-archimedean field (like $\mathbb{C}_p$).

The classical projective line $\mathbb{P}^1(\mathbb{C}_v) = \mathbb{C}_v \cup \{\infty\}$ is non-compact and totally disconnected.

The **Berkovich projective line** $\mathbb{P}^1_{\text{Berk}}$ is a path-connected compactification of $\mathbb{P}^1(\mathbb{C}_p)$.

The points in $\mathbb{P}^1_{\text{Berk}}$ come in four flavors:

- **Type 1**: points of $\mathbb{P}^1(\mathbb{C}_v)$
- **Type 2**: one point $\zeta(a, r)$ for each closed disk $\overline{D}(a, r)$ with $r \in |\mathbb{C}_v^\times|$
- **Type 3**: one point $\zeta(a, r)$ for each closed disk $\overline{D}(a, r)$ with $r \notin |\mathbb{C}_v^\times|$
- **Type 4**: point corresponding to decreasing chains of disks $D_1 \supset D_2 \supset \cdots$ with empty intersection
Pullback Measures

The action of a (nonconstant) rational function \( \phi \in \mathbb{C}_v(z) \) on \( \mathbb{P}^1(\mathbb{C}_v) \) extends continuously to \( \mathbb{P}^1_{\text{Berk}} \).

Let \( \mathcal{M} \) be the space of (finite real) signed Borel measures on \( \mathbb{P}^1_{\text{Berk}} \). Given \( \mu \in \mathcal{M} \), the **pullback measure** \( \phi^*(\mu) \) is the signed Borel measure such that

\[
\int_{\mathbb{P}^1_{\text{Berk}}} f(\zeta) \, d(\phi^*(\mu))(\zeta) = \int_{\mathbb{P}^1_{\text{Berk}}} \sum_{\xi \in \phi^{-1}(\zeta)} (\deg_\xi \phi) \cdot f(\xi) \, d\mu(\zeta).
\]

- If \( \mu \) is a probability measure, then so is \( \frac{1}{\deg \phi} \phi^*(\mu) \).
- \( (\phi \circ \psi)^*(\mu) = \psi^*(\phi^*(\mu)) \).
The Laplacian

Let $\mathcal{M}_0 \subseteq \mathcal{M}$ be the set of signed Borel measures $\mu \in \mathcal{M}$ such that $\mu(\mathbb{P}^1_{\text{Berk}}) = 0$.

There is a certain space $\mathcal{P}$ of “nice enough” functions $f : \mathbb{P}^1_{\text{Berk}} \to [-\infty, \infty]$ and a linear Laplacian operator $\Delta : \mathcal{P} \to \mathcal{M}_0$, satisfying

\[ \Delta f = 0 \text{ iff } f \text{ is constant.} \]

\[ \int_{\mathbb{P}^1_{\text{Berk}}} f \Delta g = \int_{\mathbb{P}^1_{\text{Berk}}} g \Delta f. \]

\[ \text{For each } \mu \in \mathcal{M}_0 \text{ and each } \zeta \in \mathbb{P}^1_{\text{Berk}}, \text{ there is an associated potential function } g_{\mu} \in \mathcal{P} \text{ such that } \Delta(g_{\mu}) = \mu. \]

\[ \phi^*(\Delta f) = \Delta(f \circ \phi). \]

\[ \text{If } f_n \to f \text{ uniformly on } \mathbb{P}^1_{\text{Berk}} \text{ (and } |\Delta f_n| \text{ is uniformly bounded on } \mathbb{P}^1_{\text{Berk}}), \text{ then } \Delta f_n \to \Delta f \text{ weakly.} \]
The Invariant/Canonical/Equilibrium Measure

Theorem (Favre & Rivera-Letelier; Baker & Rumely; Autissier, Chambert-Loir, & Thuiller, mid 2000s)

Let \( \phi \in \mathbb{C}_v(z) \) be a rational function of degree \( d \geq 2 \).
Then there is a unique probability measure \( \mu = \mu_\phi \) on \( \mathbb{P}^1_{\text{Berk}} \) such that:

- \( \phi^*(\mu) = d \cdot \mu \), and
- \( \mu(E_\phi) = 0 \),

where \( E_\phi \subseteq \mathbb{P}^1(\mathbb{C}_v) \) is the (type 1) exceptional set of \( \phi \).

FYI:
There is a point \( \zeta \in \mathbb{P}^1_{\text{Berk}} \) for which \( \mu(\{\zeta\}) > 0 \) if and only if \( \zeta \) is type 2, \( \mu = \delta_\zeta \), and \( \phi \) has potentially good reduction, attained by moving \( \zeta \) to \( \zeta(0,1) \).
Definition of the Equilibrium Measure

Given \( \phi \in C_v(z) \) of degree \( d \geq 2 \), choose any \( \zeta \in \mathbb{P}^1_{\text{Berk}} \) that is not type 1.

It’s easy to write down an explicit bounded function \( g \in \mathcal{P} \) such that \( \Delta g = d^{-1}\phi^*\delta_\zeta - \delta_\zeta \). For each \( n \geq 1 \), set

\[
\mu_n = d^{-n}(\phi^n)^*\delta_\zeta \quad \text{and} \quad f_n = \sum_{i=0}^{n-1} d^{-i} \cdot g \circ \phi^i.
\]

Then \( \Delta f_n = \sum_{i=0}^{n-1} d^{-i} \Delta (g \circ \phi^i) = \sum_{i=0}^{n-1} d^{-i} (\phi^i)^* (\Delta g) \)

\[
= \sum_{i=0}^{n-1} \left[ d^{-i-1}(\phi^{i+1})^* (\delta_\zeta) - d^{-i}(\phi^i)^* (\delta_\zeta) \right]
\]

\[
= d^{-n}(\phi^n)^* (\delta_\zeta) - \delta_\zeta = \mu_n - \delta_\zeta
\]

On the other hand, \( \{f_n\} \) converges uniformly to some function \( f \).
Thus, setting \( \mu_\phi = \Delta f + \delta_\zeta \), we immediately obtain
\( \mu_n \to \mu_\phi \) weakly, and \( \phi^* \mu_\phi = d \cdot \mu_\phi \).
The Energy Pairing

If $\mu, \nu \in M_0$ are nice enough measures, with $\mu = \Delta f$ and $\nu = \Delta g$, then their energy pairing is

$$ (\mu, \nu) = - \int_{\mathbb{P}^1_{\text{Berk}}} f \, d\Delta(\nu) = - \int_{\mathbb{P}^1_{\text{Berk}}} g \, d\Delta(\mu). $$

If furthermore $\mu(\mathbb{P}^1_{\text{Berk}}) = 0$, then $(\mu, \mu) \geq 0$, with equality iff $\mu = 0$.

Idea: If $\mu, \nu \in M$ are probability measures, then

$$ (\mu - \nu, \mu - \nu) \geq 0 $$

quantifies how different $\mu$ and $\nu$ are.

Warning: Those facts may fail if $\mu$ has delta-masses at type 1 points.
Energy, Canonical Heights, and Equidistribution

Let $K$ be a number field, with set of places $M_K$.
Let $\phi \in K(z)$ with $\deg \phi \geq 2$.

For each $v \in M_K$, let $\mu_{\phi,v}$ be the $v$-adic invariant measure on $\mathbb{P}^1_{\text{Ber},v}$.

Then for any $x \in \mathbb{P}^1(K)$, the canonical height of $x$ satisfies

$$\hat{h}_\phi(x) = [K : \mathbb{Q}]^{-1} \sum_{v \in M_K} \frac{n_v}{2} (\delta_x - \mu_{\phi,v}, \delta_x - \mu_{\phi,v}).$$

More generally, for any nonempty finite $\text{Gal}(\overline{K}/K)$-invariant set $X \subseteq \mathbb{P}^1(\overline{K})$, we have

$$\frac{1}{\#X} \sum_{x \in X} \hat{h}_\phi(x) = [K : \mathbb{Q}]^{-1} \sum_{v \in M_K} \frac{n_v}{2} (\nu_X - \mu_{\phi,v}, \nu_X - \mu_{\phi,v}),$$

where $\nu_X = \frac{1}{\#X} \sum_{x \in X} \delta_x$. 
Applying Equidistribution to Uniform Boundedness

If \( \phi \) has a large set \( X \) of \( K \)-rational preperiodic points, or even of small height points, then \( \hat{h}_\phi(X) := \frac{1}{\#X} \sum_{x \in X} \hat{h}_\phi(x) \) is zero, or at least small.

So let’s aim for a contradiction when \( N = \#X \) is large, by bounding each local term \((\nu_X - \mu_{\phi,v}, \nu_X - \mu_{\phi,v})\) from below. That is, find a good lower bound for \((\nu_X - \mu_{\phi,v}, \nu_X - \mu_{\phi,v})\), where 
\[
\nu_X = \frac{1}{\#X} \sum_{x \in X} \delta_x.
\]

When \( \phi \) is a monic polynomial and \( x \) and \( y \) are both preperiodic, that simplifies to finding a good lower bound for
\[
\sum_{x \neq y \in X} -\log |x - y|_v.
\]
Sketch of Proof of RB's 2004 Theorem

(For ease, we assume $\phi$ is monic and all places are non-archimedean.)

At each $v \in M_K$, given $z_1, \ldots, z_N$ preperiodic, prove Lemma 1:

$$-\sum_{i \neq j} \log |z_i - z_j|_v \geq (d - 1)(N \log d \ N)r_{\phi, v},$$

where $r_{\phi, v} = \inf \{(\delta_x - \mu_{\phi, v}, \delta_y - \mu_{\phi, v}) : x \neq y \} \leq 0$.

At the place $w \in M_K$ maximizing $r_{\phi, w}$, prove Lemma 2: we can partition $\mathbb{P}^1(K)$ into two pieces so that given $z_1, \ldots, z_N$ preperiodic and all in a single piece,

$$-\sum_{i \neq j} \log |z_i - z_j|_v \geq \left(-\frac{N^2}{d - 1} + (d - 1)(N \log_d N)\right)r_{\phi, w}$$
Finishing the Proof

Set $R_v := n_v r_{\phi,v}$.

Let $w \in M_K$ be the place at which $R_w < 0$ is most negative, and partition $K_{\phi,w} = U \sqcup V$.

Given $z_1, \ldots, z_N \in K$ preperiodic and lying in $U$ (resp., $V$) at $w$,

$$0 = \sum_{v \in M_K} n_v \sum_{i \neq j} -\log |z_i - z_j|_v \geq \sum_{v \text{ bad}} n_v \sum_{i \neq j} -\log |z_i - z_j|_v$$

$$\geq -\frac{N^2}{d-1} R_w + \sum_{v \text{ bad}} (d - 1)(N \log_d N) \log R_v$$

$$\geq \left[ -\frac{N^2}{d-1} + s(d - 1)(N \log_d N) \right] \log R_w.$$  

If $N \geq O_d(s \log s)$, we get $0 > 0$.  

Contradiction!
Generalizing to Rational Functions

The same proof works, if we:

- Remove the restriction that $z_1, \ldots, z_N$ are preperiodic,
- Replace $-\log |z_i - z_j|_v$ by $(\delta z_i - \mu_{\phi,v}, \delta z_j - \mu_{\phi,v})$,
- Somehow prove analogues of Lemmas 1 and 2.

For Lemma 1:

- I proved Lemma 1 for polynomials by rewriting the sum as $-\log |\det A|_v$, where $A$ was an $N \times N$ Vandermonde matrix, and using the dynamics of $\phi$ to do some appropriate row reduction.
- Juan Rivera-Letelier [2011] proved an analogue for rational functions by a completely different strategy, involving smoothing certain discontinuous potential functions.
Lemma 2 for Rational Functions

A point $\zeta \in \mathbb{P}^1_{\text{Berk}}$ is a barycenter if $(\delta_\zeta - \mu_\phi, \delta_\zeta - \mu_\phi) \geq 0$ is minimized.

Equivalently, no connected component $U$ of $\mathbb{P}^1_{\text{Berk}} \setminus \{\zeta\}$ has $\mu_\phi(U) > 1/2$.

In ongoing joint work with Juan Rivera-Letelier, we are proving that Lemma 2 will work with $U$ being such a component of maximal mass $\mu_\phi(U)$, and with $V$ being $\mathbb{P}^1_{\text{Berk}} \setminus U$. 