Which bilinear recursions are theta sequences?

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[Unrelated prelude -- thanks to Rob Benedetto for Wednesday chat:
what is smallest h>0 that occurs as canonical height of x_0 where
x_n = f(x_{n-1}) with some rational function f of degree 2?
Conj.: about 0.003606, for f(x) = (14 - 22 x) / (28 - 57 x + 20 x^2)
and x_0 = oo (or 7/11), when x_1, x_2, x_3, ... x_11, x_12, ... are

```
  0, 1/2, 2/3, 3/5, 4/5, 3/4, 5/7, 28/41, 41/64, 32/105,
  8043/13766, 15514282/20631263, ...
```

and second smallest seems to be x_1!
(alternative starting points: 7/11 --> 0, 13/20 --> 1/2)]

(See http://faculty.uml.edu/jpropp/somos.html)

Somos [1982] 1989: Define s_n by s_1 = s_2 = s_3 = s_4 = 1, and for n>4

```
   s_n = (s_[n-1] s_[n-3] + s_[n-2]^2) / s_[n-4];
```

or equivalently, there's a bilinear recurence

```
   s_[n-4] s_n = s_[n-1] s_[n-3] + s_[n-2]^2:

  n    1  2  3  4  5  6  7   8   9   10    11    12     13  ...
 s_n   1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, ...
```

e.g. s_10 = (7*59 + 23^2) / 3 = (413 + 529) / 3 = 942 / 3 = 314
yes, they're all integers! Can also extend recursion to the left:

```
  ..., 314, 59, 23, 7, 3, 2, 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, ...
```

Apparently

```
  s_n = exp(c (2n-5)^2 + \epsilon_n)
```

where c = 0.0255... and \epsilon_n is quasiperiodic. That's
the behavior "at infinity"; it also seems to be true that for each k
the sequence {s_n mod k} is periodic, e.g. s_n is even iff n is 4 mod 5.
[NB this is not obvious from the recurrence because of the divisions!]

Moreover, s_n satisfies further bilinear recursions:

```
  s_n s_[n-5] +    s_[n-1] s_[n-4] -  5 s_[n-2] s_[n-3] = 0
  s_n s_[n-6] -    s_[n-2] s_[n-4] -  5 s_[n-3]^2 = 0
  s_n s_[n-7] -  5 s_[n-2] s_[n-5] -  4 s_[n-3] s_[n-4] = 0
  s_n s_[n-8] - 25 s_[n-3] s_[n-5] +  4 s_[n-4]^2 = 0
  s_n s_[n-9] - 20 s_[n-3] s_[n-6] - 29 s_[n-4] s_[n-5] = 0
  ...
```

in other words, the sequences {{s_[n-m] s_[n+m] | n in Z} : m in Z}
span a 2-dimensional space (note we must use the s[n] indexed by

all of Z, not just the positive indices) as do the sequences
{{s_[n-m] s_[n+m+1] | n in Z} : m in Z}.

Much the same seems to work for any initial choices of s_1,s_2,s_3,s_4
(though there are degenerate examples for which {s_n} is a geometric
sequence and the dimension is only 1), and any coefficients:

s_[n-4] s_n = A_1 s_[n-1] s_[n-3] + A_2 s_[n-2]^2

There may be some denominators, but only those contained in
s_1 through s_4; indeed if we write everything symbolically
in terms of s_1, ..., s_4 we get only "Laurent polynomials" =
polynomials in s_1, ..., s_4 and 1/s_1, ..., 1/s_4 over Z[A_1,A_2].
In particular, this lets us extend past s[n-4]=0. [See above comment
re modular periodicity.] As before we get a 2-sided sequence,
though not necessarily symmetrical; and apparently

s_n = exp(c n^2 + d n + \epsilon_n)

with \epsilon_n quasiperiodic.

Turns out that s_n can always be described in terms of the arithmetic
of an elliptic curve. For instance, for the original Somos_4 we have
the following description: let E be the elliptic curve y^2 + y = x^3 - x,
and P the rational point (0,0). It is known that E is the curve of minimal
conductor (namely 37) that has infinitely many rational points, and every
rational point is kP for some integer k. Let's list the first few:

1 (0, 0)
2 (1, 0)
3 (-1, -1)
4 (2, -3)
5 (1/4, -5/8)
6 (6, 14)
7 (-5/9, 8/27)
8 (21/25, -69/125)
9 (-20/49, -435/343)
10 (161/16, -2065/64)
11 (116/529, -3612/12167)
12 (1357/841, 28888/24389)
13 (-3741/3481, -43355/205379)

the denominators must be d^2, d^3; call d the "denominator of kP".
The first few are

1, 1, 1, 1, 2, 1, 3, 5, 7, 4, 23, 29, 59, ...

Looks familiar? s_n is the denominator of (2k-5) P !
Moreover, our c is h(P)/2, and the oscillation epsilon_n
comes from the location of kP on the real locus of E.
Indeed we have the analytic formula

s[n] = exp(Q(2n-5)) * sum_{j \in Z} q^(n^2) z^((2n-5)j)

for some specific transcendental q and z and an even quadratic polynomial Q
with transcendental coefficients (that's how the connection with E was
first surmised). Much the same is true in general, except that (even

excluding degenerate sequences like geometric and Fibonacci):

- @ when the sequence is not symmetrical, we'll have an asymmetric arithmetic progression on E, {P_0 + kP};

- @ The factor exp(Q) in the analytic formula for s_n will have a "random" parabola for Q, symmetrical about some irrational n;

- @ The leading coefficient may include a finite linear combination of log(p)'s in addition to the canonical height of P (and likewise for the constant coefficient, whose interesting contribution is the height pairing of P_0 and P;

- @ The denominators might have to be signed, and s_n may also have finitely many factors p^(Q(n)+epsilon_p(n)) with Q quadratic and epsilon_p periodic (when there's bad reduction at p and P is not on the principal component).

Try ..., 2, 10, -4, 2, 3, 1, -2, 2, 1, 5, 2, 12, -26, 34, ...
satisfying   s_[n-4] s_n = - s_[n-1] s_[n-3] + s_[n-2]^2:
[E = 3442C: Y^2 + XY + Y = X^3 - X^2 - 1191 X + 16095,
discriminant 2^17 1721.]

(Somos-4 also has the property 2n-5 | 2n'-5 ==> s_n | s_n', i.e. it's an "elliptic divisibility sequence"; e.g. s_8 = 59 | s_22 because 13 | 39. But many bilinear recursions don't.)

Why? Because the space of sections of M*Theta on an elliptic curve has dimension M for M>0. If we define s_n as Theta(q,z) then each s_[n-m] s_[n+m] comes from a section of 2*Theta, so there's a 2-dimensional space of choices. Multiplying by exp(Q) scales the quadratic recurrence, so we have a total of 6 parameters for such a recurrence: one for q, two for the arithmetic progression in C^*/q^Z, and three for Q. This exactly matches the dimension of the space of sequences we're trying to account for (four initial s_n and two coefficients A,B).

What if we try another variation: instead of "width 4", try "width 5":

```
s_[n-5] s_n  =  A_1 s_[n-1] s_[n-3]  +  A_2 s_[n-2] s_[n-3]
```

For example, if we take A_1 = A_2 = s_1 = s_2 = s_3 = s_4 = s_5 = 1, we get

```
... 274, 83, 37, 11, 5, 3, 2, 1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, ...
```

(symmetric about the n=3 term, not about the "(2.5)th" term)
This time we get only odd-width recurrences:

```
s_n s_[n-5] -   s_[n-1] s_[n-4] -   s_[n-2] s_[n-3] = 0  [defining]
s_n s_[n-7] +   s_[n-2] s_[n-5] - 7 s_[n-3] s_[n-4] = 0
s_n s_[n-9] - 7 s_[n-3] s_[n-6] - 8 s_[n-4] s_[n-5] = 0
```

but if we multiply every other a_n by the same constant the recurrence still holds, and one choice -- which turns out to be a fourth root of 3/2 -- also yields recurrences of width 4, and thus of all larger widths as well, and subsumes this into the theta world. This too is to be expected from the parameter counts, once we notice the additional parity-dependent parameter, for a total of 7 theta parameters to match the 5+2 = 7 recursion parameters. Here the arithmetic description turns out to be as follows: let

```
  E:  y^2 + xy = x (x-1) (x+2)   [conductor 102]
  P:  (x,y) = (2,2)
```

the rational points are generated by P and the torsion point (0,0). The multiples start

```
  (2, 2)
  (1, -1)
  (8, -28)
  (9, 24)
  (50/49, 20/343)
  (121/64, -1881/512)
  (2738, -144670)
  (6889/3249, 415415/185193)
   ...
```

and s_n^2 is the *numerator* of (n-3)P if n is odd, or twice the numerator if n is even (taking x = 1/0 for n=3); e.g. 121 = 11^2, 2738 = 2*37^2, 6889 = 83^2.

What about width 6? Let's try... again with the standard initialization

```
  s_1 = s_2 = ... = s_6 = 1, and for n>4

  s_[n-6] s_n = s_[n-1] s_[n-5] + s_[n-2] s_[n-4] + s_[n-3]^2


  ..., 421, 75, 23, 9, 5, 3, 1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, ...
```

This time, no new recursion of width 7 or 8, but yes at 9:

```
  s_n s_[n-9] = - s_[n-1] s_[n-8] - s_[n-2] s_[n-7]
              + s_[n-3] s_[n-6] + 34 s_[n-4] s_[n-5]
```

and likewise at 10 (coefficients 1, -1, 0, -15, 19, -34), and then 11, 12, ... Again this happens for generic coefficients and initial data (3+6 = 9 choices). So here the sequences

```
   {{s_[n-m] s_[n+m] | n in Z} : m in Z}
```

span a 4-dimensional space, as do {{s_[n-m] s_[n+m+1] | n in Z} : m in Z}. This suggests that we're dealing with exp(quadratic) * Theta(P0+nP) where this time it's a Theta function in two variables! There are 3 parameters for the quadratic, 3 for the quasiperiod lattice, and 2 each for P0 and P, total 10; but subtract 1 for the condition that the linear dependence among {s_[n-m] s_[n+m]} for m=0,1,2,3,4 should not involve the m=3 term.

Of course that's not a proof, but eventually one was found, though this requires the generalization of elliptic curves to Jacobians of curves of genus 2.

Likewise for width 7, thanks to the parity-dependent scaling.

Width 8 ...

```
  1, 1, 1, 1, 1, 1, 1, 1,
  4, 7, 13, 25, 61, 187, 775, 5827,
  14815, 420514/7, 28670773/91, 6905822101/2275, ...
```

no integrality or extra recursions etc. And indeed there's no way to find a big enough space of thetas. E.g. unrestricted 2-variable thetas do have 10 parameters as above, but now we need 12 (for 8 initial conditions and 4 coefficients). 3-variable thetas have 3 + 6 + 3 + 3 = 15, but generically their first relation is at width 16, and to shrink this to 8 we must impose 4 conditions, so we end up with a space of theta sequences of dimension 15-4 = 11 which is still not enough.

However...

These are not the simplest width-8 recurrences. We could certainly uses s_n s_[n-8] = s[n-4]^2 or = s[n-3] s[n-5] but that's trivial because it's linear in log s_n. However, try

```
  s_n s_[n-8]  =  s[n-4]^2 + s[n-3] s[n-5]
```

with initial condition s_1 = s_2 = ... = s_8 = 1:

```
   1, 1, 1, 1, 1, 1, 1, 1,
   2, 2, 2, 3, 6, 8, 10, 21,
   30, 62, 134, 247, 367, 983, 3327, 5247, ...
```

and now we do get new recursions at with 17, 18, 19, ... e.g.

```
   width 17:  1 1 0 0 -9 -9 0 -17 17
   width 18:  1 -1 0 -9 -17 0 0 9 73 -8
   width 19:  1 0 -9 -8 -9 17 81 72 17 -153
```

suggesting a three-variable theta sequence. So does periodicity modulo small integers, etc. Likewise for any *three-term* bilinear recurrence of this kind. We can't prove it yet, though I'm told that integrality is known via cluster algebras etc. But it is suggested by a parmeter count together with a nonetrivial result in algebraic geometry, the "Fay trisecant formula", which works only for Jacobians of curves (so special once we're at width 10 and up)! But I don't think anybody has found it explicitly yet, either with rational coefficients or numerical approximations to the periods. Likewise for 4-term sequences as long as the separations sum to zero when appropriately signed -- quadrisecant formula for certain Pryms. Can we prove or at least construct?