Admissible Covers and Rescaling Limits

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after Matthieu Arfeux

Origin of the problem

- Rescaling limits appear in the work of Epstein and have been studied by Kiwi, DeMarco and others.
- The approach we will use is inspired by the work of Epstein on the deformation space $\mathrm{Def}_A^B(f)$ and by work of Selinger and Koch on Thurston's theorem.
- Our contribution is to use admissible covers between marked stable curves.

The moduli space rat_D

- $\mathbb{S} := \mathbb{P}^1(\mathbb{C})$ is the Riemann sphere.
- Rat_D is the space of rational maps $f : \mathbb{S} \to \mathbb{S}$ of degree D.
- The group of Möbius transformations $\mathrm{Aut}(\mathbb{S}) = \mathrm{Rat}_1$ acts on Rat_D by conjugation.
- The moduli space rat_D is the quotient orbifold $\mathrm{Rat}_D/\mathrm{Aut}(\mathbb{S})$.
- The moduli space rat_D is not compact.

Rescaling limits

- Let $(\tau_k \in \mathrm{rat}_D)$ be a divergent sequence.
- A rescaling for (τ_k) of period $q \geq 1$ is a sequence of representatives $(f_k \in \mathrm{Rat}_d)$ such that

$$f_k^{\circ q} \to g$$

for some rational map g with $\deg g \geq 2$ (the convergence is locally uniform outside a finite subset of $\mathbb{S}$).

Example

- $\tau_\varepsilon = [f_\varepsilon] \in \mathrm{rat}_3$ with $f_\varepsilon(z) = z^2 + \varepsilon/z$ and $\varepsilon \to 0$.
- (f_ε) is a rescaling of period 1.
- Set $\delta = \sqrt[3]{\varepsilon}$ and

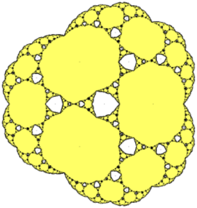
$$g_\varepsilon(w) = \frac{1}{\delta} f_\varepsilon(\delta \cdot w) = \delta \cdot \left(w^2 + \frac{1}{w}\right).$$

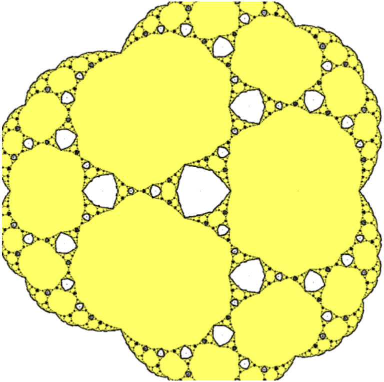
- Then,

$$g_\varepsilon^{\circ 2}(w) = \varepsilon \cdot \left(w^2 + \frac{1}{w}\right)^2 + \frac{w}{1+w^3}.$$

- (g_ε) is a rescaling of period 2.
- 0 is a multiple fixed point of the limit $w \mapsto w/(1+w^3)$.

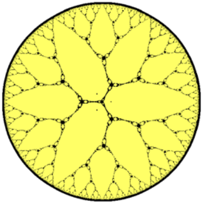
Example

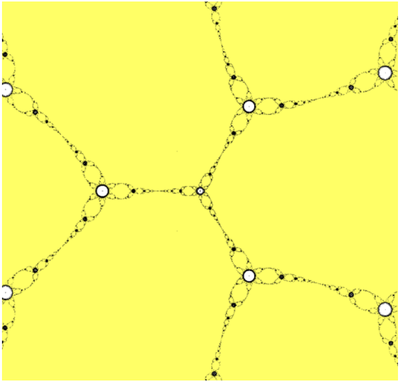




$$\sqrt[3]{\varepsilon} = .5$$

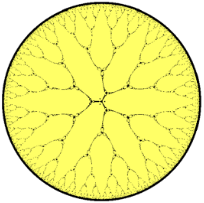
Example

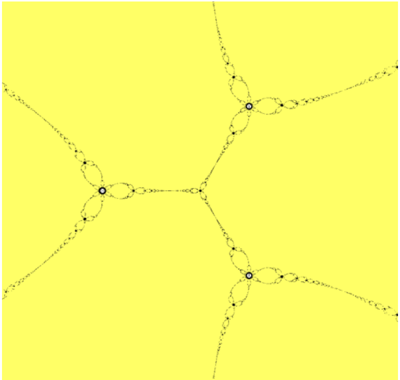




$$\sqrt[3]{\varepsilon} = .2$$

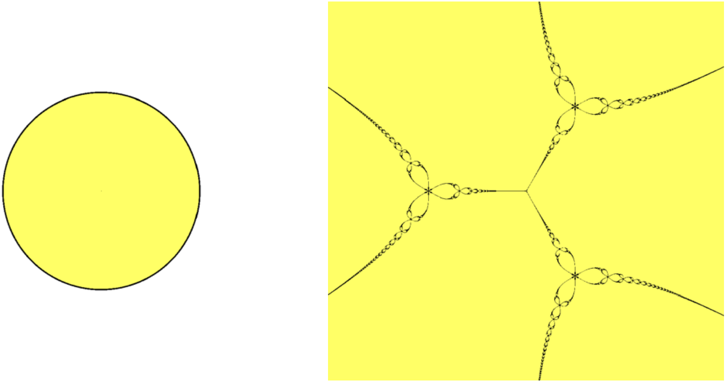
Example





$$\sqrt[3]{\varepsilon} = .1$$

Example



Julia sets for $z \mapsto z^2$ and $w \mapsto w/(1 + w^3)$.

Questions

- How many essentially distinct rescaling limits can there be?
- How can one explain the presence of a multiple fixed point?

Moduli space of marked Riemann spheres

- I is a finite set containing at least three points.
- Two injective maps $x_1 : I \to \mathbb{S}$ and $x_2 : I \to \mathbb{S}$ are equivalent when there is a Möbius transformation $M : \mathbb{S} \to \mathbb{S}$ such that $x_2 = M \circ x_1$.
- $\mathrm{Mod}(I)$ is the set of equivalence classes of injective maps $x : I \to \mathbb{S}$.
- $\mathrm{Mod}(I)$ may be endowed with the structure of a quasiprojective variety of dimension $|B| - 3$.

Sarah Koch's space

- $f \in \mathrm{Rat}_D$, $\mathcal{V}(f)$ is the set of critical values of f.
- $B, C \subset \mathbb{S}$ are finite sets with $\mathcal{V}(f) \subseteq B$, $|B| \geq 3$ and $C = f^{-1}(B)$.
- $\mathcal{K}_B(f)$ is the set of pairs $(\mathbf{y}, \mathbf{z}) \in \mathrm{Mod}(B) \times \mathrm{Mod}(C)$ for which there are triples $(F, y, z) \in \mathrm{Rat}_D \times \mathbb{S}^B \times \mathbb{S}^C$ with $[y] = \mathbf{y}$, $[z] = \mathbf{z}$ and:

$$\begin{array}{ccc} C & \xrightarrow{z} & \mathbb{S} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ B & \xrightarrow[y]{} & \mathbb{S} \end{array} \qquad \text{with} \qquad \deg_{z(c)} F = \deg_c f.$$

Sarah Koch's space

Theorem

- *The set* $\mathcal{K}_B(f)$ *is a smooth submanifold of* $\mathrm{Mod}(B) \times \mathrm{Mod}(C)$*.*
- *The dimension of* $\mathcal{K}_B(f)$ *is* $|B| - 3$*.*
- *The projection* $\mathcal{K}_B(f) \to \mathrm{Mod}(B)$ *is a finite cover.*
- *The projection* $\mathcal{K}_B(f) \to \mathrm{Mod}(C)$ *is a proper embedding.*

Adam Epstein's space

- $A, B, C \subset \mathbb{S}$ are finite sets with $|A| \geq 3$, $A \subseteq B \cap C$, $B \supseteq \mathcal{V}(f)$ and $C = f^{-1}(B)$.
- $\mathcal{E}_A^B(f)$ is the set of pairs $(\mathbf{y}, \mathbf{z}) \in \mathcal{K}_B(f)$ which admits representatives $y \in \mathbb{S}^B$ and $z \in \mathbb{S}^C$ such that $y|_A = z|_A$:

$$\begin{array}{ccc} C & \xrightarrow{\;z\;} & \mathbb{S} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F} \\ B & \xrightarrow[\;y\;]{} & \mathbb{S} \end{array} \quad \text{with} \quad \deg_{z(c)} F = \deg_c f \quad \text{and} \quad y|_A = z|_A.$$

Adam Epstein's space

- The conjugacy class of F is determined by the point of $\mathcal{E}_A^B(f)$.
- The space $\mathcal{E}_A^B(f)$ parameterizes conjugacy classes of rational maps marked by the dynamics of f on A.

Theorem

Assume f is not a flexible Lattès example.

- *The set $\mathcal{E}_A^B(f)$ is a smooth submanifold of $\mathcal{K}_B(f)$.*
- *The dimension of $\mathcal{E}_A^B(f)$ is $|B| - |A|$.*

Compactification of $\mathrm{Mod}(I)$

- $\overline{\mathrm{Mod}}(I)$ is the Deligne-Mumford compactification of $\mathrm{Mod}(I)$.
- An I-tree of spheres is a pair (T, Z) where
  - T is a tree whose leaves are the points of I and whose nodes have valence ≥ 3 and
  - Z is a collection of maps $z_v : I \to \mathbb{S}$ indexed by the nodes of T with $z_v(i_1) = z_v(i_2)$ if and only if i_1 and i_2 are in the same component of $T - v$.
- $\overline{\mathrm{Mod}}(I)$ parameterizes equivalence classes of I-trees spheres.

Compactification of $\mathcal{K}_B(f)$

- $\overline{\mathcal{K}_B}(f)$ is the closure of $\mathcal{K}_B(f)$ in $\overline{\mathrm{Mod}}(B) \times \overline{\mathrm{Mod}}(C)$.
- An admissible cover $F : (T_C, Z) \to (T_B, Y)$ is a pair $(\hat{f}, \{f_v\})$ where
  - $\hat{f} : T_C \to T_B$ is a weighted tree map whose restriction to C coincides with f,
  - $f_v : \mathbb{S} - z_v(C) \to \mathbb{S} - y_{\hat{f}(v)}(B)$ is a cover and
  - $f_v\big(z_v(e)\big) = f_w\big(z_w(e)\big)$ and $\deg_{z_v(e)} f_v = \deg_{z_w(e)} f_w$ if v and w are linked by an edge e.
- $\overline{\mathcal{K}_B}(f)$ parameterizes equivalence classes of admissible covers.

Compactification of $\mathcal{E}_A^B(f)$

- $\overline{\mathcal{E}_A^B}(f)$ is the closure of $\mathcal{E}_A^B(f)$ in $\overline{\mathcal{K}_B}(f)$.
- To each point of $\overline{\mathcal{E}_A^B}(f)$, one may associate a dynamical cover is a triple (F, ι_B, ι_C) where
  - $F : (T_C, Z) \to (T_B, Y)$ is an admissible cover,
  - $\iota_B : (T_B, Y) \to (T_A, X)$ is a contraction,
  - $\iota_C : (T_C, Z) \to (T_A, X)$ is a contraction.

$$\begin{array}{ccc} & & (T_C, Z) \\ & \overset{\iota_C}{\swarrow} & \downarrow F \\ (T_A, X) & \xleftarrow[\iota_B]{} & (T_B, Y). \end{array}$$

Back to rescaling limits

- Let (F, ι_B, ι_C) represent a point in $\overline{\mathcal{E}_A^B}(f)$.
- Rescaling limits correspond to cycles of spheres for which the first return map has degree ≥ 2.

Proposition (Kiwi)

There are at most $2D - 2$ rescaling limits which are not postcritically finite.

Proposition (Arfeux)

If $e \in T_A$ is a periodic edge linking vertices v and w, the product of multipliers at $x_v(e)$ and $x_w(e)$ is 1.