Bers, Brown and Lyapunov

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École Polytechnique

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Plan

0. Prologue : bifurcation currents for rational maps.
1. Stability/bifurcation dichotomy for Kleinian groups.
2. Lyapunov exponent of a surface group representation
3. The degree of a projective structure.

Joint work (in progress) with Bertrand Deroin (Orsay)

Prologue : bifurcation currents for rational maps.

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For every λ, f_λ admits a natural invariant probability measure μ_λ (Brolin-Lyubich measure).
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Definition (DeMarco)

The bifurcation current is $T_{\mathrm{bif}} = dd^c_\lambda(\chi(f_\lambda))$

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- Equidistribution of special subvarieties. (D.-Favre, Bassanelli-Berteloot) Natural sequences of hypersurfaces associated with bifurcations equidistribute towards T_{bif}.
- Formulas for Lyapunov exponent.
  Example : the Manning-Przytycki formula : if f_λ is a monic polynomial of degree d then

$$\chi(f_\lambda) = \log d + \sum_{c \text{ critical}} G_{f_\lambda}(c).$$

Sullivan's dictionary

Aim : translate these concepts into the context of families of subgroups of $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PSL}(2, \mathbb{C})$.

1. Stability/bifurcation dichotomy for Kleinian groups

- $\mathrm{Aut}(\mathbb{P}^1) \simeq \mathrm{PSL}(2, \mathbb{C})$ via $\frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Fix $\|\cdot\|$ on $\mathrm{SL}(2, \mathbb{C})$.

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- A Möbius transformation $\gamma(z) = \frac{az+b}{cz+d} \neq \mathrm{id}$ has a type

elliptic	model : $z \mapsto e^{i\theta} z$	$\mathrm{tr}^2(\gamma) \in [0,4)$
parabolic	model : $z \mapsto z+1$	$\mathrm{tr}^2(\gamma) = 4$
loxodromic	model : $z \mapsto kz$	$\mathrm{tr}^2(\gamma) \notin [0,4]$

1. Stability/bifurcation dichotomy for Kleinian groups

Let G be a finitely generated group, Λ a complex manifold, and $\rho = (\rho_\lambda)_{\lambda\in\Lambda} : \Lambda \times G \to \mathrm{PSL}(2,\mathbb{C})$ be a holomorphic family of representations of G (i.e. it is holomorphic in λ and a homomorphism in g).

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Standing assumptions :

(R1) the family is non-trivial

(R2) ρ_{λ_0} is faithful for some λ_0

(R3) for every λ, ρ_λ is non-elementary (i.e. does not have a finite orbit on $\mathbb{H}^3 \cup \mathbb{P}^1$).

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(or sometimes

(R3') there is λ_0, s.t. ρ_{λ_0} is non-elementary.)

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Theorem (Sullivan, and also Bers, Marden, etc.)

Let $(\rho_\lambda)_{\lambda\in\Lambda}$ *be as above, and* $\Omega \subset \Lambda$ *be a connected open subset. The following are equivalent :*

1. $\forall\lambda \in \Omega$, ρ_λ *is discrete ;*
2. $\forall\lambda \in \Omega$, ρ_λ *is faithful ;*
3. *for every* $g \in G$, $\rho_\lambda(g)$ *does not change type as* λ *ranges in* Ω *;*
4. *for all* λ, $\lambda' \in \Omega$, ρ_λ *and* $\rho_{\lambda'}$ *are quasiconformally conjugate on* $\mathbb{P}^1$*, i.e. there exists a qc homeo* $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ *s.t.* $\forall g \in G$, $\rho_{\lambda_0}(g) \circ \phi = \phi \circ \rho_{\lambda_1}(g)$.

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Such a family is said to be stable on Ω.

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Corollary

For every $t \in [0, 4]$,

$$\overline{\bigcup_{g \in G} \{\lambda,\ \mathrm{tr}^2(\rho_\lambda(g)) = t\}} \supset \mathrm{Bif}\,.$$

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Note : Bif has non-empty interior (Margulis-Zassenhaus lemma) so Stab is not dense in this setting.

2. Lyapunov exponent of a surface group representation

For the remainder of the talk $G = \pi_1(X, \star)$ is the fundamental group of a compact connected surface of genus ≥ 2, endowed with Riemann surface structure.

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Note : everything should work for (hyperbolic) surfaces of finite type (technically much more difficult)

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Let $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$ be a non-elementary representation.

For v a unit tangent vector at $\star$, let $\gamma_{\star,v}$ the corresponding unit speed half geodesic. For $t > 0$ close the path $\gamma_{\star,v}|_{[0,t]}$ by a path of length $\leq \mathrm{diam}(X)$ returning to $\star$. We get a loop $\widetilde{\gamma}_t$.

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Definition-Proposition

For a.e. $v \in T^1_\star X$, the limit

$$\chi_{\mathrm{geodesic}}(\rho) = \lim_{t\to\infty} \frac{1}{t} \log \|\rho([\widetilde{\gamma}_t])\|$$

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"The Lyapunov exponent of ρ associated to geodesic flow on X"

Example (Fuchsian representation)

View X as $\Gamma \setminus \mathbb{H}^2$, where Γ is a Fuchsian group. Identify G and Γ via ρ_{Fuchs}.
Then $\chi_{\mathrm{geodesic}}(\rho_{\mathrm{Fuchs}}) = \frac{1}{2}$.

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Indeed, let 0 be the lift of $\star$. In $\mathbb{H}^2$, travel a geodesic issued from 0 for time t, and close it by a short path joining its endpoint to $\gamma(0)$ for some $\gamma \in \Gamma$. Then

$$\log \|\gamma\| \simeq \frac{1}{2} \log d_{\mathbb{H}^2}(0, \gamma(0)) = \frac{t}{2} + O(1).$$

2. Lyapunov exponent of a surface group representation

Let $(\rho_\lambda)_{\lambda\in\Lambda}$ be a holomorphic family of representations of $G = \pi_1(X)$ into $\mathrm{PSL}(2,\mathbb{C})$, satisfying $(R1-3)$.

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$\lambda \mapsto \chi_{\rm geodesic}(\lambda)$ *is a continuous positive psh function on* Λ*.*

Let $T_{\rm bif} = \frac{1}{2} dd^c(\chi_{\rm geodesic})$ be the bifurcation current.

Theorem

$\mathrm{Supp}(T_{\rm bif})$ *is the bifurcation locus of the family.*

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Let γ be a closed geodesic on X. It defines a conjugacy class in G so $\mathrm{tr}^2(\rho([\gamma])$ is well defined. For $t \in \mathbb{C}$, let

$$Z(\gamma, t) = \left\{\lambda \in \Lambda,\ \mathrm{tr}^2(\rho_\lambda([\gamma]) = t\right\}.$$

(interesting values : $t = 4,\ t = 4\cos^2(\theta)$, $\theta \notin \pi\mathbb{Q}$.)

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For every $t \in \mathbb{C}$, if (γ_n) is a random sequence of geodesics with length $\to \infty$, then

$$\frac{1}{4\,\mathrm{length}(\gamma_n)}[Z(\gamma_n, t)] \underset{n\to\infty}{\longrightarrow} T_{\mathrm{bif}}.$$

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(to be made more precise later)

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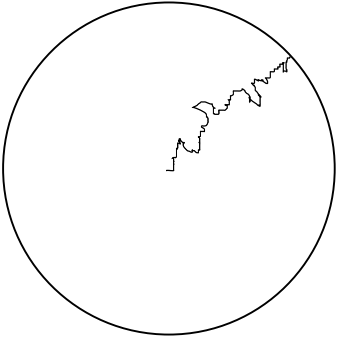
(loose definition) Brownian motion on X is the data for every $x \in X$ of a probability measure W_x on the set of continuous paths $\omega : [0, \infty) \to X$, satisfying :

1. the law of $\omega(t)$ assuming $\omega(0) = x$ is given by the distribution of heat at time t , starting from δ_x at $t = 0$.
2. Markov property : the law of $\omega(T + t)$ given $\omega(T) = y$ is the law of $\omega(t)$ given $\omega(0) = y$.

2. Some ingredients of proof

On $\mathbb{H}^2$ a typical Brownian path from 0 escapes at speed $\frac{t}{2}$ to the boundary, i.e. for W_0 a.e. ω, $\lim_{t\to\infty} \frac{1}{t} d_{\mathbb{H}^2}(0, \omega(t)) = \frac{1}{2}$ (normalization : heat kernel associated to $\frac{1}{2}\Delta$)



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We establish the following

Proposition

Given a Brownian path ω from $\star$, close the Brownian path $\omega|_{[0,t]}$ with some path of length $\leq \mathrm{diam}(X)$ joining $\omega(t)$ to $\star$. Let $\widetilde{\omega}_t$ be the corresponding loop.
For $W_\star$ a.e. ω, the limit

$$\chi_{\mathrm{Brown}}(\rho) = \lim_{t\to\infty} \frac{1}{t} \log \left\| \rho\left([\widetilde{\omega}_t]\right) \right\|$$

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Then we get that $\chi_{\mathrm{Brown}} = \frac{1}{2}\chi_{\mathrm{geodesic}}$ and $T_{\mathrm{bif}} = dd^c\chi_{\mathrm{Brown}}$.

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Let μ be a proba measure on G. Define

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We proved in a previous work that if μ satisfies certain moment and non-degeneracy conditions, then $T_{\mathrm{bif},\mu} = dd^c \chi_\mu$ satisfies $\mathrm{Supp}(T_{\mathrm{bif},\mu}) = \mathrm{Bif}$ and some equidistribution properties.

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Here, for $\mu =$Furstenberg's measure, there exists τ s.t. for every ρ

$$\chi_\mu(\rho) = \tau \chi_{\mathrm{Brown}}(\rho)$$

so we can use our previous work.

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A typical element in Γ relative to μ^n (n^{th} step of the random walk associated to μ) is primitive and satisfies :

$$\log \|\gamma\| \simeq \frac{\tau}{4} n \text{ and } \log \ell(\gamma) \simeq \frac{\tau}{2} n \text{ (translation length)}$$

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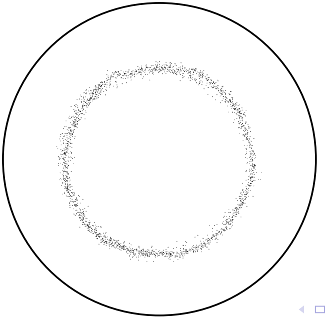
(get something like a Gaussian measure concentrated around the circle of radius $\simeq \frac{\tau}{2} n$)



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Every loop in X is homotopic to a unique closed geodesic so we can project the measure μ^n to a proba measure m_n on the set of closed geodesics on X. This is a Gaussian-like measure concentrated around primitive geodesics of length $\simeq \frac{\tau}{2} n$.

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Theorem

Put $m = \prod m_n$. For every $t \in \mathbb{C}$, if (γ_n) is a m-random sequence of closed geodesics, then

$$\frac{1}{4\,\mathrm{length}(\gamma_n)}[Z(\gamma_n, t)] \underset{n\to\infty}{\longrightarrow} T_{\mathrm{bif}}.$$

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A projective structure σ on the Riemann surface X is the data of a locally injective map

$$\mathrm{dev}(\sigma) : \mathbb{H}^2 \to \mathbb{P}^1 \text{ (the developing map of } \sigma)$$

satisfying the equivariance property

$$\mathrm{dev} \circ \gamma = \rho(\gamma) \circ \mathrm{dev}$$

for some representation ρ of $\Gamma \simeq G$. By definition ρ is the holonomy map $\mathrm{hol}(\sigma)$.

3. Projective structures on X

- We say $\sigma \sim \sigma'$ if there exists $A \in \mathrm{PSL}(2, \mathbb{C})$ s.t. $\mathrm{dev}(\sigma') = A \circ \mathrm{dev}(\sigma)$. Then $\mathrm{hol}(\sigma') = A \circ \mathrm{hol}(\sigma) \circ A^{-1}$.

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  Hence we get a map

  $$P(X) = \text{proj. str.}/\text{conjugacy} \xrightarrow{\text{hol}} \mathrm{Hom}(G, \mathrm{PSL}(2,\mathbb{C}))/\text{conjugacy}$$

- It turns out that

  $$P(X) \simeq \{\text{holom. quad. diff. on } X\} \simeq \mathbb{C}^{3g-3} \text{ (Schwarzian param.)}$$

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  and hol is a proper holomorphic embedding.

- We thus get a natural holomorphic family of representations (mod. conjugacy) associated to X (notice that χ_{Brown} is insensitive to conjugacies so $\chi_{\mathrm{Brown}}(\mathrm{hol}(\sigma))$ is well-defined).

3. Projective structures on X

Example

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- For a general σ, hol(σ) may be discrete or not. dev(σ) can cover $\mathbb{P}^1$ many times.

3. Projective structures on X

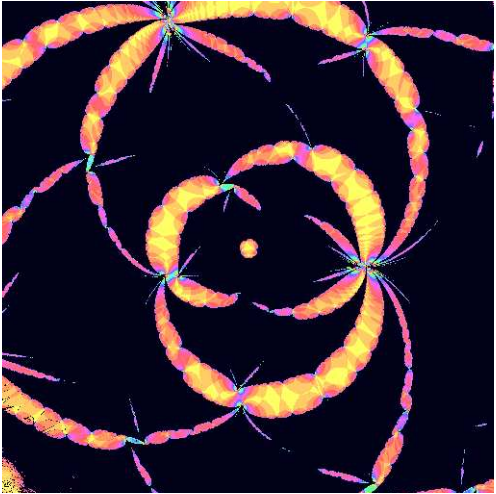

FIGURE: Bers slice and stability components (Yasushi Yamashita)

3. The degree of a projective structure

Definition-Proposition

Let σ be a projective structure on X. Fix $z \in \mathbb{P}^1$. Then for any sequence $p_n \in \mathbb{H}^2$ and $r_n \to \infty$ the sequence

$$\frac{1}{\mathrm{Vol} B_{\mathbb{H}^2}(p_n, r_n)} \# \, \mathrm{dev}^{-1}(z) \cap B_{\mathbb{H}^2}(p_n, r_n)$$

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For a Fuchsian or QF structure, $\delta = 0$.

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Corollary

δ is a continuous psh function on $P(X)$ and $dd^c\delta = \frac{1}{\pi} T_{\mathrm{bif}}$.

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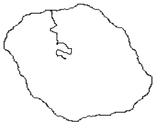

For a QF group, consider the Brownian motion starting from some point 0 in the component uniformizing X. The hitting measure on the boundary is the harmonic measure viewed from 0.

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Conversely the degree formula gives an independent proof of Makarov's theorem for QF groups. This is similar to the case of polynomials with connected Julia sets

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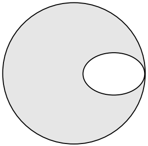
These two notions of polynomial convexity differ.



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Note : $\{\delta = 0\} \neq \overline{B(X)}$.

3. Proof of the degree formula/existence of the degree

Based on the study of the suspension of ρ :

$$X_\rho = \mathbb{H}^2 \times \mathbb{P}^1 \Big/ \Gamma \otimes \rho(G) \cdots$$

....to be continued on whiteboard.