On Mordell-Lang in Algebraic Groups of Unipotent Rank 1

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(work in progress)

Abstract. In the previous ICERM workshop, Tom Scanlon raised the question of whether the (classical, i.e., non-dynamic) Mordell-Lang conjecture remains true in algebraic groups of unipotent rank 1 (with additional hypotheses on the closed subvariety X). I will discuss some initial work in progress on this question, focusing on the Lang exceptional set of X.

Conventions and Basic Definitions

For this talk:

- $\mathbb{N} = \{0, 1, 2, \ldots\}$;
- unless otherwise specified, all fields are assumed to have characteristic 0 .
- a variety over a field k is an integral scheme, separated and of finite type over $\operatorname{Spec} k$. A morphism of varieties over k is a morphism of schemes over k .
- an algebraic group over a field k is a geometrically integral variety over k with a group structure given by morphisms over k .

Algebraic Groups

The following facts about the structure of (commutative) algebraic groups will be useful here.

Theorem (Chevalley, 1960). *Let G be an algebraic group over a perfect field k. Then G has a unique closed normal subgroup H such that H is a linear group (group subvariety of $\mathrm{GL}_n(k)$) and G/H is an abelian variety.*

Theorem (Serre). *A commutative linear algebraic group over an algebraically closed field k of characteristic zero is isomorphic to a product*

$$\mathbb{G}_{\mathrm{a}}^{\alpha} \times \mathbb{G}_{\mathrm{m}}^{\mu} .$$

The isomorphism is not in general unique, but $\alpha, \mu \in \mathbb{N}$ are.

A commutative algebraic group over an algebraically closed field k of characteristic zero is an abelian variety if and only if $\alpha = \mu = 0$ (as in the above two theorems), and is a semiabelian variety if and only if $\alpha = 0$.

In the latter case:

$$0 \longrightarrow \mathbb{G}_{\mathrm{m}}^{\mu} \longrightarrow G \longrightarrow A \longrightarrow 0 .$$

(over more general fields, the first factor need not be split).

Mordell-Lang and Examples

Theorem (Faltings, V., McQuillan). *Let k be a number field, let X be a closed subvariety of a semiabelian variety G over k, and let Γ be a subgroup of $G(\bar{k})$ of finite rank (i.e., it is the division group of a finitely generated subgroup Γ_0 of itself). If X is not the translate of a group subvariety of G by an element of Γ, then $X(\bar{k}) \cap \Gamma$ is not Zariski dense in X.*

If, instead, G is a commutative algebraic group of unipotent rank 1, then are there any conditions on X that will imply that the same conclusion holds?

Question. *Let G be a commutative algebraic group over a number field with $\alpha = 1$ (i.e., whose linear part is isomorphic to $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}^\mu$ for some μ), and let X be a closed subvariety of G. What conditions on X ensure that $X \cap \Gamma$ is not Zariski dense for any finitely generated subgroup Γ of $X(\bar{k})$?*

[The same question for the division group of such Γ is much harder.]
Obviously, X should not be a translate of a subgroup of G.
Other examples:

(i). $G = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$, $\Gamma = \mathscr{O}_k \times \mathscr{O}_k^*$, $X = \{(t,u) : t = u\}$.
(ii). $G = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$, $\Gamma = \mathscr{O}_k \times \mathscr{O}_k^*$, $X = \{(t,u) : t = u^2\}$.
(iii). $G = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$, $\Gamma = \mathscr{O}_k \times \mathscr{O}_k^*$, $X = \{(t,u) : t^2 = u\}$.
(ii′). $G = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$, $\Gamma = \mathscr{O}_k \times \mathscr{O}_k^*$, $X = \{(t,u) : t = 3u^2 + 4u + 6\}$.

The common thread in these examples is that there is a nontrivial character $\chi\colon \mathbb{G}_\mathrm{m} \to \mathbb{G}_\mathrm{m}^\mu$ such that the pull-back of X to $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}^\mu$ contains a regular section.

Reductions Let G be a commutative algebraic group over a number field k.

- We may assume that the linear part is split, so that there is a short exact sequence
$$0 \longrightarrow \mathbb{G}_\mathrm{a} \longrightarrow G \longrightarrow G' \longrightarrow 0$$
with G' semiabelian.
- We may assume that X dominates G', and that it is a prime divisor in G.
- We may assume that X is not fibered by subgroups of G; i.e., there is no nontrivial algebraic subgroup H of G such that X is the pull-back of a closed subset of G/H via $G \to G/H$.

Why unipotent rank 1?

Exceptional Sets

This section describes the Lang exceptional set and the Kawamata locus. The former is defined for any algebraic variety X (and therefore does not use any group structure on X or any containing variety). The latter is specific to closed subvarieties of group varieties.

General theme: These sets are where you expect to find dense subsets of rational or integral points.

Definition. Let X be a complete variety over a field k. Then the Lang exceptional set of X is the Zariski closure of the union of the images of all nonconstant rational maps from $\mathbb{G}_\mathrm{m}$ or abelian varieties over extension fields of k, to X.

In the above, we may assume that the extension field of k is algebraic.

In addition, if X is a (closed) subvariety of a semiabelian variety, then we may also assume that the rational map $G \dashrightarrow X$ is a morphism.

(I'm ignoring the possibility that G is a more general algebraic group.)

This motivates the following definition:

Definition. Let X be a closed subvariety of a commutative group variety G over a number field k. Then the Lang-like exceptional set $\operatorname{Exc}'(X)$ is the Zariski closure of the union of the images of all nonconstant morphisms from $\mathbb{G}_\mathrm{m}$ or abelian varieties over extension fields of k, to X.

Again, we may assume that the extension fields are algebraic.

(We don't need $\mathbb{G}_\mathrm{a}$ here.)

We may write

$$\operatorname{Exc}'(X) = \operatorname{Exc}'_\mathrm{T}(X) \cup \operatorname{Exc}'_\mathrm{A}(X) ,$$

where $\operatorname{Exc}'_\mathrm{T}(X)$ and $\operatorname{Exc}'_\mathrm{A}(X)$ are the Zariski closures of the unions of images of morphisms from $\mathbb{G}_{\mathrm{m},L}$ ($L \supseteq k$) to X and from abelian varieties over extension fields of k to X, respectively.

The Kawamata locus is a similar set:

Definition 1. Let X be a closed subvariety of a commutative group variety G over an algebraically closed field k of characteristic zero. Then the Kawamata locus of X is the union $Z(X)$ of all nontrivial translated group subvarieties of G contained in X.

When G is a semiabelian variety, $Z(X)$ is known to be closed:

Theorem (Kawamata Structure Theorem). *Let X be a closed irreducible subset of a semiabelian variety G over an algebraically closed field k of characteristic zero. Then $Z(X)$ is closed, and is a proper subset of X unless X is fibered by subgroups of G.*

Also, if G is semiabelian, then $Z(X)$ equals the Lang-like exceptional set, as a trivial consequence of the following theorem.

Theorem. *Any morphism from one semiabelian variety to another is a translate of a group homomorphism.*

In general, for G and X as in Definition 1, we can write

$$Z(X) = Z_\mathrm{U}(X) \cup Z_\mathrm{T}(X) \cup Z_\mathrm{A}(X) ,$$

where $Z_\mathrm{U}(X)$, $Z_\mathrm{T}(X)$, and $Z_\mathrm{A}(X)$ are the unions of translated group subvarieties of G, isomorphic to $\mathbb{G}_\mathrm{a}$, $\mathbb{G}_\mathrm{m}$, and abelian varieties, respectively, contained in X.

It is easy to see that if G has unipotent rank 1 then $Z_\mathrm{U}(X)$ is closed, because the map $\pi\colon G \to G/\mathbb{G}_\mathrm{a}$ is smooth, hence open, so the set $\pi(G \setminus X)$ is open, and $Z_\mathrm{U}(X)$ is the (closed) pull-back of its complement. (Note that all maps $\mathbb{G}_\mathrm{a} \to G/\mathbb{G}_\mathrm{a}$ are constant, so the above argument suffices to characterize $Z_\mathrm{U}(X)$.)

——— Why these sets are important. ———

The Theorem

It is possible to show that $Z_A(X)$ is closed for commutative G with unipotent rank 1. This starts by looking at abelian subvarieties of G. This, in turn, starts with some simple cases.

[Added later: you can do this in general, because you only need a surjection of the Ext groups, below.]

Throughout this section, G is a commutative group variety over an algebraically closed field of characteristic zero.

Lemma. *Let G be a commutative group variety over an algebraically closed field of characteristic zero. Then the set of abelian subvarieties of G forms a directed set under inclusion.*

Proof. Let B_1 and B_2 be abelian subvarieties of G. Then addition gives a group homomorphism

$$B_1 \times B_2 \to G$$

whose image is an abelian subvariety of G containing both B_1 and B_2. □

Lemma. *Let G be a commutative group variety for which there exists an exact sequence*

$$0 \to \mathbb{G}_m \to G \to A \to 0 \,,$$

with A an abelian variety. Then there is a largest abelian subvariety B of G.

Proof. It is well known that the set of extensions G (as above) is canonically and functorially isomorphic to $\operatorname{Pic}^0 A$, in such a way that, if G corresponds to $\mathscr{M} \in \operatorname{Pic}^0 A$, then G is isomorphic as an abstract variety to the variety $\mathbb{P}(\mathscr{M} \oplus \mathscr{O}_A)$, minus the (disjoint) sections corresponding to the projections

$$\mathscr{M} \oplus \mathscr{O}_A \twoheadrightarrow \mathscr{M} \qquad \text{and} \qquad \mathscr{M} \oplus \mathscr{O}_A \twoheadrightarrow \mathscr{O}_A \,.$$

Let this difference be denoted $\mathbb{P}'(\mathscr{M})$.

Now let B be an abelian subvariety of G, and let $\pi_B \colon B \to A$ denote its projection to A. Then the product $G \times_A B$ is isomorphic to $\mathbb{P}'(\pi_B^* \mathscr{M})$, and the diagonal map $B \to G \times_A B$ gives a regular section of this scheme over B. Existence of this section implies that $\pi_B^* \mathscr{M}$ is trivial. Thus $\mathscr{M}\big|_{\pi_B(B)}$ is torsion in $\operatorname{Pic}^0(\pi_B(B))$.

Now the earlier lemma implies that the set

$$\{\pi_B(B) : B \text{ is an abelian subvariety of } G\}$$

of abelian subvarieties of A is also directed under inclusion, so it has a largest element B' (take an element of largest dimension). Then the lemma will be proved if we can show that the set of abelian subvarieties of G that dominate B' is finite. If B dominates B', then $B \to B'$ is an isogeny, and it will suffice to show that the degree of the isogeny is bounded by the order of $\mathscr{M}\big|_{B'}$ in $\operatorname{Pic}^0(B')$.

To see this, we note that if the degree of $B \to B'$ does not divide the order of $\mathscr{M}\big|_{B'}$, then $B \to B'$ factors as

$$B \xrightarrow{\alpha} B'' \xrightarrow{\beta} B' \,,$$

with $\deg \alpha > 1$ and $\beta^*(\mathscr{M}\big|_{B'})$ trivial. But then $B \to G$ also factors through α, a contradiction. □

Lemma. *Let G be a commutative group variety for which there exists an exact sequence*

$$0 \to \mathbb{G}_a \to G \to A \to 0 \,,$$

with A an abelian variety. Then there is a largest abelian subvariety B of G.

Proof. As before, let B' be the largest element of the set

$$\{\pi_B(B) : B \text{ is an abelian subvariety of } G\} \,,$$

and let B be an abelian subvariety of G that maps onto it. Since $\mathbb{G}_a$ is not proper and has no nontrivial torsion subgroups, $B \to B'$ must have degree 1, and we are done. □

Lemma. *Let G be a commutative group variety for which there exists an exact sequence*

$$0 \to \mathbb{G}_a \times \mathbb{G}_m^\mu \to G \to A \to 0$$

for some $\mu \in \mathbb{N}$ and some abelian variety A. Then there is a largest abelian subvariety B of G.

Proof. It suffices to show that G is isomorphic to a product over A of $\mu + 1$ group varieties of the form considered in the previous two lemmas.

Let $G' = G/\mathbb{G}_a$. Then G' is a semiabelian variety, and it is known that it can be written as a product over A of μ group varieties of the form considered in the first of the two lemmas.

The remainder of the lemma is proved by induction on μ. The case $\mu = 0$ is trivial. For the inductive step, apply the contravariant exact sequence in $\operatorname{Ext}(\cdot, \mathbb{G}_a)$ to the short exact sequence

$$0 \to \mathbb{G}_m \to G'' \to G''' \to 0 \,,$$

with G'' and G''' semiabelian. It gives the sequence

$$0 = \operatorname{Hom}(\mathbb{G}_m, \mathbb{G}_a) \to \operatorname{Ext}(G''', \mathbb{G}_a) \xrightarrow{\gamma} \operatorname{Ext}(G'', \mathbb{G}_a) \to \operatorname{Ext}(\mathbb{G}_m, \mathbb{G}_a) = 0 \,.$$

Therefore γ is an isomorphism, so by induction $\operatorname{Ext}(A, \mathbb{G}_a) \to \operatorname{Ext}(G', \mathbb{G}_a)$ is bijective, and we are done. □

Now we can prove:

Theorem. *Let X and G be as above, and assume that X is not fibered by abelian subvarieties of G. Then $Z_A(X)$ is a proper closed subset of X.*

Proof. Let B be the largest abelian subvariety of G. Then all algebraic subgroups involved in $Z_A(X)$ are contained in B.

Let $\pi \colon G \to G/B$ be the quotient map. This is a fiber bundle with fiber B. Work of Serre on equivariant completions of commutative algebraic groups implies that there are completions $\overline{G}$ and $\overline{G/B}$ of G and G/B, respectively, such that π extends as a fiber bundle to a morphism $\overline{\pi} \colon \overline{G} \to \overline{G/B}$.

Let $\overline{X}$ be the closure of X in $\overline{G}$, and let V be the (closed) subset $\overline{\pi}(\overline{X}) \subseteq \overline{G/B}$.

One can define a Kawamata locus in this context (take the union of the Kawamata loci of each fiber). The intersection of this set with X is none other than $Z_A(X)$.

Bogomolov's proof of the Kawamata Structure Theorem extends readily to this situation. □

Obstructions to actually proving Mordell-Lang in this context

New ideas needed:

- Is $Z_T(X)$ a closed and proper subset of X?
- There is no canonical height on $\mathbb{G}_{\mathrm{a}}$.

Next Steps

The following special cases are currently open (and could be looked at next):

1. No $\mathbb{G}_{\mathrm{m}}$ part...
2. Exceptional set when there is a $\mathbb{G}_{\mathrm{m}}$ part.