Puiseux series dynamics and leading monomials of escape regions

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Workshop on Moduli Spaces Associated to Dynamical Systems
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Critical points of $f_{a,v}$ are $\pm a$.
Critical value: $v = f_{a,v}(+a)$.
Co-critical value $-2a$, since $v = f_{a,v}(+a) = f_{a,v}(-2a)$.

Periodic Curves

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What is the Euler characteristic of the smooth compactification of $\mathcal{S}_p$?

> Requires to compute the number N_p of "escape regions". (De Marco and Schiff, De Marco-Pilgrim).

Escape Regions

The connectedness locus

$$\mathcal{C}(\mathcal{S}_p) = \left\{ f_{a,v} \in \mathcal{S}_p \mid f_{a,v}^n(-a) \not\to \infty \right\}$$

is compact.

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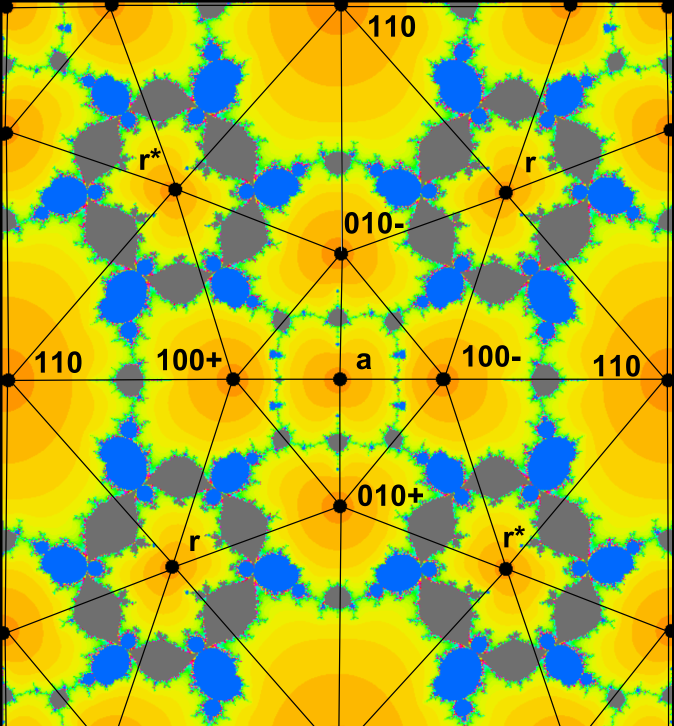
A escape region $\mathcal{U}$ is a connected component of $\mathcal{E}(\mathcal{S}_p)$:

- $\mathcal{U}$ is conformally isomorphic to punctured disk.
- The puncture is at $\infty_{\mathcal{U}}$.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Asymptotics of critical periodic orbit

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Bonifant and Milnor:

> Do the leading terms of $a_j - a$, for $j = 1, \dots, p-1$ determine $\mathcal{U}$ uniquely?

Leading monomials

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Leading monomial

$$\mathbf{m}_j = c_{k_0} \left(\frac{1}{a}\right)^{k_0/\mu}.$$

Theorem. *The leading monomial vector*

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Corollary. (Bonifant,-,Milnor) Assume that the leading monomial vector of $\mathcal{U}$ is

$$(c_1 a^{-k_1/\mu}, \ldots, c_{p-1} a^{-k_{p-1}/\mu}).$$

Then, for all j,

$$a_j \in \mathbb{Q}(c_1, \ldots, c_{p-1})((a^{-1/\mu})).$$

Equations

For an escape region $\mathcal{U}$,

$$v = a_1 = (+a \text{ or } -2a) + a \cdot \sum_{k \geq k_0} c_k \left(\frac{1}{a}\right)^{k/\mu}$$

is a solution of

$$f_{a,v}^{p}(+a) = +a \qquad (*)$$

in some extension of $\mathbb{Q}((1/a))$.

Field

Put t instead of $1/a$ to get $\mathbb{Q}((t))$ and study solutions of (*) in the algebraic closure of $\mathbb{Q}((t))$:

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Complete it to get $\mathbb{L}$.

Non-Archimedean problem

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$$\omega_j^+ = \begin{cases} +1 + c \cdot t^{k/m} + h.o.t. \\ -2 + d \cdot t^{\ell/n} + h.o.t. \end{cases}$$

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The corresponding leading monomial are

$$\mathbf{m}(\omega_j^+ - \omega^+) = \begin{cases} c \cdot t^{k/m} \\ -3 \end{cases}$$

respectively.

Theorem. *The leading monomial vector*

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determines the periodic parameter ν uniquely.

Dynamical Space

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is a disjoint union of finitely many dynamical balls of level n.

Each level $n+1$ ball is contained in, and maps onto, a level n ball.

Branches

Maximal open balls in a closed ball D are parametrized by $\mathbb{Q}^a$.

If ψ_ν maps D onto D' by degree d, then it induces a polynomial map of degree d in $\mathbb{Q}^a$.

Computing $|\omega_j^+ - \omega^+|$.

Fact. If B is a maximal open ball of a dynamical ball D_ℓ of level ℓ, then B contains at most one dynamical ball of level $\ell + 1$.

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Consequence. If $D_\ell(\omega^+) = D_\ell(\omega_j^+)$ but $D_{\ell+1}(\omega^+) \neq D_{\ell+1}(\omega_j^+)$, then $|\omega_j^+ - \omega^+|$ is the diameter of $D_\ell(\omega^+) = D_\ell(\omega_j^+)$.

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The parameters of level n

$$\{|\psi_\nu^{n-1}(\omega^+)| \leq 1\}$$

are a disjoint union of closed balls $\mathcal{D}_n$ called level n parameter balls.

Parameter ball, dynamical balls and branch dynamics

Proposition. Assume $\mathcal{D}_\ell$ is a level ℓ parameter ball. Then:

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- The ψ_ν action on maximal open balls is also independent of $\nu \in \mathcal{D}_\ell$.

If p_ℓ is the smallest integer q such that

$$\omega^+ \in \psi_\nu^{q-1}(D_\ell(\nu)),$$

then every periodic parameter in $\mathcal{D}_\ell$ has period at least p_ℓ.

Centers

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Proof. The map $T : \mathcal{D}_\ell \to \mathcal{D}_\ell$ defined by

$$T(\nu) = \left(\psi_\nu^{p_\ell - 1}|_{D_\ell(\nu)}\right)^{-1}(\omega^+)$$

is a strict contraction. □

Such ν is called the center of $\mathcal{D}_\ell$.

Level $\ell + 1$ correspondence

Proposition. Let B be a maximal open ball of a parameter ball $\mathcal{D}_\ell$ and consider any $\nu \in \mathcal{D}_\ell$.

B contains a parameter ball of level $\ell + 1$
if and only if
B contains a dynamical ball of level $\ell + 1$.

In this case, the level $\ell + 1$ balls are unique.

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That is, ν, ν' belong to B
Thus ν, ν' belong to the unique level $\ell + 1$ parameter ball contained in B.
Hence, $\mathcal{D}_{\ell+1}(\nu) = \mathcal{D}_{\ell+1}(\nu')$.

Homework

Given a sequence

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With centers

$$\nu_0, \nu_1, \cdots .$$

Which lie in

$$L_0((t^{1/m_0})) \subset L_1((t^{1/m_1})) \subset \cdots$$

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Compute L_k.