Puiseux series dynamics and leading monomials of escape regions

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Workshop on Moduli Spaces Associated to Dynamical Systems
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Monic centered cubic polynomials with marked critical points:

\[ f_{a,v} : z \mapsto (z - a)^2(z + 2a) + v \]

where \((a, v) \in \mathbb{C}^2\).

After identification of \((a, v)\) with \((-a, -v)\) one obtains the moduli space of cubic polynomials with marked critical points.

Critical points of \(f_{a,v}\) are \(\pm a\).

Critical value: \(v = f_{a,v}(a)\).

Co-critical value \(-2a\), since \(v = f_{a,v}(a) = f_{a,v}(-2a)\).
Parameter space

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What is the Euler characteristic of the smooth compactification of $S_p$?

Requires to compute the number $N_p$ of “escape regions”. (De Marco and Schiff, De Marco-Pilgrim).
Escape Regions

The connectedness locus

\[ C(S_p) = \{ f_{a,v} \in S_p \mid f_{a,v}^n(-a) \not\to \infty \} \]

is compact.

The escape locus

\[ E(S_p) = \{ f_{a,v} \in S_p \mid f_{a,v}^n(-a) \to \infty \} \]

is open and every connected component is unbounded.

An escape region \( U \) is a connected component of \( E(S_p) \):

\( U \) is conformally isomorphic to punctured disk.

The puncture is at \( \infty \).
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A escape region \( \mathcal{U} \) is a connected component of \( E(S_p) \):

\( \mathcal{U} \) is conformally isomorphic to punctured disk.

The puncture is at \( \infty_{\mathcal{U}} \).
Asymptotics of critical periodic orbit

For $f_{a,v} \in U$, let

$$a_0 = +a \mapsto a_1 = v \mapsto \cdots \mapsto a_{p-1} \mapsto a_0.$$
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Dynamical space picture. After conjugacy, $f_{a,v}(az)/a$, we have:
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a_j = \begin{cases} 
  a + o(a) & \text{or} \\
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Bonifant and Milnor:

Do the leading terms of $a_j - a$, for $j = 1, \ldots, p - 1$ determine $U$ uniquely?
There exists $\mu \geq 1$ and a local coordinate $\zeta$ for $\mathcal{U}$ near $\infty$ such that

$$a = \frac{1}{\zeta^\mu}.$$
Leading monomials

There exists $\mu \geq 1$ and a local coordinate $\zeta$ for $U$ near $\infty$ such that

$$a = \frac{1}{\zeta^\mu}.$$ 

Then,

$$\frac{a_j - a}{a} = \text{holomorphic}(\zeta) = \sum_{k \geq k_0} c_k \zeta^k = \sum_{k \geq k_0} c_k \left(\frac{1}{a}\right)^{k/\mu}.$$
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Leading monomial

$$m_j = c_{k_0} \left(\frac{1}{a}\right)^{k_0/\mu}.$$
**Theorem.** *The leading monomial vector*

\[ \vec{m} = (m_1, \ldots, m_{p-1}, 0) \]

*determines the escape region uniquely.*
Theorem. The leading monomial vector

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Corollary. (Bonifant, -, Milnor) Assume that the leading monomial vector of \( \mathcal{U} \) is

\[ (c_1 a^{-k_1/\mu}, \ldots, c_{p-1} a^{-k_{p-1}/\mu}). \]

Then, for all \( j \),

\[ a_j \in \mathbb{Q}(c_1, \ldots, c_{p-1})(a^{-1/\mu}). \]
Equations

For an escape region $\mathcal{U}$,

$$v = a_1 = (+a \text{ or } -2a) + a \cdot \sum_{k \geq k_0} c_k \left(\frac{1}{a}\right)^{k/\mu}$$

is a solution of

$$f^p_{a,v}(+a) = +a \quad (*)$$

in some extension of $\mathbb{Q}((1/a))$. 
Put $t$ instead of $1/a$ to get $\mathbb{Q}((t))$ and study solutions of (*) in the algebraic closure of $\mathbb{Q}((t))$: 

$$\mathbb{Q}^a \ll t \gg = \bigcup \mathbb{Q}^a((t^{1/m})).$$
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Complete it to get $\mathbb{L}$. 
Non-Archimedean problem

For \( \nu \in \mathbb{L} \), consider

\[
\psi_\nu(z) = t^{-2}(z - 1)(z + 2) + \nu \in \mathbb{L}[z].
\]
Non-Archimedean problem

For $\nu \in \mathbb{L}$, consider

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$\nu$ is periodic parameter if $\omega^+ = +1$ is periodic under $\psi_\nu$. 
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In this case:

$$\omega^+ = +1 \Longleftrightarrow \omega_1^+ = \nu \Longleftrightarrow \cdots \Longleftrightarrow \omega_{p-1}^+ \Longleftrightarrow \omega^+. $$
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$$\omega^+ = +1 \mapsto \omega_1^+ = \nu \mapsto \cdots \mapsto \omega_{p-1}^+ \mapsto \omega^+.$$ 

It follows,

$$\omega_j^+ = \begin{cases} +1 + c \cdot t^{k/m} + h.o.t. \\ -2 + d \cdot t^{\ell/n} + h.o.t. \end{cases}$$
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\end{cases}
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The corresponding **leading monomial** are

\[
\mathbf{m}(\omega_j^+ - \omega^+) = \begin{cases} 
c \cdot t^{k/m} \\
-3
\end{cases}
\]

respectively.
Theorem. The leading monomial vector

$$\vec{m} = (m_1, \ldots, m_{p-1}, 0)$$

determines the periodic parameter $\nu$ uniquely.
The filled Julia set of $\psi_\nu$ is

$$K(\psi_\nu) = \{z \in \mathbb{L} | \psi_\nu^n(z) \not\to \infty\}.$$
Dynamical Space

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\(|\nu| \leq 1 \implies \psi_\nu^n(z) \to \infty \) when \(|z| > 1\).
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Level 0 dynamical ball $D_0 = \{|z| \leq 1\}$. 
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Level $n$ set

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Each level $n + 1$ ball is contained in, and maps onto, a level $n$ ball.
Maximal open balls in a closed ball $D$ are parametrized by $\mathbb{Q}^a$.

If $\psi$ maps $D$ onto $D'$ by degree $d$, then it induces a polynomial map of degree $d$ in $\mathbb{Q}^a$. 
Computing $|\omega_j^+ - \omega^+|$.

**Fact.** If $B$ is a maximal open ball of a dynamical ball $D_\ell$ of level $\ell$, then $B$ contains at most one dynamical ball of level $\ell + 1$. 
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**Fact.** If $B$ is a maximal open ball of a dynamical ball $D_\ell$ of level $\ell$, then $B$ contains at most one dynamical ball of level $\ell + 1$.

**Consequence.** If $D_\ell(\omega^+) = D_\ell(\omega_j^+)$ but $D_{\ell+1}(\omega^+) \neq D_{\ell+1}(\omega_j^+)$, then $|\omega_j^+ - \omega^+|$ is the diameter of $D_\ell(\omega^+) = D_\ell(\omega_j^+)$. 
Parameter space

The level 0 parameter ball is

\[ \mathcal{D}_0 = \{ |\psi_{\nu}(\omega^+) = \nu| \leq 1 \}. \]
Parameter space

The level 0 parameter ball is

\[ \mathcal{D}_0 = \{ |\psi_v(\omega^+) = v| \leq 1 \} . \]

The parameters of level \( n \)

\[ \{ |\psi_n^{-1}(\omega^+)| \leq 1 \} \]

are a disjoint union of closed balls \( \mathcal{D}_n \) called level \( n \) parameter balls.
Proposition. Assume $D_\ell$ is a level $\ell$ parameter ball. Then:

$$D_\ell = D_\ell(\nu) \text{ for all } \nu \in D_\ell.$$
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The level $\ell + 1 - j$ dynamical ball $\psi^\nu_{\nu}(\mathcal{D}_\ell(\nu))$ is independent of $\nu \in \mathcal{D}_\ell$. 
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The $\psi_\nu$ action on maximal open balls is also independent of $\nu \in D_\ell$. 
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The $\psi_\nu$ action on maximal open balls is also independent of $\nu \in \mathcal{D}_\ell$.

If $p_\ell$ is the smallest integer $q$ such that

$$\omega^+ \in \psi^{q-1}_\nu(\mathcal{D}_\ell(\nu)),$$

then every periodic parameter in $\mathcal{D}_\ell$ has period at least $p_\ell$. 
Proposition. There exists a unique $\nu$ in $D_\ell$ which is periodic with period $p_\ell$. 

Proof. The map $T : D_\ell \to D_\ell$ defined by

$$T(\nu) = (\psi_{p_\ell} - 1 \nu|_{D_\ell(\nu)}) - 1(\omega + 1)$$

is a strict contraction. □

Such $\nu$ is called the center of $D_\ell$. 

Centers
Proposition. There exists a unique \( \nu \) in \( D_\ell \) which is periodic with period \( p_\ell \).

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\]

is a strict contraction.

Such \( \nu \) is called the center of \( D_\ell \).
Level $\ell + 1$ correspondence

**Proposition.** Let $B$ be a maximal open ball of a parameter ball $D_\ell$ and consider any $\nu \in D_\ell$.

- $B$ contains a parameter ball of level $\ell + 1$ if and only if
- $B$ contains a dynamical ball of level $\ell + 1$.

In this case, the level $\ell + 1$ balls are unique.
Leading monomials determine parameter balls

Assume $\nu, \nu'$ have the same leading monomial vector.
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Let us prove by induction on $\ell$.

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$$D_\ell(\nu) = D_\ell(\nu')$$

Take $\nu_\ell$ the center of level $\ell$.

There is a unique maximal open ball $B$ in $D_\ell(\nu_\ell)$ whose orbit is compatible with the leading monomial vector.
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Take $\nu_\ell$ the center of level $\ell$.

There is a unique maximal open ball $B$ in $D_\ell(\nu_\ell)$ whose orbit is compatible with the leading monomial vector.

That is, $\nu, \nu'$ belong to $B$

Thus $\nu, \nu'$ belong to the unique level $\ell + 1$ parameter ball contained in $B$.

Hence, $D_{\ell+1}(\nu) = D_{\ell+1}(\nu')$. 
Given a sequence
\[ \mathcal{D}_0 \supset \mathcal{D}_1 \supset \cdots . \]
With centers
\[ \nu_0, \nu_1, \cdots . \]
Which lie in
\[ L_0((t^{1/m_0})) \subset L_1((t^{1/m_1})) \subset \cdots \]
Branner and Hubbard tell us how to compute \( m_k \).
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With centers

$$\nu_0, \nu_1, \cdots$$

Which lie in

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Compute $L_k$. 

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**Homework**