Moduli Spaces Associated to Dynamical Systems Workshop
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Rational maps with prescribed critical values

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The questions

Two rational maps $f, g : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ are sad to be isomorphic (or covering equivalent) if there is a Möbius transformation M such that $f \circ M = g$.

$$\begin{array}{ccc} \overline{\mathbb{C}} & \stackrel{M}{\to} & \overline{\mathbb{C}} \\ g \searrow & & \swarrow f \\ & \overline{\mathbb{C}} & \end{array}$$

Clearly f and g share the same degree and the same critical value set.

Questions. Given a degree d and a set of $V \subset \overline{\mathbb{C}}$

1. Enumerate the degree d isomorphism classes realizing V as the critical value set.
2. Give a combinatorial description of these classes.
3. Compute the coefficients of representatives.
4. Study the bifurcations with V.
5. Is this helpful for implementing Thurston's algorithm?

Motivations

1. two dimensional-quantum chromodynamics and the related string theories
2. Subgroups of symmetric groups
3. Graph theory, combinatorics
4. Algebraic geometry, singularity theory, Gromov-Witten invariants, complements of discriminants in diverse moduli spaces
5. Holomorphic dynamics

Enumeration results

A simple remark: the enumeration does not depend on the position of V, but only on the collection $\mathbf{X} = \{X(v)\}_{v \in V}$ (the passport) of the **branching types** over the points of V, where

$$X(v) = 1^{m_1} 2^{m_2} \cdots d^{m_d}$$

with m_i denoting the number of points above v of local degree i. Clearly

$$m_1 + 2 \cdot m_2 + \cdots + d \cdot m_d = d \ .$$

Any orientation preserving homeomorphism $h : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ induces a bijection between isomorphism classes over V and over $h(V)$ with the same passport.

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{\exists H} & \overline{\mathbb{C}} \\ f \downarrow & & \downarrow \exists F \\ \overline{\mathbb{C}}, V & \xrightarrow[h]{} & \overline{\mathbb{C}}, h(V) \end{array}$$

Enumeration with simple critical values

1. (Lyashko-Looijenga) For V a set of $d-1$ distinct points in $\mathbb{C}$, the number of isomorphism classes of polynomials with V as the critical value set is equal to d^{d-3}:

$$1, 1, 1, 4, 25, 216, 2401, \cdots .$$

2. (Hurwitz) For V a set of $2d-2$ distinct points in $\overline{\mathbb{C}}$, the number of isomorphism classes of rational maps with V as the critical value set is equal to $\frac{(2d-2)!}{d!} d^{d-3}$:

$$1, 1, 4, 120, 8400, 1088640, 228191040, \cdots .$$

3. Real polynomials with real critical points and simple critical values : with generating function $\sec x + \tan x$ (heard from Thurston, known result?) :

$$1, 1, 1, 2, 5, 16, 61, 272, \cdots .$$

4. Real rational maps of degree d with real critical points and simple critical values : $\leq 2d-2\,!$.

Enumerations

• Zvonkin & Lando: a formula for every polynomial passport
• Ekedahl, Lando, Shapiro & Vainshtein: a formula for $h_{g,X(\infty)}$, where g is the genus of the covering surface, $X(\infty)$ is the branching type over ∞, all other critical values are simple.

Ideas of the proof of Lyashko-Looijenga's theorem

$\mathcal{P}$:= the set of monic centered polynomials $f(z)$ of degree d ($\approx \mathbb{C}^{d-1}$);
$\mathcal{D}$:= the set of monic polynomials $d(t)$ of degree $d-1$ ($\approx \mathbb{C}^{d-1}$), and

$$\mathcal{LL} : \mathcal{P} \to \mathcal{D}, f(z) \mapsto d(t) := \prod_{v \text{ a critical value of } f} (t - v).$$

In particular, $f(z)$ has multiple critical values iff $d(t)$ has multiple roots.
Theorem: $\mathcal{LL}; \mathcal{P} \to \mathcal{D}$ *is polynomial, quasi-homogeneous,*
$\mathcal{LL}^{-1}(\mathbf{0}) = \{\mathbf{0}\}$,
a covering over the complement of the discriminant in $\mathcal{D}$,
with covering multiplicity=d^{d-2},
the number of isomorphism classes of polynomials with simple critical values is $d^{d-2}/d = d^{d-3}$.

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Actually, $\mathcal{LL}(f) = t \mapsto \text{discriminant}(f(z) - t, z)/(-d^d)$.
For example,

$$\text{discriminant}(z^3 + p_2 z + p_3 - t, z)/(-27) = t^2 - 2p_3 t + \left(\frac{4p_2^3}{27} + p_3^2\right)$$

$$\text{discriminant}(z^4 + p_2 z^2 + p_3 z + p_4 - t, z)/(-256) =$$

$$t^3 + \frac{1}{2}(p_2^2 - 6p_4)t^2 + \frac{1}{16}(p_2^4 + 9p_2 p_3^2 - 16p_2^2 p_4 + 48p_4^2)t +$$

$$\frac{1}{256}(4p_2^3 p_3^2 + 27p_3^4 - 16p_2^4 p_4 - 144p_2 p_3^2 p_4 + 128p_2^2 p_4^2 - 256p_4^3)$$

p_i has weight i.

The rest follows from (quasi-homogeneous) Bezout's theorem.
For the non-generic polynomial branching types:
1. if there is only one multiple cr. value, find a vector space representation and do the same trick;
2. if there are more, lift $\mathcal{LL}$ to a map from the space of ordered critical sets into the space of (centered) ordered critical value sets and find a transversality there.

(Arnold) Rational maps with at most three poles can be treated similarly.

Otherwise, the space of meromorphic functions with n poles from curves of genus g onto $\overline{\mathbb{C}}$ is fibered over the moduli space of the curves of genus g with n distinguished points.
Intersection theory (instead of Bezout's theorem) in the moduli space is needed, and before that the moduli space must be compactified and its fibered space must be completed.
The Deligne-Mumford compactification gives the enumeration (with a fixed branching type at poles).

Combinatorics, pullback of a CV polygon

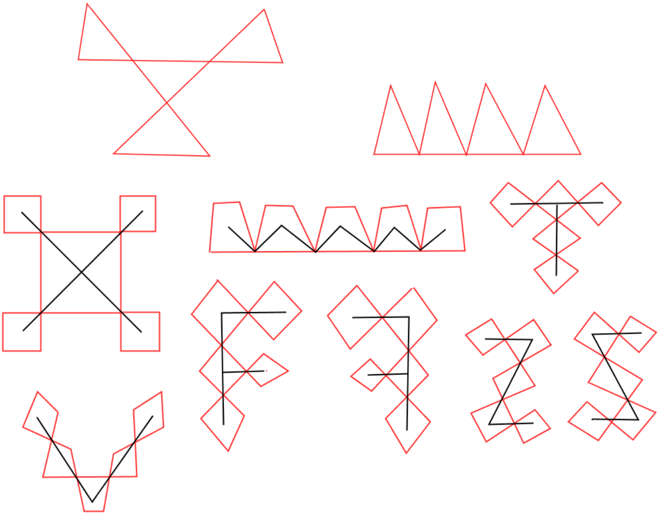
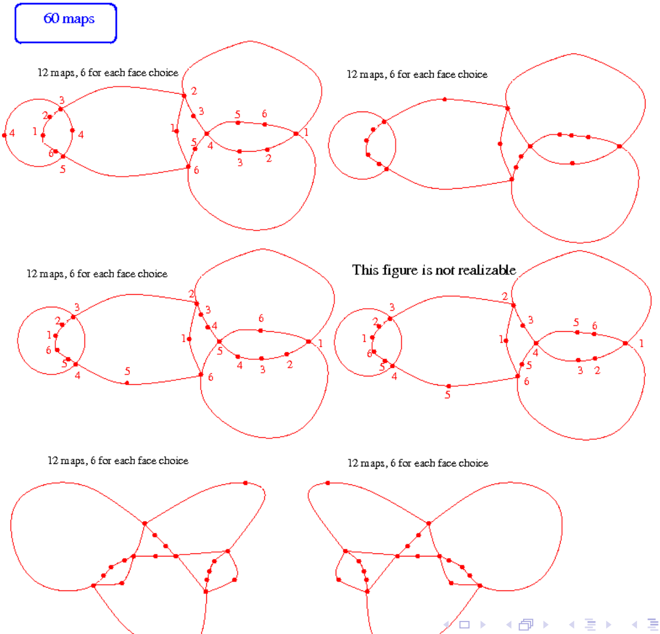
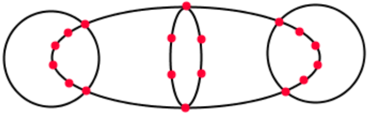


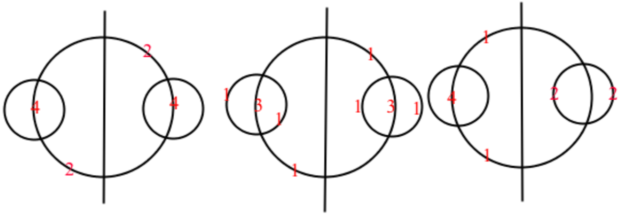
Figure: Degree 4 and 5 polynomial combinatorics

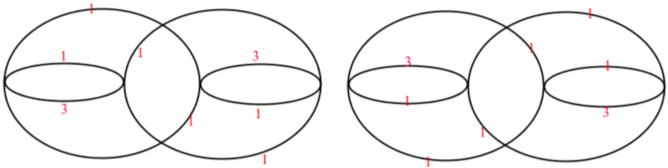
Quartic rational maps, 120 isomorphism classes



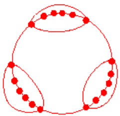
36 maps, 3 for each face choice of each figure





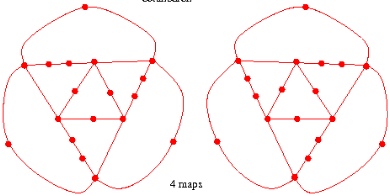


24 maps

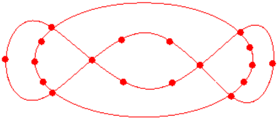


2 maps

octahedron



4 maps



6 maps

3 for each choice of a face

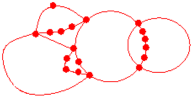
there are no other dotting patterns

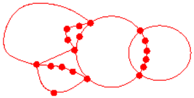
due to the outer piece with 4

critical accesses and the center

piece with two critical accesses

6 maps (symmetric with face choice)





6 maps (symmetric with face choice)

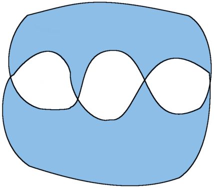
Underlying 4-valent planar graphs

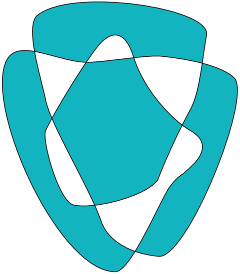
Theorem (W. Thurston, 2010) *A 4-valent connected planar graph* Γ *is homeomorphic to the pullback of a CV polygon of some rational map if and only if*

1. *(global balance) In an alternating coloring of the complementary faces, there are equal numbers of white and blue faces;*
2. *(local balance) For any oriented simple closed curve drawn in the graph that keeps blue faces on the left and white on the right (except at the corners), there are strictly more blue faces than white faces on the left side.*

A 4-valent planar graph with the above conditions will be said **balanced**.

Counter examples, globally imbalanced





4 blue-green regions
6 white regions
Two regions same-color regions
share at most 3 vertices

Counter examples, locally imbalanced

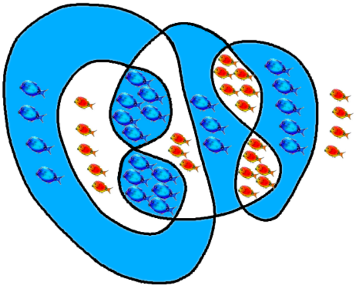
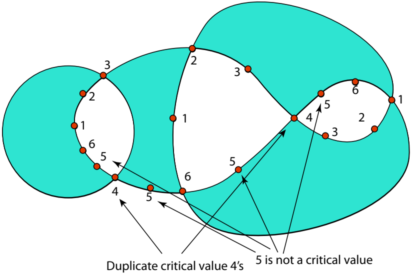


Figure: In each face of this diagram, the number of shown fish plus the number of corners equal to 8 (the degree). There are the same number of blue faces and white faces, so the graph is globally balanced. But the right half of the diagram violates the local balance condition.

Proof.

1. This condition is equivalent to the possibility of distributing the 2-valent dotted vertices, which is a marriage problem or graph flow problem in graph theory.
2. The dots can be consistently labelled, due to a cohomology argument with coefficients in $\mathbb{Z}/(|V| \cdot \mathbb{Z})$.
3. Perturb to get rid of duplicate critical points.

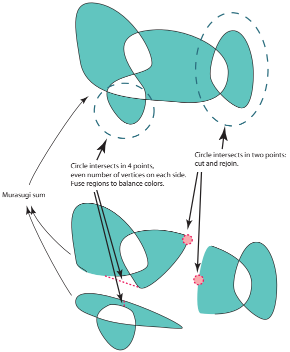


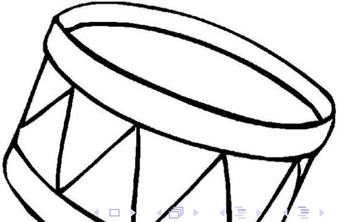
Note how the numbering goes clockwise around each white face and counterclockwise around each blue face. There's a duplicate critical value: the two vertices labeled 4 go to the same point, and none of the vertices labeled 5 are at a crossing point, so 5 is not a critical value. With such a diagram, one can always perturb it to make the duplicate critical values distinct (in two different ways to turn one of label 4 into 5, or more if there are more duplications), and eliminate all dots with a label that is not on a critical point, to get valid labelings.

Structures and decompositions

Following ideas of W.Thurston, in parallel to Martin Bridgeman's link projection classification, Tomasini obtains:

Theorem. *Every balanced graph can be decomposed, after cutting along essential Jordan curves intersecting the graph at 2 or 4 points, into hyperbolic graphs by pinching simultaneously Turksheads in opposite colored components.*





Computing coefficients and study bifurcations

The scilab code of H.H. Rugh uses analytic continuation of achieve partially this goal...

Implementing Thurston's algorithm?

Real maps with real postcritical set ?
The general case?
See mathematica code of Kathryn Lindsey....