Variations on a Theme: Fields of Definition, Fields of Moduli, Automorphisms, and Twists

Michelle Manes (mmanes@math.hawaii.edu)

ICERM Workshop
Moduli Spaces Associated to Dynamical Systems
17 April, 2012

Definitions

Definition

Let $\phi \in \mathrm{Rat}_d^N$. A field K'/K is a *field of definition* for ϕ if $\phi^f \in \mathrm{Rat}_d^N(K')$ for some $f \in \mathrm{PGL}_{N+1}$.

Definitions

Definition

Let $\phi \in \mathrm{Rat}_d^N$. A field K'/K is a *field of definition* for ϕ if $\phi^f \in \mathrm{Rat}_d^N(K')$ for some $f \in \mathrm{PGL}_{N+1}$.

Definition

Let $\phi \in \mathrm{Rat}_d^N$, and define

$$G_\phi = \{\sigma \in G_K \mid \phi^\sigma \text{ is } \overline{K} \text{ equivalent to } \phi\}.$$

The *field of moduli* of ϕ is the fixed field $\overline{K}^{G_\phi}$.

Definitions

- The field of moduli of ϕ is the smallest field L with the property that for every $\sigma \in \mathrm{Gal}(\overline{K}/L)$ there is some $f_\sigma \in \mathrm{PGL}_{N+1}$ such that $\phi^\sigma = \phi^{f_\sigma}$.

Definitions

- The field of moduli of ϕ is the smallest field L with the property that for every $\sigma \in \mathrm{Gal}(\overline{K}/L)$ there is some $f_\sigma \in \mathrm{PGL}_{N+1}$ such that $\phi^\sigma = \phi^{f_\sigma}$.
- The field of moduli for ϕ is contained in every field of definition.

Definitions

- The field of moduli of ϕ is the smallest field L with the property that for every $\sigma \in \text{Gal}(\overline{K}/L)$ there is some $f_\sigma \in \text{PGL}_{N+1}$ such that $\phi^\sigma = \phi^{f_\sigma}$.
- The field of moduli for ϕ is contained in every field of definition.
- Equality???

FOD = FOM criterion

Proposition (Hutz, M.)

Let $\xi \in M_d^N(K)$ be a dynamical system with $\mathrm{Aut}_\phi = \{\mathrm{id}\}$, and let $D = \sum_{j=0}^{N} d^N$.

If $\gcd(D, N+1) = 1$, then K is a field of definition of ξ.

FOD = FOM criterion

Idea: If $[\phi] \in M_d^N(K)$, then you get a cohomology class

$$f : \mathrm{Gal}(\bar{K}/K) \to \mathrm{PGL}_{N+1}$$
$$\sigma \mapsto f_\sigma$$

FOD = FOM criterion

Idea: If $[\phi] \in M_d^N(K)$, then you get a cohomology class

$$f : \mathrm{Gal}(\bar{K}/K) \to \mathrm{PGL}_{N+1}$$
$$\sigma \mapsto f_\sigma$$

Twists of $\mathbb{P}^N$ are in 1-1 correspondence with cocyles:

$$i : \mathbb{P}^N \to X$$
$$\sigma \mapsto i^{-1} i^\sigma$$

$$[\phi] \in M_d^N(K) \rightsquigarrow \text{cocycle } c_\phi \rightsquigarrow X_{c_\phi}$$
$$K \text{ is FOD for } \phi \iff c_\phi \text{ trivial } \iff X_{c_\phi}/K$$

When $\gcd(D, N+1) = 1$, we can find a K-rational zero-cycle on X_{c_ϕ}.

FOD = FOM criterion

If $N = 1$, then $D = d + 1$, and the test is on $\gcd(d + 1, 2)$.

Corollary (Silverman)

If d is even, then the field of moduli is a field of definition.

FOD = FOM criterion

If $N = 1$, then $D = d + 1$, and the test is on $\gcd(d + 1, 2)$.

Corollary (Silverman)

If d is even, then the field of moduli is a field of definition.

Result in $\mathbb{P}^1$ doesn't require $\mathrm{Aut}(\phi) = \mathrm{id}$.

Proof requires knowledge of the possible automorphism groups and "cohomology lifting."

Example (Silverman)

$\phi(z) = i\left(\frac{z-1}{z+1}\right)^3$. So $\mathbb{Q}(i)$ is a field of definition for ϕ.

Let σ represent complex conjugation, then

$$\phi^\sigma = \phi^f \qquad \text{for} \qquad f = -\frac{1}{z}.$$

Hence, $\mathbb{Q}$ is the field of moduli for ϕ.

K is a field of definition for ϕ iff $-1 \in N_{K(i)/K}(K(i)^*)$.

Normal Form for M_2

Lemma (Milnor)

Let $\phi \in \mathrm{Rat}_2$ *have multipliers* $\lambda_1, \lambda_2, \lambda_3$.

1. *If not all three multipliers are* 1, ϕ *is conjugate to a map of the form:*
$$\frac{z^2 + \lambda_1 z}{\lambda_2 z + 1}.$$
2. *If all three multipliers are* 1, ϕ *is conjugate to:*
$$z + \frac{1}{z}.$$

Normal Form for M_2

Lemma (Milnor)

Let $\phi \in \mathrm{Rat}_2$ *have multipliers* $\lambda_1, \lambda_2, \lambda_3$.

1. *If not all three multipliers are* 1, ϕ *is conjugate to a map of the form:*
$$\frac{z^2 + \lambda_1 z}{\lambda_2 z + 1}.$$
2. *If all three multipliers are* 1, ϕ *is conjugate to:*
$$z + \frac{1}{z}.$$

Possible that $\phi \in K(z)$ but the conjugate map is not.

Arithmetic Normal Form for M_2

Theorem (M., Yasufuku)

Let $\phi \in \mathrm{Rat}_2(K)$ *have multipliers* $\lambda_1, \lambda_2, \lambda_3$.

1. *If the multipliers are distinct or if exactly two multipliers are* 1, *then* $\phi(z)$ *is conjugate over* K *to*

$$\frac{2z^2 + (2-\sigma_1)z + (2-\sigma_1)}{-z^2 + (2+\sigma_1)z + 2 - \sigma_1 - \sigma_2} \in K(z),$$

where σ_1 *and* σ_2 *are the first two symmetric functions of the multipliers.*

Furthermore, no two distinct maps of this form are conjugate to each other over $\overline{K}$.

Arithmetic Normal Form for M_2

Theorem (M., Yasufuku)

2. *If* $\lambda_1 = \lambda_2 \neq 1$ *and* $\lambda_3 \neq \lambda_1$ *or if* $\lambda_1 = \lambda_2 = \lambda_3 = 1$*, then* ψ *is conjugate over* K *to a map of the form*

$$\phi_{k,b}(z) = kz + \frac{b}{z}$$

with $k = \frac{\lambda_1+1}{2}$*, and* $b \in K^*$*.*

Furthermore, two such maps $\phi_{k,b}$ *and* $\phi_{k',b'}$ *are conjugate over* $\overline{K}$ *if and only if* $k = k'$*; they are conjugate over* K *if in addition* $b/b' \in (K^*)^2$*.*

Arithmetic Normal Form for M_2

Theorem (M., Yasufuku)

3. *If* $\lambda_1 = \lambda_2 = \lambda_3 = -2$, *then* ϕ *is conjugate over* K *to*

$$\theta_{d,k}(z) = \frac{kz^2 - 2dz + dk}{z^2 - 2kz + d},$$

with $k \in K, d \in K^*$, *and* $k^2 \neq d$.

All such maps are conjugate over $\overline{K}$. *Furthermore,* $\theta_{d,k}(z)$ *and* $\theta_{d',k'}(z)$ *are conjugate over* K *if and only if*

ugly, but easily testable condition

$\phi \in \mathrm{Hom}_d^1$

Aut(ϕ) is conjugate to one of the following:

1. Cyclic group of order n: $\mathcal{C}_n = \langle \zeta_n z \rangle$.
2. Dihedral group of order $2n$: $D_n = \left\langle \zeta_n z, \frac{1}{z} \right\rangle$.
3. Tetrahedral group: $A_4 = \left\langle -z, \frac{1}{z}, i\left(\frac{z+1}{z-1}\right) \right\rangle$.
4. Octahedral group: $S_4 = \left\langle iz, \frac{1}{z}, i\left(\frac{z+1}{z-1}\right) \right\rangle$.
5. Icosahedral group:

$$A_5 = \left\langle \zeta_5 z, -\frac{1}{z}, \frac{\left(\zeta_5 + \zeta_5^{-1}\right) z + 1}{z - \left(\zeta_5 + \zeta_5^{-1}\right)} \right\rangle .$$

$\phi \in \mathrm{Hom}_d^2$

1. Diagonal Abelian Groups (Cyclic Group of order n):

$$H = \begin{pmatrix} \zeta_n^a & 0 & 0 \\ 0 & \zeta_n^b & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \gcd(a,n) = 1 \text{ or } \gcd(b,n) = 1.$$

Proposition

Let r be the number of solutions to $x^2 \equiv 1 \mod n$. There are $n + r/2 - \varphi(n)/2$ representations of C_n of the form

$$\begin{pmatrix} \zeta_n & 0 & 0 \\ 0 & \zeta_n^a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\phi \in \mathrm{Hom}_d^2$

2. Subgroups of the form

$$\left\langle \begin{pmatrix} \zeta_p & 0 & 0 \\ 0 & a_i & b_i \\ 0 & c_i & d_i \end{pmatrix} \right\rangle,$$

where the lower right 2×2 matrices come from embedding the PGL_2 automorphism groups.

$\phi \in \mathrm{Hom}_d^2$

3. Subgroups that don't come from embedding PGL_2. (Lots of them.)

$$\left\langle \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0, \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\rangle$$

Higher Dimensions

Higher Dimensions

This slide intentionally left blank.

Computing the Absolute Automorphism Group

Algorithm (Faber, M., Viray)

Input:

- a nonconstant rational function $\phi \in K(z)$,
- an $\mathrm{Aut}_\phi(\bar{K})$-invariant subset $T = \{\tau_1, \ldots, \tau_n\} \subset \mathbb{P}^1(E)$ with $n \geq 3$.

Output: the set $\mathrm{Aut}_\phi(\bar{K})$

Computing the Absolute Automorphism Group

Algorithm (Faber, M., Viray)

- create an empty list L.
- for each triple of distinct integers $i, j, k \in \{1, \ldots, n\}$:
  - compute $s \in \mathrm{PGL}_2(\bar{K})$ by solving the linear system
  $$s(\tau_1) = \tau_i, \quad s(\tau_2) = \tau_j, \quad s(\tau_3) = \tau_k.$$
  - if $s \circ \phi = \phi \circ s$: append s to L.
- return L.

Computing the Automorphism Group for a Given Map

Proposition (Faber, M., Viray)

Let K be a number field and let $\phi \in K(z)$ a rational function of degree $d \geq 2$. Define S_0 to be the set of rational primes given by

$$S_0 = \{2\} \cup \left\{ p \textit{ odd} : \frac{p-1}{2} \,\Big|\, [K:\mathbb{Q}] \ \textit{ and } \ p \mid d(d^2-1) \right\},$$

and let S be the (finite) set of places of K of bad reduction for ϕ along with the places that divide a prime in S_0. Then $\mathrm{red}_v : \mathrm{Aut}_\phi(K) \to \mathrm{Aut}_\phi(\mathbb{F}_v)$ is a well-defined injective homomorphism for all places v outside S.

Realizing Maps with a Given Automorphism Group

Given a finite subgroup $\Gamma \in \mathrm{PGL}_2$, Doyle & McMullen give a way to construct all rational maps

$$\phi \in \bigcup_{2 \leq d \leq n} \mathrm{Rat}_d \text{ with } \Gamma \subseteq \mathrm{Aut}(\phi).$$

$$\text{inv. hom. one-form} \overset{1-1}{\longleftrightarrow} \text{inv. hom. rational map}$$

$$Fdx + Gdy \overset{1-1}{\longleftrightarrow} \phi = -\frac{G}{F}$$

Realizing Maps with a Given Automorphism Group

It is enough to find all (relative) invariant homogeneous polynomials, i.e. for each $\gamma \in \Gamma$ there is a character χ:

$$\gamma_* F = F(\gamma \bar{x}) = \chi(\gamma) F(\bar{x}).$$

Realizing Maps with a Given Automorphism Group

It is enough to find all (relative) invariant homogeneous polynomials, i.e. for each $\gamma \in \Gamma$ there is a character χ:

$$\gamma_* F = F(\gamma \bar{x}) = \chi(\gamma) F(\bar{x}).$$

$\lambda = (xdy - ydx)/2$. Every invariant one-form has the form:

$$F\lambda + dG,$$

where $\deg F + 2 = \deg G$,
F and G invariant with the same character.

Two useful gadgets

Molien Series: Given a finite group Γ (and character χ), outputs the power series

$$\sum_{k=0}^{\infty} \dim \left(K[\bar{x}]_k^{\Gamma}\right) t^k.$$

Reynolds Operator: Given a finite group Γ, (character χ), and all homogeneous monomials of a given degree, outputs all (relative) Γ-invariants of that degree.

Example

$$\Gamma = \mathcal{C}_4 = \left\langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

Example

$$\Gamma = \mathcal{C}_4 = \left\langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

Molien Series:

$$1 + t^2 + t^4 + t^6 + 3t^8 + 3t^{10} + 3t^{12} + 3t^{14} + 5t^{16} + O(t^{18})$$

Example

$$\Gamma = \mathcal{C}_4 = \left\langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

Molien Series:

$$1 + t^2 + t^4 + t^6 + 3t^8 + 3t^{10} + 3t^{12} + 3t^{14} + 5t^{16} + O(t^{18})$$

Invariants of degree ≤ 8:

xy	x^2y^2	x^3y^3
x^4y^4	x^8	y^8

Example

$$\Gamma = \mathcal{C}_4 = \left\langle \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

Molien Series:

$$1 + t^2 + t^4 + t^6 + 3t^8 + 3t^{10} + 3t^{12} + 3t^{14} + 5t^{16} + O(t^{18})$$

Invariants of degree ≤ 8:

$$\begin{array}{ccc} xy & x^2y^2 & x^3y^3 \\ x^4y^4 & x^8 & y^8 \end{array}$$

Some maps with $\Gamma \subseteq \mathrm{Aut}(\phi)$:

$$\phi_1(z) = \frac{z^4 + 16}{z^3} \qquad \phi_2(z) = \frac{z^9 + 9z}{z^8 - 1}$$

Example

$$\Gamma = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle \qquad \text{(also cyclic of order 4).}$$

Example

$$\Gamma = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle \qquad \text{(also cyclic of order 4).}$$

Invariants of degree ≤ 8:

$$\begin{array}{ll} x^2+y^2 & x^8+12x^6y^2-20x^4y^4+12x^2y^6+y^8 \\ x^4+2x^2y^2+y^4 & x^8-4x^6y^2+22x^4y^4-4x^2y^6+y^8 \\ x^6+3x^4y^2+3x^2y^4+y^6 & x^7y-7x^5y^3+7x^3y^5-xy^7 \end{array}$$

Some maps with $\Gamma \subseteq \mathrm{Aut}(\phi)$:

$$\phi_1(z) = -\frac{z^7+24z^6+3z^5-40z^4+3z^3+72z^2+z+8}{8z^7-z^6+72z^5-3z^4-40z^3-3z^2+24z-1}$$

$$\phi_2(z) = -\frac{z\left(3z^6-39z^4+73z^2-13\right)}{13z^6-73z^4+39z^2-3}$$

Exact Automorphism Groups?

Proposition (Hutz, M.)

Let

$$A_4 = \left\langle -z, \frac{1}{z}, i\left(\frac{z+1}{z-1}\right)\right\rangle \text{ and } S_4 = \left\langle iz, \frac{1}{z}, i\left(\frac{z+1}{z-1}\right)\right\rangle.$$

If $\phi \in \mathbb{Q}(z)$ *satisfies* $A_4 \subseteq \mathrm{Aut}(\phi)$, *then in fact* $\mathrm{Aut}(\phi) = S_4$.

Exact Automorphism Groups?

Proposition (Hutz, M.)

Let

$$A_4 = \left\langle -z, \frac{1}{z}, i\left(\frac{z+1}{z-1}\right) \right\rangle \text{ and } S_4 = \left\langle iz, \frac{1}{z}, i\left(\frac{z+1}{z-1}\right) \right\rangle .$$

If $\phi \in \mathbb{Q}(z)$ *satisfies* $A_4 \subseteq \mathrm{Aut}(\phi)$, *then in fact* $\mathrm{Aut}(\phi) = S_4$.

Question

- How to construct maps $\phi \in K(z)$ with $\Gamma = \mathrm{Aut}(\phi)$ (or decide there are none)?
- How to construct maps $\phi \in K(z)$ with a subgroup of $\mathrm{Aut}(\phi)$ *conjugate* to (or equal to) Γ?

Automorphisms and Twists

$$\mathrm{Twist}(\phi/K) = \left\{ \begin{array}{c} K\text{-equivalence classes} \\ \text{of maps } \psi \in \mathrm{Hom}_d^N(K) \\ \text{such that } \psi \text{ is } \overline{K}\text{-equivalent to } \phi \end{array} \right\}.$$

Twists give automorphisms of the map ϕ:

$$\begin{aligned} f\phi f^{-1} &= (f\phi f^{-1})^\sigma \\ &= f^\sigma \phi (f^{-1})^\sigma. \\ \phi &= f^{-1} f^\sigma \phi (f^\sigma)^{-1} f \end{aligned}$$

$$f^{-1} f^\sigma \in \mathrm{Aut}(\phi).$$

Uniform Bounds on Preperiodic Points for Twists

Proposition (Levy, M., Thompson)

Let $\phi \in \mathrm{Hom}_d^N(K)$. *There is a uniform bound* B_ϕ *such that for all* $\psi \in \mathrm{Twist}(\phi/K)$,

$$\#\,\mathrm{PrePer}(\psi, \mathbb{P}^N_K) \leq B_\phi.$$

Idea: The degree of the field of definition of the twisting map f is bounded by $\#\,\mathrm{Aut}(\phi)$. Apply Northcott.

Cohomology and Twists

For an object X, twists give automorphisms:

$$g_\sigma : X \xrightarrow{\sigma(i^{-1})} Y \xrightarrow{i} X.$$

A twist gives a one-cocyle:

$$\begin{aligned} g : \mathrm{Gal}(\bar{K}/K) &\to \mathrm{Aut}(X) \\ \sigma &\mapsto i \circ \sigma(i^{-1}) \end{aligned}$$

Cohomology and Twists

For an object X, twists give automorphisms:

$$g_\sigma : X \xrightarrow{\sigma(i^{-1})} Y \xrightarrow{i} X.$$

A twist gives a one-cocyle:

$$\begin{aligned} g : \mathrm{Gal}(\bar{K}/K) &\to \mathrm{Aut}(X) \\ \sigma &\mapsto i \circ \sigma(i^{-1}) \end{aligned}$$

Does every one-cocyle come from a twist?

Cohomology and Twists

For an object X, twists give automorphisms:

$$g_\sigma : X \xrightarrow{\sigma(i^{-1})} Y \xrightarrow{i} X.$$

A twist gives a one-cocyle:

$$\begin{aligned} g : \mathrm{Gal}(\bar{K}/K) &\to \mathrm{Aut}(X) \\ \sigma &\mapsto i \circ \sigma(i^{-1}) \end{aligned}$$

Does every one-cocyle come from a twist?

For algebraic varieties, yes. For morphisms, sometimes.

$$\mathrm{Twist}(\phi/K) = \left\{ \begin{array}{c} \xi \in H^1\left(\mathrm{Gal}(\bar{K}/K), \mathrm{Aut}(\phi)\right) : \\ \xi \text{ becomes trivial in } H^1\left(\mathrm{Gal}(\bar{K}/K)\right), \mathrm{PGL}_{N+1} \end{array} \right\}.$$

Describing Twists

Question

Given $\phi \in \mathrm{Rat}_d$, can we write an explicit formula for all twists of ϕ?

Describing Twists

Question

Given $\phi \in \mathrm{Rat}_d$, can we write an explicit formula for all twists of ϕ?

- Done for Rat_2 by Arithmetic Normal Form Theorem.

Describing Twists

Question

Given $\phi \in \mathrm{Rat}_d$, can we write an explicit formula for all twists of ϕ?

- Done for Rat_2 by Arithmetic Normal Form Theorem.
- If $\mathrm{Aut}(\phi) = \{\zeta_n z\}$, then we have an isomorphism

$$K^*/K^{*n} \to \mathrm{Twist}(\phi/K)$$

$$b \mapsto \left[\frac{\phi\left(z\sqrt[n]{b}\right)}{\sqrt[n]{b}} \right].$$

Describing Twists

Question

Given $\phi \in \mathrm{Rat}_d$, can we write an explicit formula for all twists of ϕ?

- Done for Rat_2 by Arithmetic Normal Form Theorem.
- If $\mathrm{Aut}(\phi) = \{\zeta_n z\}$, then we have an isomorphism

$$K^*/{K^*}^n \to \mathrm{Twist}(\phi/K)$$
$$b \mapsto \left[\frac{\phi\left(z\sqrt[n]{b}\right)}{\sqrt[n]{b}}\right].$$

- More general presentation of $\mathcal{C}_n$? Other automorphism groups? Higher dimensions?