Totally Marked Rational Maps

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Rational maps of degree $d \geq 2$. (Mostly $d = 2$.)

Let K be an algebraically closed field of characteristic $> d$, or characteristic zero, let $\mathbb{P}^1 = \mathbb{P}^1(K)$, and let K_0 be the smallest subfield: $K_0 = \mathbb{Q}$ or $\mathbf{F}_p$.

Definition. A rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ is:

fixed point marked if we are given an ordered list

$$(z_1, z_2, \ldots, z_{d+1})$$

of its fixed points (not necessarily distinct);

critically marked if we are given an ordered list

$$(c_1, c_2, \ldots, c_{2d-2})$$

of its critical points (not necessarily distinct); and is

totally marked if we are given both.

Moduli Spaces: the quadratic case.

Collapsing the space Rat_d of all degree d rational maps under the action of $\mathrm{Aut}(\mathbb{P}^1)$ by conjugation, we obtain the corresponding moduli space rat_d. Similarly, for marked maps we obtain marked moduli spaces

$$\begin{array}{ccc} \mathrm{rat}_d^{\mathrm{tm}} & \longrightarrow & \mathrm{rat}_d^{\mathrm{fm}} \\ \downarrow & & \downarrow \\ \mathrm{rat}_d^{\mathrm{cm}} & \longrightarrow & \mathrm{rat}_d \end{array}$$

The unmarked space rat_2 is isomorphic to K^2. The surfaces $\mathrm{rat}_2^{\mathrm{fm}}$ and $\mathrm{rat}_2^{\mathrm{cm}}$ each have one singular point (at the class of $z \mapsto z + 1/z$ or $z \mapsto 1/z^2$ respectively).

Theorem 1. The totally marked moduli space $\mathrm{rat}_2^{\mathrm{tm}}$ is isomorphic to the smooth affine surface $V \subset K^3$ defined by the equation

$$x_1 + x_2 + x_3 \ + \ x_1\, x_2\, x_3 \ = \ 0 \, .$$

Some properties of this construction:

(1) There are 12 obvious automorphisms of V, and correspondingly 12 obvious automorphisms of $\mathrm{rat}_2^{\mathrm{fm}}$.

[**Example:** *Renumbering the first two fixed points in* $\mathrm{rat}_2^{\mathrm{fm}}$ *corresponds to the involution*

$$(x_1, x_2, x_3) \leftrightarrow (-x_2, -x_1, -x_3) \quad \text{of} \quad V.]$$

(2) The x_h and the fixed point multipliers λ_h are related by:

$$\lambda_h = 1 + x_j x_k \,, \qquad x_h^2 = 1 - \lambda_j \lambda_k \,,$$

where $\{h, j, k\}$ is any permutation of $\{1, 2, 3\}$.

(3) The subfield $K' = K_0(\{x_j\}) \subset K$ generated by the x_j is precisely the smallest field such that there is a representative rational map with all fixed points and critical points in $\mathbb{P}(K')$.

Examples.

For $x_1 = x_2 = x_3 = 0$ we obtain the conjugacy class of $f(z) = z + 1/z$, with $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

For $(x_1, x_2, x_3) = (1, 1, -1)$ we obtain the conjugacy class of $f(z) = z^2$, with $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 2)$.

For $x_1 = x_2 = x_3 = \pm\sqrt{-3}$ we obtain the conjugacy class of $f(z) = 1/z^2$, with $\lambda_1 = \lambda_2 = \lambda_3 = -2$.

Thus, in this last case $K' = K_0(\sqrt{-3})$.

[For further details, see “Hyperbolic Components”, Stony Brook IMS preprint: ims12-02, §9.]

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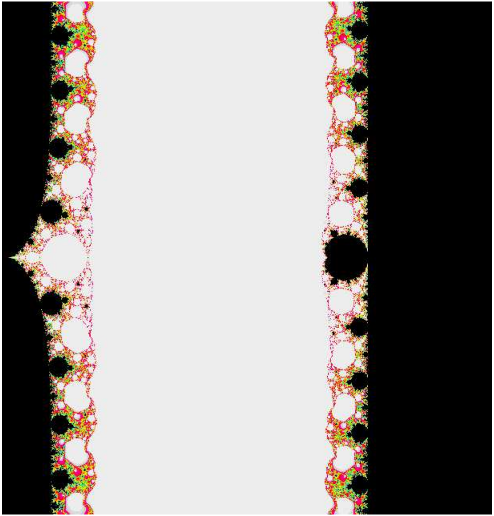
Now let K be the field of complex numbers $\mathbb{C}$.

Parameter space example: A 2-dimensional slice through Rat_2, centered at $z \mapsto 1/z^2$.



Maps for which both critical points converge to the same attracting period 2 orbit are colored white.

Problem: To study hyperbolic components in $\mathrm{rat}_2^{\mathrm{tm}}$, and their closures.

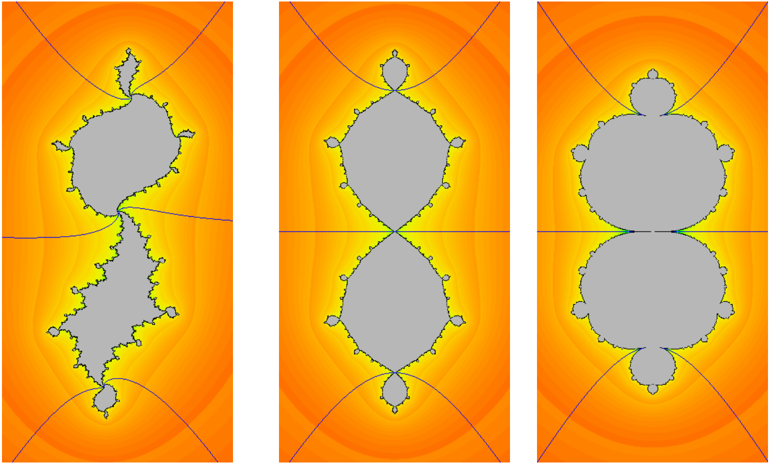
One motivation for the study of $\mathrm{rat}_2^{\mathrm{tm}}$ is that it provides a uniform and non-singular environment for studying hyperbolic components in the family of quadratic rational maps.

The "simplest" examples are the hyperbolic components centered at $f(z) = z^2$, with some choice of marking.
(For example if z_1 and z_2 are the two attracting fixed points, then we can number so that c_1 is in the basin of z_1 and c_2 is in the basin of z_2.)

Long Digression. Since it is similar, and easier to understand, I will first consider an analogous family of **cubic polynomials**.

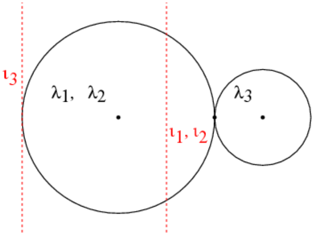
$\mathcal{H} = \{$monic centered cubic polynomials with 2 attracting fixed points$\}$.

Two Julia sets with $f \in \mathcal{H}$, and one with $f \in \partial\mathcal{H}$:



A typical point of $\mathcal{H}$. The center point. The "bad" point in $\partial\mathcal{H}$.

For maps in $\overline{\mathcal{H}}$, we can distinguish between upper and lower critical points, and between upper, lower and middle fixed points (with multipliers λ_1, λ_2, λ_3 respectively).



For maps in $\mathcal{H}$ the first two mulipliers λ_1, λ_2 lie in the unit disk. Hence the corresponding residue indices

$$\iota_j = \frac{1}{1-\lambda_j}$$

lie in the half-plane $\Re(\iota_j) > 1/2$. Since $\iota_1 + \iota_2 + \iota_3 = 0$, it follows that $\Re(\iota_3) < -1$, which implies that λ_3 lies in the disk $\mathbb{D}_{1/2}(3/2)$.

We can choose any λ_1 and λ_2 in $\mathbb{D}$, and solve uniquely for

$$\lambda_3 = \frac{3 - 2\lambda_1 - 2\lambda_2 + \lambda_1\lambda_2}{2 - \lambda_1 - \lambda_2}. \qquad (1)$$

(Compare equation (2) below.) In fact this is also true for any λ_1 and λ_2 in $\overline{\mathbb{D}}$, **unless** $\lambda_1 = \lambda_2 = 1$.

Moduli Space.

The moduli space $\mathrm{poly}_3^{\mathrm{fm}}$ for fixed point marked cubic polynomials can be identified with the smooth affine surface

$$3 - 2(\lambda_1 + \lambda_2 + \lambda_3) + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) = 0 . \qquad (2)$$

Proof. Every monic degree 3 polynomial with marked fixed points z_j has the form

$$f(z) = z + \prod(z - z_j) .$$

The moduli space $\mathrm{poly}_3^{\mathrm{fm}}$ can be obtained from the set of all $(z_1, z_2, z_3) \in \mathbb{C}^3$ by the identifications

$$(z_1, z_2, z_3) \sim (-z_1, -z_2, -z_3) \sim (z_1+c, z_2+c, z_3+c) \text{ for any } c .$$

Let $\delta_h = z_j - z_k$ where (h, j, k) is to be any cyclic permutation of $(1, 2, 3)$, so that $\delta_1 + \delta_2 + \delta_3 = 0$. Then a brief computation shows that the fixed point multipliers are given by

$$\lambda_h = 1 - \delta_j\delta_k ,$$

or in other words $1 - \lambda_h = \delta_j \, \delta_k$. It follows easily that

$$\sum_{h_1 < h_2} (1 - \lambda_{h_1})(1 - \lambda_{h_2}) \; = \; \delta_1 \, \delta_2 \, \delta_3 (\delta_1 + \delta_2 + \delta_3) \; = \; 0 \, ,$$

which is equivalent to the required equation (2).

Conversely, if we are given the λ_h satisfying (2), then the triple $(\delta_1, \, \delta_1, \, \delta_3)$ is uniquely determined up to sign. In fact the two-fold products $\delta_j \delta_k$ determine $\delta_1 \, \delta_2 \, \delta_3$ up to sign. If this three-fold product is non-zero, then a choice of sign determines all of the $\delta_h = (\delta_1 \, \delta_2 \, \delta_3)/(\delta_j \, \delta_k)$, while the case $\delta_h = 0 \; \Leftrightarrow \; \delta_j + \delta_k = 0$ is straightforward. □

Similarly one can show that

$$\delta_h^2 \; = \; \lambda_j + \lambda_k - 2 \, .$$

In particular, $|z_j - z_k| \; = \; \sqrt{|\lambda_j + \lambda_k - 2|}$.

Lemma. The closure $\overline{\mathcal{H}}$ of our hyperbolic component in $\mathrm{poly}_3^{\mathrm{fm}}$ is the semi-algebraic set consisting of all points $(\lambda_1, \lambda_2, \lambda_3) \in \overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \overline{\mathbb{D}}_{1/2}(3/2)$ satisfying equation (2).

Corollary 1. The set of all points in $\overline{\mathcal{H}}$ with $(\lambda_1, \lambda_2) \neq (1, 1)$ is homeomorphic to $\overline{\mathbb{D}} \times \overline{\mathbb{D}} \setminus \{(1, 1)\}$; while the set with $(\lambda_1, \lambda_2) = (1, 1)$ is homeomorphic to the disk $\overline{\mathbb{D}}_{1/2}(3/2)$.

Corollary 2. $\overline{\mathcal{H}}$ is **not** homeomorphic to a closed 4-dimensional ball.

The proof will show that $\pi\big(\partial\mathcal{H} \setminus \mathrm{point}\big) \neq 0$.

[**Remark 1.** It is easiest to prove Corollary 2 by first considering the analogous problem for **monic** cubic maps. (See the following pages.)

Remark 2. The bad behavior at the point $\lambda_1 = \lambda_2 = \lambda_3 = 1$ is related to the fact that this triple fixed point is a singular point of the variety defined by equation (2).

Remark 3. We could try understanding the situation over the triple fixed point by one or two blow-ups. However, this doesn't work unless we first resolve the singularity, for example by passing to the space of **monic** cubic polynomials with marked fixed points. Let:

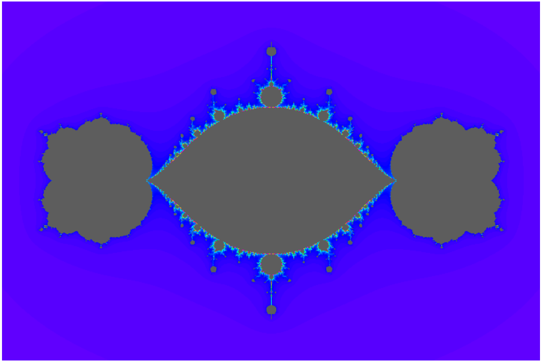
$$1 - \lambda_1 = \delta_2\delta_3 , \quad 1 - \lambda_2 = \delta_1\delta_3 , \quad 1 - \lambda_3 = \delta_1\delta_2$$

as above. Now blow up at $\delta_1 = \delta_2 = \delta_3 = 0$ by setting $\delta_2 = \delta_1 s$ or $\delta_1 = \delta_2 t$, where $s = 1/t$ ranges over $\widehat{\mathbb{C}}$. Then the λ_j can be expressed as polynomial functions, either of (δ_1, s) or of (δ_2, t) (or of either pair when $s \neq 0, \infty$).]

Now consider the same problem for the parametrized family of monic cubic polynomials with a marked fixed point at zero:

$$f(z) = z^3 + az^2 + \lambda z .$$

Theorem 2. The closure of the corresponding hyperbolic component $\widetilde{\mathcal{H}}$ in this family is a closed topological 4-ball.

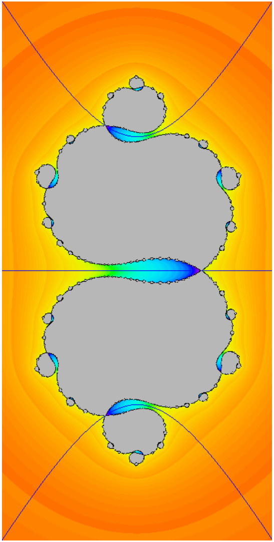


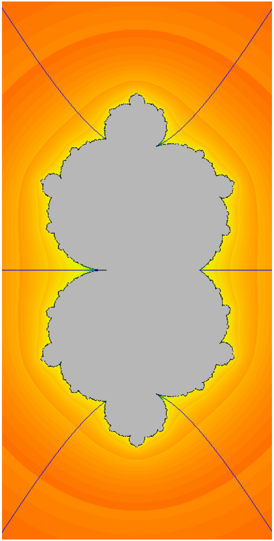
The a-plane for $\lambda = \lambda_3 = 3/2$.

[Remark 4. If we conjugate by the 180° rotation $z \mapsto -z$, so that $(a, \lambda_3) \mapsto (-a,\ \lambda_3)$ then the two critical points are interchanged, and the first two (upper and lower) fixed points are interchanged. For most points of $\overline{\mathcal{H}}$, either $\lambda_1 \neq \lambda_2$ so that $(\lambda_1\,,\ \lambda_2\,,\ \lambda_3)\ \neq\ (\lambda_2\,,\ \lambda_1\,,\ \lambda_3)$ or else we are in the symmetry locus $a = 0$, so that $-f(-z) = f(z)$. However, in the special case where the upper and lower fixed points crash together, so that $\lambda_1 = \lambda_2 = 1$, these two maps represent the same point of $\mathrm{poly}_3^{\mathrm{fm}}$ but different points of the monic family. This is the essential difference between these two families!

The following page shows one Julia set towards the right of the central hyperbolic component in the figure, and one at its right hand tip.

Each of these is distinct from its image under 180° rotation within the monic family, but in the moduli space $\mathrm{poly}_3^{\mathrm{fm}}$ the right hand one is identified with its rotated image.]





Hyperbolic Julia set, $a = 1.35$ Parabolic Julia set, $a = \sqrt{2}$.

On the right, the upper and lower fixed points have crashed together. hence the image under 180° rotation represents the same element of $\mathrm{rat}_2^{\mathrm{fm}}$.

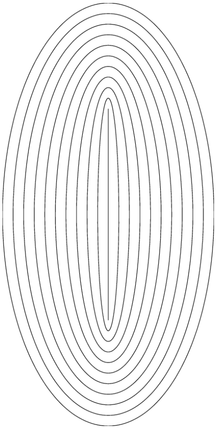
Proof Outline for Theorem 2.

Let $r_j = \Re(\iota_j)$. Recall that $\iota_1 + \iota_2 + \iota_3 = 0$, and that

$$r_1\,,\ r_2 \ \geq\ 1/2\,, \qquad \text{hence} \qquad r_3 \ \leq\ -1\,.$$

If we fix ι_3, then the difference $\Delta = \iota_1 - \iota_2$ varies over the strip $|\Re(\Delta)| \leq |r_3| - 1$. **We must also add two ideal points with $\Im(\Delta) = \pm\infty$ to this strip,** corresponding to the limit as λ_1 and λ_2 both tend to $+1$.

This strip, together with the two points at infinity, is homeomorphic to the region bounded by an ellipse in the plane. Think of this ellipse as being thin for $|r_3|$ near one and fat for $|r_3|$ large.



Family of ellipses filling out the plane.

[Looking only at $\partial\mathcal{H}$, we get an ellipse (respectively a line segment) for each ι_3 with real part < -1 (or $= -1$), hence a copy of $\mathbb{R}^2$ for each choice of $\Im(\iota_3) \in \mathbb{R}$.

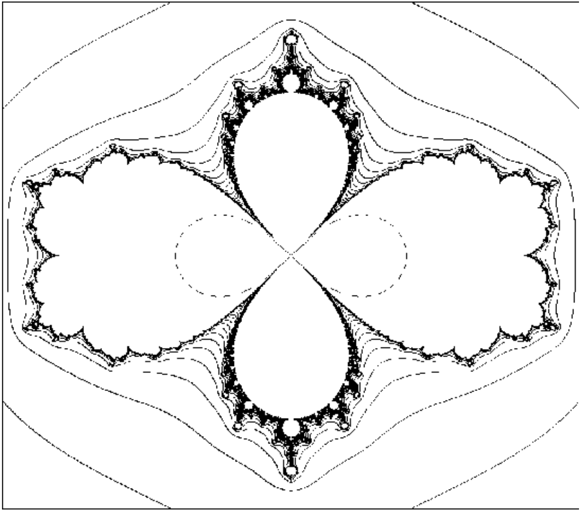
Thus $\partial\mathcal{H}$ with $(1, 1, 1)$ removed is homeomorphic to $\mathbb{R}^2 \times \mathbb{R}$.

It follows that $\partial\mathcal{H}$ is homeomorphic to a 3-sphere.

Similarly, $\overline{\mathcal{H}}$ is homeomorphic to a closed 4-ball. (Think of the family of ellipses filling out the plane as the contour map for a mountain, which reprents a 3-dimensional slice through $\mathcal{H}$.)

However, the corresponding argument for $\mathrm{poly}_3^{\mathrm{fm}}$ breaks down, since the limit as $\Delta \to +\infty$ and as $\Delta \to -\infty$ must be identified.

There is one such identification for each ι_3 with $\Re(\iota_3) \leq -1$.



The a-plane for $\lambda_3 = 1$.

These identified limit points fill out a 2-dimensional set bounded by a lemniscate in $\partial\mathcal{H} \cong S^3$. This lemniscate is dotted in the figure.

Any path in S^3 joining a point in the left lobe to the identified point in the right lobe represents a non-zero element of

$$\pi_1\left(\partial\mathcal{H}^{\mathrm{fm}} \setminus \{\text{triple fixed point}\}\right).$$

Remarks.

There are also regions bounded by upper and lower lemniscates in the preceding figure. These represent boundary points of $\mathcal{H}$ which have an attracting fixed point, and hence remain distinct in $\mathrm{rat}^{\mathrm{fm}}$.

This argument would work equally well in the space of fixed point marked monic centered cubic polynomial maps.

One could also use the space $\mathrm{poly}_3^{\mathrm{tm}}$ of totally marked polynomial maps in place of the monic family; but the result would be more complicated since the projection

$$\mathrm{poly}_3^{\mathrm{tm}} \rightarrow \mathrm{poly}_3^{\mathrm{fm}}$$

is ramified over the entire unicritical locus $\lambda = a^2/3$, which has a substantial intersection with $\mathcal{H}$. (This problem doesn't occur for quadratic rational maps, since the two critical points can never come together.)]

The corresponding quadratic rational example.

Now consider quadratic rational maps with two attracting fixed points.

In the moduli space $\mathrm{rat}_2^{\mathrm{fm}}$ *with marked fixed points, the hyperbolic component* $\mathcal{H}$ *for which the first two fixed points are attracting has a nasty closure, with* $\pi_1(\partial\mathcal{H} \smallsetminus \mathrm{point}) \neq 0$.

However, the closure of a corresponding component $\widetilde{\mathcal{H}} \subset \mathrm{rat}_2^{\mathrm{tm}}$ in the totally marked case is homeomorphic to a closed 4-dimensional ball with a single boundary point removed.

First consider the hyperbolic component $\mathcal{H} \subset \mathrm{rat}_2^{\mathrm{fm}}$ with marked fixed points.
The space rat_2^{fp} can be identified with the affine surface

$$\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3 + 2 = 0 . \qquad (3)$$

Again we want $|\lambda_1|, |\lambda_2| < 1$. Hence the real parts $r_j = \Re(\iota_j)$ satisfy $r_j > 1/2$. But now

$$\iota_1 + \iota_2 + \iota_3 = 1 , \qquad \text{hence} \quad r_3 < 0 .$$

It follows that λ_3 must belong to the half-space $\Re(\lambda_3) > 1$.
Now $\overline{\mathcal{H}}$ is the set of all

$$(\lambda_1, \lambda_2, \lambda_3) \quad \in \quad \overline{\mathbb{D}} \times \overline{\mathbb{D}} \times \{\Re(\lambda_3) \geq 1\}$$

which satisfy equation (3).

The discussion of the space $\overline{\mathcal{H}}$ in $\mathrm{rat}_2^{\mathrm{tm}}$ is almost the same as the discussion of the corresponding component for monic cubic polynomials. One just has to substitute $\Re(\iota_3) \le 0$ in place of $\Re(\iota_3) \le -1$.

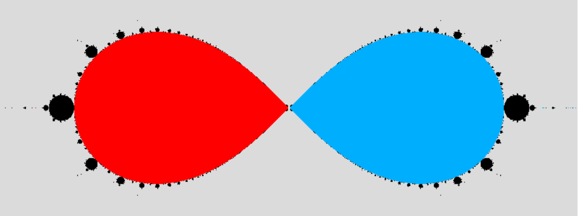
However there is one key difference: The point $\iota_3 = 0$ must be deleted, since it would correspond to $\lambda_3 = \infty$.

Thus, instead of $\partial\mathcal{H} \smallsetminus (\text{triple point})$ being homeomorphic to $\mathbb{R}^3$, it is homeomorphic to $\mathbb{R}^3$ with a line segment

$$\iota_3 = 0\,, \quad -\infty \le \Delta \le +\infty$$

removed, where $\Delta = \iota_1 - \iota_2$.

Therefore $\partial\mathcal{H}$ is non-compact, homeomorphic to S^3 with a line segment removed.



The plane of totally marked quadratic raional maps with a parabolic fixed point $z_1 = z_2$.

In (x_1, x_2, x_3)-coordinates, this plane is defined by: with

$$x_1 + x_2 \;=\; x_3 \;=\; 0 \,,$$

hence $$\lambda_1 \;=\; \lambda_2 \;=\; 1\,, \quad \lambda_3 \;=\; 1 + x_1 x_2 \;=\; 1 - x_1^2\,.$$

In the quotient space $\mathrm{rat}_2^{\mathrm{fm}}$, the points with coordinate x_1 and $-x_1$ are identified. In particular, the red and blue lobes (the intersection of this plane with $\partial\mathcal{H}$) are identified with each other.

It follows, as in the $\mathrm{poly}_3^{\mathrm{fm}}$ *polynomial case, that* $\partial\mathcal{H} \smallsetminus (\text{triple point})$ *is not simply connected.*

Higher Degrees ?

Theorem. *The space* $\mathrm{Rat}_d^{\mathrm{fm}}$ *of fixed point marked rational maps is a smooth complex manifold of dimension* $2d+1$*.*

Proof. Let $U \subset \mathrm{Rat}_d^{\mathrm{fm}}$ be the open subset consisting of all points $(f\,;\ z_1,\, z_2\,,\ \ldots\,,\ z_{d+1})$ such that the fixed points $z_1,\, z_2\,,\ \ldots\,,\ z_{d+1}$ are all finite.

Then $f(z)$ can be written uniquely as a quotient $p(z)/q(z)$ where $q(z)$ is a monic polynomial of degree d and $p(z)$ is a polynomial of degree $\le d$.

The fixed point equation $p(z)/q(z) = z$ takes the form

$$z\,q(z) - p(z) \;=\; (z-z_1)\,(z-z_2)\cdots(z-z_{d+1}) \;=\; 0\,.$$

Thus we can choose the polynomial $q(z)$ and the fixed points z_j independently, and solve for

$$p(z) \;=\; z\,q(z) \;-\; \prod_1^{d+1}(z-z_j)\,.$$

Here $p(z)$ and $q(z)$ must have no common zeros

$\iff$ the $q(z_j)$ must all be non-zero. $\square$

However, the corresponding moduli space $\mathrm{rat}_d^{\mathrm{fm}}$ will always have singular points at some (but not all) of the places where two or more fixed points come together.

Theorem. In either $\mathrm{rat}_d^{\mathrm{fm}}$ or $\mathrm{rat}_d^{\mathrm{tm}}$, any hyperbolic component in the connectedness locus is an open topological $(4d-4)$-cell.

In the fixed point marked case, the space $\mathrm{rat}_d^{\mathrm{fm}}$ can have singularities only where there are multiple fixed points.

However, I have no information about the singularities, if any, of $\mathrm{rat}_d^{\mathrm{tm}}$ (except as implied by the theorem above), and no information about the closures $\overline{\mathcal{H}}$.

For further details, again see "Hyperbolic Components", Stony Brook IMS preprint: ims12-02 (or arXiv:1205.2668) .