Automorphic Forms: A Brief Introduction

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Key Ingredient: Discrete Subgroup of a Topological Group

Suppose that G is a *topological group*:

- G is a group and also a topological space.
- The product and inverse maps are continuous.

Let Γ be a *discrete* subgroup of G.

Experience suggests:

$\boxed{\text{The study of (left) }\Gamma\text{-invariant functions on }G\text{ is of interest.}}$

Slight generalization: Study functions that satisfy

$$f(\gamma g) = \chi(\gamma)\, f(g)$$

where χ is a character of Γ.

Examples

- $G = \mathbb{R}$, $\Gamma = \mathbb{Z}$. Fourier expansions on $\mathbb{Z}\backslash\mathbb{R} \cong S^1$.
- $G = SL(2,\mathbb{R})$, $\Gamma = SL(2,\mathbb{Z})$. Get the classical theory of modular forms (including Maass forms).
  Notes:
  1. Classical modular forms are functions on the upper half plane H. Link:
     $$G/SO(2,\mathbb{R}) = H \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G \mapsto \tfrac{ai+b}{ci+d} \in H.$$
     Modular forms may be thought of as functions on G that have a property of the form
     $$f(g\kappa) = \chi(\kappa)f(g) \qquad \text{for all } \kappa \in K := SO(2,\mathbb{R})$$
     for a suitable character χ.
  2. Since $\Gamma\backslash G$ is not compact, one most impose additional conditions on the functions:
     - ★ Growth condition.
     - ★ Right K-finiteness.
     - ★ Invariance with respect to G-invariant differential operators.

Examples (Continued)

- The adeles of $\mathbb{Q}$, $\mathbb{A}_{\mathbb{Q}}$, consists of tuples
$$(a_\infty, a_2, a_3, a_5, \ldots)$$
such that $a_v \in \mathbb{Q}_v$ for $v = \infty, 2, 3, \ldots$ and $a_v \in \mathbb{Z}_v$ for almost all v. Topological ring. Then $\mathbb{Q}$, embedded diagonally, sits discretely in $\mathbb{A}_{\mathbb{Q}}$.
- More generally, let F a global field, and similarly define $\mathbb{A}_F$, the *adeles* of F. Let $G = \mathbb{A}_F$, and $\Gamma = F$. Then Γ is a discrete subgroup of G. Get Fourier expansion on $F\backslash\mathbb{A}_F$; analogous to expansion on $\mathbb{Z}\backslash\mathbb{R}$.
- $G = \mathbb{A}_F^\times$, $\Gamma = F^\times$. A continuous function ξ on G that is Γ-invariant is called a *Hecke character*.

L-Functions

The main examples above have something in common: One can attach an *L-function* to a nice Γ-invariant function.

- Hecke attached an L-function $L(s, \xi)$ of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

  to each Hecke character ξ. This includes the Riemann zeta function as a special case.
- Hecke and Maass showed how to attach to f the *L-function* $L(s, f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$, where $a(n)$ are the Fourier coefficients of f.

All these L-functions satisfy the fundamental properties

1. They are Euler products: $\sum_{n=1}^{\infty} = \prod_{p \text{ prime}}$.
2. The series, defined for $\Re(s)$ sufficiently large, have meromorphic continuation to the full complex plane and satisfy a functional equation under $s \mapsto 1 - s$.

Connection to p-adic Integrals

- Tate showed how to use adelic and p-adic integrals to establish (and better understand) the properties of the Hecke L-functions $L(s, \xi)$.

  Key point (Global to Local): Tate's global (adelic) integral can be expressed in terms of integrals over the p-adic groups $F_v^\times$ where v runs over the places of F.
- Jacquet and Langlands did the same thing for modular and Maass forms (over any number field F).

General Case

- Let $\underline{G}$ be a reductive algebraic group defined over F, and $G = \underline{G}(\mathbb{A}_F)$. Then $\Gamma = G(F)$ sits discretely inside G. Functions on the quotient (with similar additional conditions involving smoothness, growth, etc.) are called *automorphic forms on G*.
- Given an automorphic form f, roughly speaking, one considers the vector space V_π spanned by the space of functions

$$g \mapsto f(gg_1)$$

as g_1 varies over G and calls this *the automorphic representation* of G attached to f. The group G acts by the right regular representation π. More carefully, one must do something different at the archimedean places.

The Langlands Conjectures, I

Conjecture

Given an automorphic representation π of G, there is a family of Dirichlet series

$$L(s,\pi,\rho) = \sum_{n=1}^{\infty} \frac{A_{\pi,\rho}(n)}{n^s}$$

(Langlands L-functions), originally defined and absolutely convergent for $\Re(s)$ sufficiently large, each having meromorphic continuation to all complex s and functional equation under $s \mapsto 1-s$.

1. Each $L(s,\pi,\rho)$ is an Euler product.
2. Here ρ is a complex analytic representation of the L-group of $\underline{G}$.
3. The functional equation takes L into its contragredient L-function.
4. Defining the exact coefficients at the ramified places are not given in full generality. They are understood for GL_m thanks to the local Langlands correspondence.

The Langlands Conjectures, II

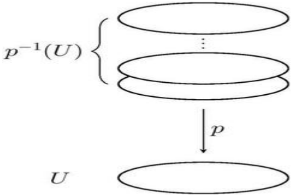
Conjecture

Automorphic forms on different groups are related ("Langlands functoriality").

In studying these conjectures, integrals on p-adic groups arise that may be expressed in terms of characters of representations of complex Lie groups. Aspects of combinatorial representation theory play an important role in the theory.

Beyond the Langlands Conjectures?

Fix an integer $n > 1$. There are also groups that are *n-fold covers* of $G(\mathbb{A}_F)$ (if F contains enough roots of unity), called *metaplectic groups*.



Can one formulate similar conjectures in those cases?

Key observation: In computations, crystal graphs arise!

Concluding Remark

"*It is a deeper subject than I appreciated and, I begin to suspect, deeper than anyone yet appreciates. To see it whole is certainly a daunting, for the moment even impossible, task.*" (Robert Langlands, writing about the theory of automorphic forms.)