Alcovepath models and Macdonald polynomials, ICERM 25.02.2013 (1)

Alcoves and hexagons

$$W = \{\text{alcoves}\}$$

$\mathfrak{h}^*_{\mathbb{R}} = \{\text{hexagons}\}$ $\qquad W_0 = \left\{\begin{matrix}\text{alcoves on the} \\ \text{0-hexagon}\end{matrix}\right\}$

$s_0, s_1, \dots, s_n$ are reflections in

$\mathfrak{h}^{\alpha_0^\vee}, \mathfrak{h}^{\alpha_1^\vee}, \dots, \mathfrak{h}^{\alpha_n^\vee}$ the walls of 1; $\frac{\pi}{m_{ij}} = \mathfrak{h}^{\alpha_i^\vee} \measuredangle \mathfrak{h}^{\alpha_j^\vee}$

t_μ = translate of 1 to the μ-hexagon,

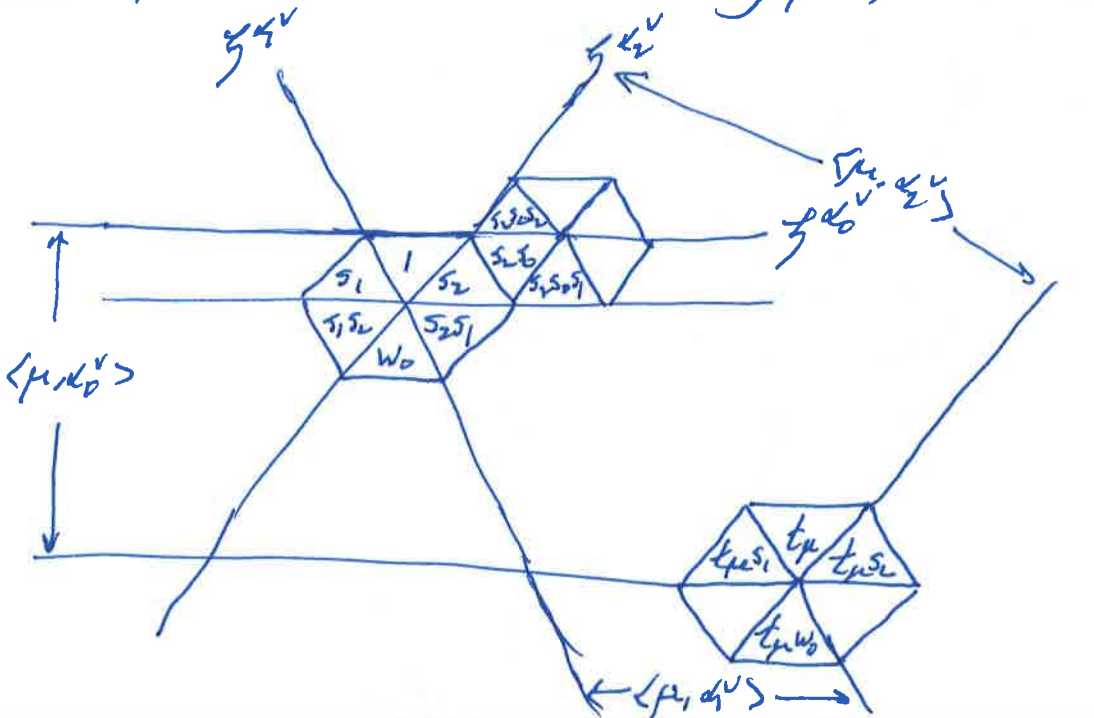
$\langle \mu, \alpha_i^\vee \rangle$ = distance from μ to $\mathfrak{h}^{\alpha_i^\vee}$

$\ell(w)$ = length of a reduced word for w,

$w = s_{i_1} \cdots s_{i_\ell}$ (min'l length path to w)

w_0 = longest element of W_0

The periodic orientation has the north star on the positive side of all hyperplanes.



Hecke algebras

The double affine braid group $\tilde{\mathcal{B}}$ is generated by

$$T_0, T_1, \ldots, T_n, \qquad X^\mu,\ \mu \in \mathfrak{h}^*_{\mathbb{Z}}, \qquad q$$

with

$$q \in Z(\tilde{\mathcal{B}}), \qquad X^\mu X^\nu = X^{\mu+\nu}, \qquad \underbrace{T_i T_j T_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{T_j T_i T_j \cdots}_{m_{ij} \text{ factors}}$$

$$T_i X^\mu = \begin{cases} X^\mu T_i, & \text{if } \langle \mu, \alpha_i^\vee \rangle = 0, \\ X^{s_i\mu} T_i^{-1}, & \text{if } \langle \mu, \alpha_i^\vee \rangle = 1. \end{cases}$$

The double affine Hecke algebra $\tilde{H}$ is $\mathbb{C}\tilde{\mathcal{B}}$ with

$$T_i^2 = (t^{\frac{1}{2}} - t^{-\frac{1}{2}}) T_i + 1, \qquad \text{for } i = 0, 1, \ldots, n.$$

$\tilde{H}$ has $\mathbb{C}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$-basis

$$\{ q^k X^\mu T_w Y^{\lambda^\vee} \mid k \in \mathbb{Z}, \mu \in \mathfrak{h}^*_{\mathbb{Z}}, w \in W_0, \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}} \}$$

where $T_w = T_{j_1} \cdots T_{j_r}$ if $w = s_{j_1} \cdots s_{j_r}$ is reduced and if $t_{\lambda^\vee} = s_{i_1} \cdots s_{i_\ell}$ is reduced then

$$Y^{\lambda^\vee} = T_{i_1}^{\varepsilon_1} \cdots T_{i_\ell}^{\varepsilon_\ell}$$

where $\varepsilon_k = \begin{cases} +1, & \text{if the } k^{\text{th}} \text{ step of } s_{i_1} \cdots s_{i_\ell} \text{ is } \overset{-\,|\,+}{\longrightarrow} \\ -1, & \text{if the } k^{\text{th}} \text{ step of } s_{i_1} \cdots s_{i_\ell} \text{ is } \overset{-\,|\,+}{\longleftarrow} \end{cases}$

Remark: $Y^{\lambda^\vee} Y^{\sigma^\vee} = Y^{\lambda^\vee + \sigma^\vee}$

Intertwiners and Macdonald polynomials

$$\tau_i = T_i + \frac{t^{-\frac{1}{2}}(1-t)}{1-y^{-\alpha_i^\vee}}, \quad \text{for } i=0,1,\dots,n.$$

and $\tau_w = \tau_{i_1}\cdots\tau_{i_\ell}$ for $w = s_{i_1}\cdots s_{i_\ell}$ reduced. Then

$$\tau_w y^{\lambda^\vee} = y^{w\lambda^\vee}\tau_w, \quad \text{for } w\in W \text{ and } \lambda^\vee \in \mathfrak{h}_{\mathbb{Z}}.$$

The <u>polynomial</u> <u>representation</u> of $\tilde{H}$ is

$$\tilde{H}\mathbb{1} = \mathbb{C}[X] = \mathrm{Ind}_H^{\tilde{H}}(\mathbb{1}) \quad \text{with basis } \{q^k X^\mu \mid k\in\mathbb{Z}, \mu\in\mathfrak{h}^*_{\mathbb{Z}}\}$$

where $\quad T_i\mathbb{1} = t^{\frac{1}{2}}\mathbb{1}$ for $i=0,1,\dots,n$.

Let $\vec{\mu} = s_{i_1}\cdots s_{i_\ell}$ be a min'l length path to the μ-hexagon.

The <u>nonsymmetric</u> and <u>symmetric</u> <u>Macdonald</u> <u>polynomials</u> are

$$E_\mu = E_\mu(q,t) = \tau_{\vec{\mu}}\mathbb{1} \quad \text{and} \quad P_\mu = P_\mu(q,t) = \mathbb{1}_0 E_\mu.$$

where $\quad \mathbb{1}_0 = \sum_{v\in W_0} t^{-\frac{1}{2}\ell(w_0 v)} T_v \quad$ so that $\mathbb{1}_0 T_i = t^{\frac{1}{2}}\mathbb{1}_0$ for $i=1,2,\dots,n$.

<u>Remarks</u> E_μ are simult. eigenvectors of $y^{\lambda^\vee}$ on $\tilde{H}\mathbb{1} = \mathbb{C}[X]$.

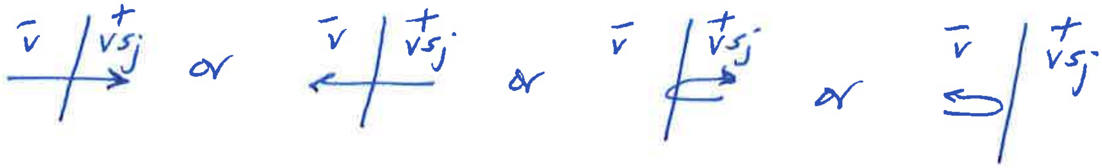
P_μ are simult. eigenvectors of $f(Y)\in\mathbb{C}[Y]^{W_0}$ on $\mathbb{1}_0\tilde{H}\mathbb{1} = \mathbb{C}[X]^{W_0}\mathbb{1}$

$P_\mu(0,t)$ = Hall-Littlewood polynomial = Macdonald spherical function

$P_\mu(0,0) = s_\mu$ = Schur function = Weyl character.

Path formulas

A step of type j is



Fix $\vec{\mu} = s_{i_1}\cdots s_{i_\ell}$ a min'l length path to the μ-hexagon.

$$P(\vec{\mu}, v) = \{\text{paths of type } (i_1,\ldots,i_\ell) \text{ beginning at } v\}$$

end(p) = end alcove of p = $t_{wt(p)}\varphi(p)$, with $wt(p)\in\mathfrak{h}^*_{\mathbb{Z}}$, $\varphi(p)\in W_0$

$f^+(p) = \{k \mid k^{th}$ step of p is $\}$

$f^-(p) = \{k \mid k^{th}$ step of p is $\}$

$$WT(p) = \left(\prod_{k\in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1-y^{\langle\beta_k^\vee,\rho\rangle}}\right)\left(\prod_{k\in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t)\,y^{\langle\beta_k^\vee,\rho\rangle}}{1-y^{\langle\beta_k^\vee,\rho\rangle}}\right)$$

where $\mathfrak{h}^{\beta_\ell^\vee},\ \ldots,\ \mathfrak{h}^{\beta_1^\vee}$ are the hyperplanes crossed by $s_{i_\ell}\cdots s_{i_1}$ and

$$y^{-\beta_k^\vee}\mathbb{1} = q^{\langle-\beta_k^\vee,\rho\rangle}\mathbb{1}.$$

Theorem (Yip-Ram)

$$E_\mu = \sum_{p\in P(\vec{\mu},1)} X^{wt(p)}\, t^{\frac{1}{2}\ell(\varphi(p))}\, WT(p) \quad \text{and}$$

$$P_\mu = \sum_{v\in W_0}\ \sum_{p\in P(\vec{\mu},v)} t^{-\frac{1}{2}\ell(w_0 v)}\, X^{wt(p)}\, t^{\frac{1}{2}\ell(\varphi(p))}\, WT(p).$$

Lift in $\tilde{H}$

Theorem In $\tilde{H}$,

$$T_v t_{\vec{\mu}} = \sum_{p \in P(\vec{\mu}, v)} X^{wt(p)} T_{\varphi(p)} \left(\prod_{k \in f^+(p)} \frac{t^{-\frac{1}{2}}(1-t)}{1-y^{-\beta_k^\vee}} \right) \left(\prod_{k \in f^-(p)} \frac{t^{-\frac{1}{2}}(1-t)\, y^{-\beta_k^\vee}}{1-y^{-\beta_k^\vee}} \right)$$