Type $A_2$, $\lambda = 3\varepsilon_1 + \varepsilon_2$, $\Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$. 

$\hat{r}_6 = s_{\alpha_{13}, -2}$

$\hat{r}_3 = s_{\alpha_{23}, 0}$

$-\mu(J)$
Type $A_2$, $\lambda = 3\varepsilon_1 + \varepsilon_2$, $\Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$.

$J = \{3, 6\}$, chain: $ld = 123 \prec t_{23} = 132 \prec t_{23}t_{13} = 231$. 
Type $A_2$, $\lambda = 3\varepsilon_1 + \varepsilon_2$, $\Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$.

$J = \{3, 6\}$, chain: $ld = 123 < t_{23} = 132 < t_{23}t_{13} = 231$.
Folded path (alcove walk): $\Gamma(J) = (\alpha_{12}, \alpha_{13}, \overline{\alpha_{23}}, \alpha_{12}, \alpha_{13}, \alpha_{12})$. 
Construction of $\lambda$-chains

Method 2. Constructing $\omega_k$-chains explicitly.

Let $\lambda = \sum c_k \omega_k$, and concatenate $\omega_k$-chains.

Example

Type $A_4$, $\omega_2$. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

$\Gamma_2 = \{ (2, 3), (2, 4), (2, 5), (1, 3), (1, 4), (1, 5) \}$. 
Construction of $\lambda$-chains

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Example

Type $A_4$, $\omega_2$. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array}

\begin{array}{c}
\Gamma_2 = \{ \\
\}.
\end{array}
Construction of $\lambda$-chains

Method 2. Constructing $\omega_k$-chains explicitly.

Let $\lambda = \sum_k c_k \omega_k$, and concatenate $\omega_k$-chains.

Example
Type $A_4$, $\omega_2$. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

\[
\Gamma_2 = \{ (2, 3) \}.
\]
Construction of $\lambda$-chains

Method 2. Constructing $\omega_k$-chains explicitly.

Let $\lambda = \sum_k c_k \omega_k$, and concatenate $\omega_k$-chains.

Example

Type $A_4$, $\omega_2$. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

$\Gamma_2 = \{ \{(2, 3), (2, 4)\} \}$. 
Construction of \( \lambda \)-chains

Method 2. Constructing \( \omega_k \)-chains explicitly.

Let \( \lambda = \sum_k c_k \omega_k \), and concatenate \( \omega_k \)-chains.

Example
Type \( A_4, \omega_2 \). The roots: \( \alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j) \).

\[
\begin{array}{cc}
1 & \\
2 & \\
3 & \\
4 & \\
5 & \\
\end{array}
\]

\[
\Gamma_2 = \{(2, 3), (2, 4), (2, 5)\}.
\]
Construction of $\lambda$-chains

Method 2. Constructing $\omega_k$-chains explicitly.
Let $\lambda = \sum_k c_k \omega_k$, and concatenate $\omega_k$-chains.

Example
Type $A_4$, $\omega_2$. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

\[ \Gamma_2 = \{ \{(2, 3), (2, 4), (2, 5), (1, 3)\} \}. \]
Construction of \( \lambda \)-chains

**Method 2.** Constructing \( \omega_k \)-chains explicitly.

Let \( \lambda = \sum_k c_k \omega_k \), and concatenate \( \omega_k \)-chains.

**Example**

Type \( A_4, \omega_2 \). The roots: \( \alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j) \).

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array}
\]

\[\Gamma_2 = \{ (2, 3), (2, 4), (2, 5), (1, 3), (1, 4) \} .\]
Construction of $\lambda$-chains

Method 2. Constructing $\omega_k$-chains explicitly.

Let $\lambda = \sum_k c_k \omega_k$, and concatenate $\omega_k$-chains.

Example
Type $A_4$, $\omega_2$. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

$\Gamma_2 = \{(2, 3), (2, 4), (2, 5), (1, 3), (1, 4), (1, 5)\}$. 
Construction of $\lambda$-chains

Method 3. The lexicographic $\lambda$-chain.
Construction of $\lambda$-chains

Method 3. The lexicographic $\lambda$-chain.
Construction of $\lambda$-chains

Method 3. The lexicographic $\lambda$-chain.
From the alcove model to tableaux

Type $A_2$: $\lambda = (5, 3, 0) = 3\omega_2 + 2\omega_1$. 

\[
\begin{array}{c|c|c}
\hline
5 & 3 & 0 \\
\hline
\end{array}
\]
From the alcove model to tableaux

Type $A_2$: $\lambda = (5, 3, 0) = \begin{array}{ccc}
\hline
| & | & | \\
\hline
\end{array} = 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.
From the alcove model to tableaux

Type $A_2$: $\lambda = (5, 3, 0) = \begin{array}{lll} & & \end{array} = 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

$\omega_2$-chain: $\Gamma_2 = ((2, 3), (1, 3))$. 

\[
\begin{array}{ll}
1 \\
2 \\
3 \\
\end{array}
\]
Type $A_2$: $\lambda = (5, 3, 0) = \begin{array}{ccc} & & \\ & 5 & \\ & & 3 \end{array} = 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

$\omega_2$-chain: $\Gamma_2 = ((2, 3), (1, 3))$.

$\omega_1$-chain: $\Gamma_1 = ((1, 2), (1, 3))$. 
From the alcove model to tableaux

Type $A_2$: $\lambda = (5, 3, 0) = \begin{array}{|c|c|c|} \hline 5 & 3 & 0 \hline \end{array} = 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

$\omega_2$-chain: $\Gamma_2 = ((2, 3), (1, 3))$.

$\omega_1$-chain: $\Gamma_1 = ((1, 2), (1, 3))$.

$\lambda$-chain: $\Gamma = \Gamma_2 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 =$
From the alcove model to tableaux

Type $A_2$: $\lambda = (5, 3, 0) = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array} = 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

$\omega_2$-chain: $\Gamma_2 = ((2, 3), (1, 3))$.

$\omega_1$-chain: $\Gamma_1 = ((1, 2), (1, 3))$.

$\lambda$-chain: $\Gamma = \Gamma_2 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 =$

$((2, 3), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) | (1, 2), (1, 3))$. 
Admissible sequence $J$ in $\mathcal{A}(\lambda)$: $J = \{3, 6, 9\}$. 
Admissible sequence $J$ in $\mathcal{A}(\lambda)$: $J = \{3, 6, 9\}$.

$$((2, 3), (1, 3)|\underline{(2, 3)}, (1, 3)|\underline{(2, 3)}, (1, 3)|\underline{(1, 2)}, (1, 3)|\underline{(1, 2)}, (1, 3))$$
Admissible sequence $J$ in $\mathcal{A}(\lambda)$: $J = \{3, 6, 9\}$.

$((2, 3), (1, 3)|\overline{(2, 3)}, (1, 3)|\overline{(2, 3), (1, 3)}|\overline{(1, 2)}, (1, 3)|\overline{(1, 2), (1, 3)})$

$J$ corresponds to the following saturated chain in Bruhat order:
Admissible sequence $J$ in $A(\lambda)$: $J = \{3, 6, 9\}$.

\[((2, 3), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) | (1, 2), (1, 3))\]

$J$ corresponds to the following saturated chain in Bruhat order:

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 \\
2 & 3 & 3 & 2 & 3 & 2 & 1 \\
3 & 2 & 2 & 1 & 3 & 3 & 1 \\
\end{array}
\]
Admissible sequence $J$ in $\mathcal{A}(\lambda)$: $J = \{3, 6, 9\}$.

$((2, 3), (1, 3)| (2, 3), (1, 3)| (2, 3), (1, 3)| (1, 2), (1, 3)| (1, 2), (1, 3))$

$J$ corresponds to the following saturated chain in Bruhat order:

$$
\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 \\
2 & 2 & 3 & 3 & 3 & 3 & 1 \\
3 & 3 & 2 & 2 & 1 & 1 & 1
\end{array}

.$$ 

The filling map $\text{fill} : \mathcal{A}(\lambda) \to \text{SSYT}(\lambda)$. 

Admissible sequence $J$ in $\mathcal{A}(\lambda)$: $J = \{3, 6, 9\}$.

$$((2,3), (1,3)| (2,3), (1,3)| (2,3), (1,3)| (1,2), (1,3)| (1,2), (1,3))$$

$J$ corresponds to the following saturated chain in Bruhat order:

$$\begin{array}{cccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 \\
2 & 2 & 3 & 3 & 3 & 1 & 1 \\
3 & 3 & 2 & 2 & 1 & 1 & 3
\end{array}$$

The filling map $\text{fill} : \mathcal{A}(\lambda) \rightarrow \text{SSYT}(\lambda)$.

$$\text{fill}(J) = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 3 \\
2 & 3 & 3
\end{array}$$
From the alcove model to LS-paths
From the alcove model to LS-paths
From the alcove model to LS-paths
Crystal operators on SSYT (in type $A$)

Example

\[ \lambda = (5, 3, 0), \quad b = \begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 3 & 3 \\
\end{array} \]

Action of $f_2$ on $b$ (changes an entry 2 to 3):

$\rightarrow$ word $(b)$

$\rightarrow$ Obtain 2-signature

$\rightarrow$ Cancel 32 pairs

$\rightarrow$ Rightmost 2 $\mapsto$ 3

Note: $f_i$ is defined by similar procedure on $i, i+1$.

There exists a similar procedure for tableaux of type $B - D$. 
Crystal operators on SSYT (in type $A$)

Example

$\lambda = (5, 3, 0)$, $b = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array}$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

Note: $f_i$ is defined by similar procedure on $i$, $i+1$. There exists a similar procedure for tableaux of type $B-D$. 
Crystal operators on SSYT (in type $A$)

Example

$$\lambda = (5, 3, 0), \quad b = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array}$$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

$\bullet$ $b \rightarrow \text{word}(b) \quad 21313223$
Crystal operators on SSYT (in type $A$)

Example

$\lambda = (5, 3, 0), \quad b = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array}$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

- $b \rightarrow \text{word}(b) \quad 21313223$
- Obtain 2-signature $\quad 233223$
Crystal operators on SSYT (in type $A$)

Example

$$\lambda = (5, 3, 0), \quad b = \begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 3 & 3 & 3 \\
\end{array}$$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

- $b \rightarrow \text{word}(b) \quad 21313223$
- Obtain 2-signature $\quad 233223$
- Cancel 32 pairs $\quad 233223$
Crystal operators on SSYT (in type $A$)

Example

$\lambda = (5, 3, 0), \ b = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array}$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

- $b \rightarrow \text{word}(b) \quad 21313223$
- Obtain 2-signature \quad 233223
- Cancel 32 pairs \quad 232323
Crystal operators on SSYT (in type $A$)

Example

$\lambda = (5, 3, 0)$, $b = \begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 3 & 3 \\
\end{array}$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

- $b \rightarrow \text{word}(b)$ \hspace{1cm} 21313223
- Obtain 2-signature \hspace{1cm} 233223
- Cancel 32 pairs \hspace{1cm} 23\underline{3}\underline{2}23

Note: $f_i$ is defined by similar procedure on $i$, $i+1$.

There exists a similar procedure for tableaux of type $B$–$D$. 
Crystal operators on SSYT (in type $A$)

Example

$$\lambda = (5, 3, 0), \quad b = \begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 3 & 3 & 3
\end{array}$$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

- $b \rightarrow \text{word}(b) \rightarrow 21313223$
- Obtain 2-signature \rightarrow 233223
- Cancel 32 pairs \rightarrow 233223
Crystal operators on SSYT (in type $A$)

Example

$\lambda = (5, 3, 0), \quad b = \begin{array}{ccccc}
1 & 1 & 2 & 2 & 3 \\
2 & 3 & 3 &   &   
\end{array}$

Action of $f_2$ on $b$ (changes an entry $2$ to $3$):

- $b \rightarrow \text{word}(b) \quad 21313223$
- Obtain 2-signature \quad 233223
- Cancel 32 pairs \quad 233223
- Rightmost $2 \mapsto 3$
Crystal operators on SSYT (in type $A$)

Example

$\lambda = (5, 3, 0), \quad b = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array}$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

- $b \rightarrow \text{word}(b) = 21313223$
- Obtain 2-signature $233223$
- Cancel 32 pairs $233223$
- Rightmost 2 $\mapsto$ 3

$f_2(b) = \begin{array}{ccc}
1 & 1 & 2 \\
3 & 3 & 3 \\
\end{array}$

Note: $f_i$ is defined by similar procedure on $i, i+1$.

There exists a similar procedure for tableaux of type $B_{-D}$. 
Crystal operators on SSYT (in type $A$)

Example

\[ \lambda = (5, 3, 0), \quad b = \begin{array}{cccc}
1 & 1 & 2 & 2 \\
2 & 3 & 3 & \\
\end{array} \]

Action of $f_2$ on $b$ (changes an entry $2$ to $3$):

- $b \rightarrow \text{word}(b)$ \quad 21313223
- Obtain $2$-signature \quad 233223
- Cancel $32$ pairs \quad 233223
- Rightmost $2 \mapsto 3$

\[ f_2(b) = \begin{array}{cccc}
1 & 1 & 2 & 2 \\
3 & 3 & 3 & \\
\end{array} \]

Note: $f_i$ is defined by similar procedure on $i, i + 1$. 
Crystal operators on SSYT (in type $A$)

Example

$$\lambda = (5, 3, 0), \quad b = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 3 & 3 \\
\end{array}$$

Action of $f_2$ on $b$ (changes an entry 2 to 3):

- $b \rightarrow \text{word}(b) \Rightarrow 21313223$
- Obtain 2-signature $\Rightarrow 233223$
- Cancel 32 pairs $\Rightarrow 233223$
- Rightmost 2 $\mapsto$ 3

$$f_2(b) = \begin{array}{ccc}
1 & 1 & 2 \\
3 & 3 & 3 \\
\end{array}$$

Note: $f_i$ is defined by similar procedure on $i, i + 1$. There exists a similar procedure for tableaux of type $B - D$. 
Crystal operators in the alcove model

Type $A_2$, $\lambda = (5, 3, 0) = \boxed{\square \square \square \square \square \square \square \square \square \square}$. 

Step 1: Construct the "folded chain" $\Gamma(\mathcal{J})$. 

$\boxed{\square \square \square \square \square \square \square \square \square \square}$
Crystal operators in the alcove model

Type $A_2$, $\lambda = (5, 3, 0) = \begin{array}{ccc} & & \\ & & \\ & & \end{array}$. 

$\lambda$-chain $\Gamma = \Gamma_2 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 =$ 

$$(((2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) \mid (1, 2), (1, 3)) \cdot$$
Crystal operators in the alcove model

Type $A_2$, $\lambda = (5, 3, 0) = \begin{array}{ccc} & & \\ & & \\ & & \\ \end{array}$.

$\lambda$-chain $\Gamma = \Gamma_2 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 =$

$$((2, 3), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) | (1, 2), (1, 3)) .$$

Let $J = \{3, 6, 9\}$ in $A(\lambda)$. 
Crystal operators in the alcove model

Type $A_2$, $\lambda = (5, 3, 0) = \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}$.

$\lambda$-chain $\Gamma = \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 =$

$((2, 3), (1, 3) \ | \ (2, 3), (1, 3) \ | \ (2, 3), (1, 3) \ | \ (1, 2), (1, 3) \ | \ (1, 2), (1, 3))$.

Let $J = \{3, 6, 9\}$ in $A(\lambda)$.

Step 1: Construct the “folded chain” $\Gamma(J)$. 
Crystal operators in the alcove model

Type $A_2$, $\lambda = (5, 3, 0) = \begin{array}{ccc}
\end{array}$.

$\lambda$-chain $\Gamma = \Gamma_2 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 =$

$((2, 3), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) | (1, 2), (1, 3))$.

Let $J = \{3, 6, 9\}$ in $A(\lambda)$.

**Step 1:** Construct the “folded chain” $\Gamma(J)$.

$((2, 3), (1, 3) | \underline{(2, 3)}, (1, 2) | (3, 2), \underline{(1, 2)} | (2, 3), (2, 1) | \underline{(2, 3)}, (3, 1))$
Step 2. Signature rule.
Step 2. Signature rule.

\[ J = \{3, 6, 9\} \]

\[ \Gamma(J) = ((2, 3), (1, 3) \mid (2, 3), (1, 2) \mid (3, 2), (1, 2) \mid (2, 3), (2, 1) \mid (2, 3), (3, 1)) \]

- For \( f_2 \) only look at \((2, 3), (2, 3), (3, 2)\) in \( \Gamma(J) \).
Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \]

\[ \Gamma(J) = ((2, 3), (1, 3) | (2, 3), (1, 2) | (3, 2), (1, 2) | (2, 3), (2, 1) | (2, 3), (3, 1)) \]

- For \( f_2 \) only look at (2, 3), (2, 3), and (3, 2) in \( \Gamma(J) \).
Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \]

\[ \Gamma(J) = (\mathbf{(2, 3)}, (1, 3) | \mathbf{(2, 3)}, (1, 2) | (3, 2), (1, 2) | (2, 3), (2, 1) | (2, 3), (3, 1)) \]

- For \( f_2 \) only look at \( (2, 3), (2, 3), \) and \( (3, 2) \) in \( \Gamma(J) \).
- Cancel pairs \( (3, 2), (2, 3) \) like before.
Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \]
\[ \Gamma(J) = ( (2, 3), (1, 3) | (2, 3), (1, 2) | (3, 2), (1, 2) | (2, 3), (2, 1) | (2, 3), (3, 1) ) \]

- For \( f_2 \) only look at \((2, 3), (2, 3), \) and \((3, 2)\) in \( \Gamma(J) \).
- Cancel pairs \((3, 2), (2, 3)\) like before.
Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \]

\[ \Gamma(J) = (\underline{(2, 3)}, (1, 3) \mid \underline{(2, 3)}, (1, 2) \mid (3, 2), (1, 2) \mid (2, 3), (2, 1) \mid (\underline{2, 3}), (3, 1)) \]

- For \( f_2 \) only look at \((2, 3), (2, 3),\) and \((3, 2)\) in \( \Gamma(J) \).
- Cancel pairs \((3, 2), (2, 3)\) *like before*.
- Consider rightmost \((2, 3)\) *like before*.
Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \]

\[ \Gamma(J) = \{(2, 3), (1, 3) | (2, 3), (1, 2) | (3, 2), (1, 2) | (2, 3), (2, 1) | (2, 3), (3, 1)\} \]

- For \( f_2 \) only look at (2, 3), (2, 3), and (3, 2) in \( \Gamma(J) \).
- Cancel pairs (3, 2), (2, 3) like before.
- Consider rightmost (2, 3) like before.
Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \]
\[ \Gamma(J) = ((2, 3), (1, 3) | (2, 3), (1, 2) | (3, 2), (1, 2) | (2, 3), (2, 1) | (2, 3), (3, 1)) \]

- For \( f_2 \) only look at \((2, 3), (2, 3)\), and \((3, 2)\) in \( \Gamma(J) \).
- Cancel pairs \((3, 2), (2, 3)\) \textit{like before}.
- Consider rightmost \((2, 3)\) \textit{like before}.
- Add corresponding position to \( J \), and remove from \( J \) the position corresponding to \((2, 3)\) to its right (if any).
Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \]
\[ \Gamma(J) = ((2, 3), (1, 3) \mid (2, 3), (1, 2) \mid (3, 2), (1, 2) \mid (2, 3), (2, 1) \mid (2, 3), (3, 1)) \]

- For \( f_2 \) only look at \((2, 3), (2, 3)\), and \((3, 2)\) in \( \Gamma(J) \).
- Cancel pairs \((3, 2), (2, 3)\) like before.
- Consider rightmost \((2, 3)\) like before.
- Add corresponding position to \( J \), and remove from \( J \) the position corresponding to \((2, 3)\) to its right (if any).
Crystal operators in the alcove model (cont.)

**Step 2.** Signature rule.

\[ J = \{3, 6, 9\}. \quad f_2(J) = \{1, 6, 9\}. \]

\[ \Gamma(J) = (\underline{2, 3}, (1, 3) \mid (2, 3), (1, 2) \mid (3, 2), (1, 2) \mid (2, 3), (2, 1) \mid (2, 3), (3, 1)) \]

- For \( f_2 \) only look at \((2, 3), (2, 3), \) and \((3, 2)\) in \( \Gamma(J) \).  
- Cancel pairs \((3, 2), (2, 3)\) *like before*.  
- Consider rightmost \((2, 3)\) *like before*.  
- Add corresponding position to \( J \), and remove from \( J \) the position corresponding to \((2, 3)\) to its right (if any).
Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

\[ J = \{3, 6, 9\}. \quad f_2(J) = \{1, 6, 9\}. \]
\[ \Gamma(J) = \{(2, 3), (1, 3) \mid (2, 3), (1, 2) \mid (3, 2), (1, 2) \mid (2, 3), (2, 1) \mid (2, 3), (3, 1)\} \]

- For \( f_2 \) only look at \((2, 3), (2, 3), \) and \((3, 2)\) in \( \Gamma(J) \).
- Cancel pairs \((3, 2), (2, 3)\) like before.
- Consider rightmost \((2, 3)\) like before.
- Add corresponding position to \( J \), and remove from \( J \) the position corresponding to \((2, 3)\) to its right (if any).

Note: In arbitrary type, \( f_i \) is defined based on the simple roots \( \alpha_i \) in \( \Gamma(J) \).
Bruhat graph for $S_3$:
Quantum Bruhat graph for $S_3$:
Realizing $\bigotimes B(\omega_i)$ as $A(\lambda)_q$

**Example** in type $A_2$. Consider

$$B(\omega_1) \bigotimes B(\omega_2) \bigotimes B(\omega_2) \bigotimes B(\omega_1) \quad \text{and} \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4, 2, 0).$$
Realizing $\bigotimes B(\omega_i)$ as $A(\lambda)_q$

**Example** in type $A_2$. Consider

$B(\omega_1) \bigotimes B(\omega_2) \bigotimes B(\omega_2) \bigotimes B(\omega_1)$ and $\lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4, 2, 0)$.

A $\lambda$-chain as a concatenation of $\omega_1$-, $\omega_2$-, $\omega_2$-, and $\omega_1$-chains:

$\Gamma = ( (1, 2), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) )$. 
Realizing $\bigotimes B(\omega_i)$ as $A(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$\left( (1, 2), (1, 3) \mid (2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) \right).$$
Realizing $\bigotimes B(\omega_i)$ as $A(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

\[
( (1, 2), (1, 3) \mid (2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) )
\]

Claim: $J$ is in $A(\lambda)_q$. 

Realizing $\bigotimes B(\omega_i)$ as $\mathcal{A}(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$( (1, 2), (1, 3) \mid (2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) ).$$

Claim: $J$ is in $\mathcal{A}(\lambda)_q$. Indeed, the corresponding path in the quantum Bruhat graph is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 3 & 3 & 1 & 1 & 2 \\
< & < & < & 1 & < & > & > & < \\
2 & 1 & 1 & 2 & 2 & 2 & 1 & 2 \\
3 & 3 & 2 & 1 & 1 & 3 & 3 & 1 \\
3 & 3 & 2 & 1 & 1 & 3 & 3 & 2 \\
\end{array}
\]
Realizing $\bigotimes B(\omega_i)$ as $A(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$( (1, 2), (1, 3) \mid (2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) )$.

Claim: $J$ is in $A(\lambda)_q$. Indeed, the corresponding path in the quantum Bruhat graph is

\[
\begin{array}{ccccccccccc}
1 & 2 & 3 & 3 & 3 & 1 & 1 & 2 & 2 & 3 \\
2 & 1 & 1 & 2 & 2 & 1 & 3 & 3 & 3 & 3 \\
3 & 3 & 2 & 1 & 2 & 3 & 3 & 2 & 1 & 2
\end{array}
\]

The corresponding element in $B(\omega_1) \bigotimes B(\omega_2) \bigotimes B(\omega_2) \bigotimes B(\omega_1)$ (column-strict filling), obtained via “fillord:”

\[
3 \bigotimes \frac{2}{3} \bigotimes \frac{1}{2} \bigotimes 3.
\]
The combinatorial $R$-matrix in type $A$

Realized by Schützenberger’s *jeu de taquin* (sliding algorithm) on two columns.
The combinatorial $R$-matrix in type $A$

Realized by Schützenberger’s jeu de taquin (sliding algorithm) on two columns.

**Example.** Realize the isomorphism

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \simeq B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2).$$
The combinatorial $R$-matrix in type $A$

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**Example.** Realize the isomorphism

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \simeq B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2).$$

Commute the last two factors as follows:

$$\begin{array}{cccc}
3 & 2 & 1 & 3 \\
3 & 2 & 1 & 3
\end{array}$$
The combinatorial $R$-matrix in type $A$

Realized by Schützenberger’s jeu de taquin (sliding algorithm) on two columns.

Example. Realize the isomorphism

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \simeq B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2).$$

Commutate the last two factors as follows:

$$\begin{array}{ccc}
3 & \otimes & 2 \\
\hline
3 & \otimes & 1 \\
\hline
3 & \otimes & 3
\end{array}
\begin{array}{ccc}
\otimes & \times & \\
\hline
2 & \otimes & 3 \\
\hline
1 & \otimes & 3
\end{array}
= 
\begin{array}{ccc}
3 & \otimes & 2 \\
\hline
3 & \otimes & 1 \\
\hline
3 & \otimes & 2
\end{array}
\begin{array}{ccc}
\otimes & \times & \\
\hline
2 & \otimes & 3 \\
\hline
1 & \otimes & 3
\end{array}$$
The combinatorial $R$-matrix in type $A$

Realized by Schützenberger's *jeu de taquin* (sliding algorithm) on two columns.

**Example.** Realize the isomorphism

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \cong B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2).$$

Commute the last two factors as follows:

\[
\begin{array}{ccc}
3 & \times & \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \\
& \times & \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \\
& = & \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \\
& \times & \begin{array}{c} 3 \\ 2 \\ 1 \end{array}
\end{array}
\]

\[\Rightarrow\]

\[
\begin{array}{ccc}
3 & \times & \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \\
& \times & \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \\
& \Rightarrow & \begin{array}{c} 2 \\ 3 \\ 1 \end{array} \\
& \times & \begin{array}{c} 1 \\ 2 \end{array}
\end{array}
\]

\[\Rightarrow\]
The combinatorial $R$-matrix in type $A$

Realized by Schützenberger’s jeu de taquin (sliding algorithm) on two columns.

**Example.** Realize the isomorphism

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \simeq B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2).$$

Commutate the last two factors as follows:

\[
\begin{array}{ccc}
3 & \otimes & \begin{array}{c}
2 \\
3
\end{array} \\
\otimes & \begin{array}{c}
1 \\
2
\end{array} & \otimes & 3 \\
\end{array} = \begin{array}{ccc}
3 & \otimes & \begin{array}{c}
2 \\
3
\end{array} \\
\otimes & \begin{array}{c}
1 \\
3
\end{array} & \otimes & 2 \\
\end{array} \quad \mapsto \quad \begin{array}{ccc}
3 & \otimes & \begin{array}{c}
2 \\
3
\end{array} \\
\otimes & \begin{array}{c}
1 \\
2
\end{array} & \otimes & 3 \\
\end{array} \quad \mapsto \quad \begin{array}{ccc}
3 & \otimes & \begin{array}{c}
2 \\
3
\end{array} \\
\otimes & \begin{array}{c}
1 \\
2
\end{array} & \otimes & 3 \\
\end{array}
\end{array}
\]
The combinatorial $R$-matrix in type $A$

Realized by Schützenberger’s jeu de taquin (sliding algorithm) on two columns.

**Example.** Realize the isomorphism

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \cong B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2).$$

Commute the last two factors as follows:

$$\begin{array}{c}
3 \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes \begin{array}{c}1 \\ 2 \end{array} \otimes 3 = 3 \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes \begin{array}{c}1 \\ 2 \end{array} \otimes 3 \\
\implies 3 \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes \begin{array}{c}1 \\ 2 \end{array} \otimes 3 \implies 3 \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes 1 \\
\implies 3 \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes \begin{array}{c}1 \\ 2 \end{array} \otimes 3 = 3 \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes \begin{array}{c}2 \\ 3 \end{array} \otimes 1.
\end{array}$$
Example. Type $G_2$. $s_2s_1s_2 \to s_1 : 1, 2, 5, 6; \ 6, 3, 2, 1$. 

Note: $\alpha_1 = e_2 - e_3$ 
$\alpha_2 = e_1 - 2e_2 + e_3$
The quantum Yang–Baxter moves via the running example. Realize the isomorphism

\[ B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \cong B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2). \]
The quantum Yang–Baxter moves via the running example.
Realize the isomorphism

\[ B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \cong B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2). \]

\[
\left( \begin{array}{c|c|c|c}
(1,2), (1,3) & (2,3), (1,3) & (2,3), (1,3) & (1,2), (1,3) \\
\end{array} \right),
\]

\[
\left( \begin{array}{c|c|c|c}
(1,2), (1,3) & (2,3), (1,3) & (1,2), (1,3) & (2,3), (1,3) \\
\end{array} \right).
\]
The quantum Yang–Baxter moves via the running example.

Realize the isomorphism

\[ B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \cong B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2). \]

\[
\left( \begin{array}{ll} (1, 2), (1, 3) & (2, 3), (1, 3) \end{array} \right) \left( \begin{array}{ll} (2, 3), (1, 3) & (1, 2), (1, 3) \end{array} \right),
\left( \begin{array}{ll} (1, 2), (1, 3) & (2, 3), (1, 3) \end{array} \right) \left( \begin{array}{ll} (1, 2), (1, 3) & (2, 3), (1, 3) \end{array} \right).
\]
The quantum Yang–Baxter moves via the running example.
Realize the isomorphism

\[ B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \cong B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2). \]

\[
( (1, 2), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) ),
( (1, 2), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) | (2, 3), (1, 3) ).
\]
The quantum Yang–Baxter moves via the running example.

Realize the isomorphism

\[ B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \cong B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2). \]
The quantum Yang–Baxter moves via the running example. Realize the isomorphism

\[ B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \simeq B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2). \]

\[
( (1, 2), (1, 3) | (2, 3), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) ),
( (1, 2), (1, 3) | (2, 3), (1, 3) | (1, 2), (1, 3) | (2, 3), (1, 3) ).
\]

The Bruhat chain corresponding to the second case:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 3 & 3 & 2 & 2 & 3 \\
< & < & < & < & > & > & > & < \\
2 & 1 & 1 & 2 & 2 & 3 & 1 & 3 \\
3 & 2 & 1 & 1 & 1 & 3 & 2 \\
\end{array}
\]
The quantum Yang–Baxter moves via the running example.

Realize the isomorphism

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \simeq B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_1) \otimes B(\omega_2).$$

$$\left( (1, 2), (1, 3) \mid (2, 3), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) \right),$$

$$\left( (1, 2), (1, 3) \mid (2, 3), (1, 3) \mid (1, 2), (1, 3) \mid (2, 3), (1, 3) \right).$$

The Bruhat chain corresponding to the second case:

\[
\begin{array}{cccccccc}
1 & 2 & < & 3 & 3 & < & 3 & 3 \\
2 & 1 & < & 1 & 2 & < & 2 & 2 \\
3 & 3 & < & 2 & 1 & > & 3 & 3 \\
3 & 1 & > & 1 & 3 & > & 1 & 3 \\
2 & 1 & < & 3 & 3 & > & 3 & 1 \\
& & & & & & & 2 \\
\end{array}
\]

So we have

$$\left( \begin{array}{ccc}
\begin{array}{c}
3 \\
2 \\
3
\end{array} & \begin{array}{c}
1 \\
2 \\
3
\end{array} & \begin{array}{c}
3 \\
2 \\
3
\end{array} \\
\end{array} \right) \mapsto \left( \begin{array}{ccc}
\begin{array}{c}
3 \\
2 \\
3
\end{array} & \begin{array}{c}
1 \\
2 \\
3
\end{array} & \begin{array}{c}
3 \\
2 \\
3
\end{array} \\
\end{array} \right).$$