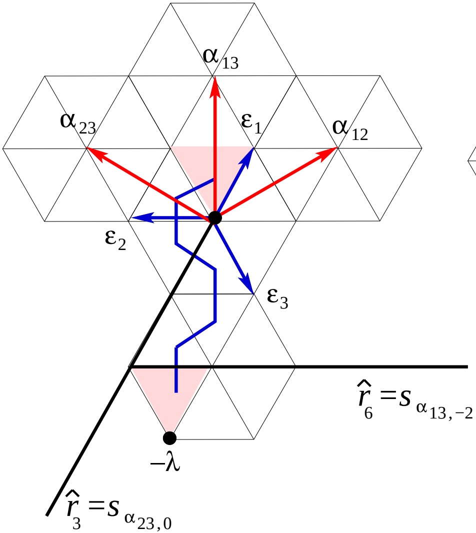
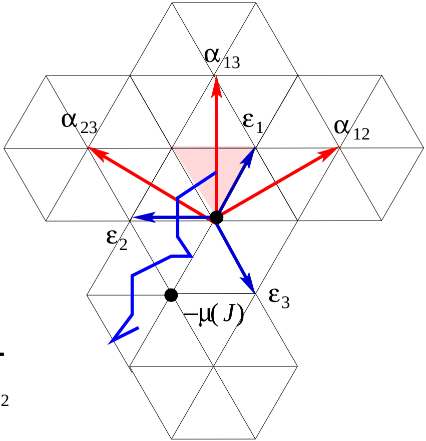
Type A_2, $\lambda = 3\varepsilon_1 + \varepsilon_2$, $\Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$.





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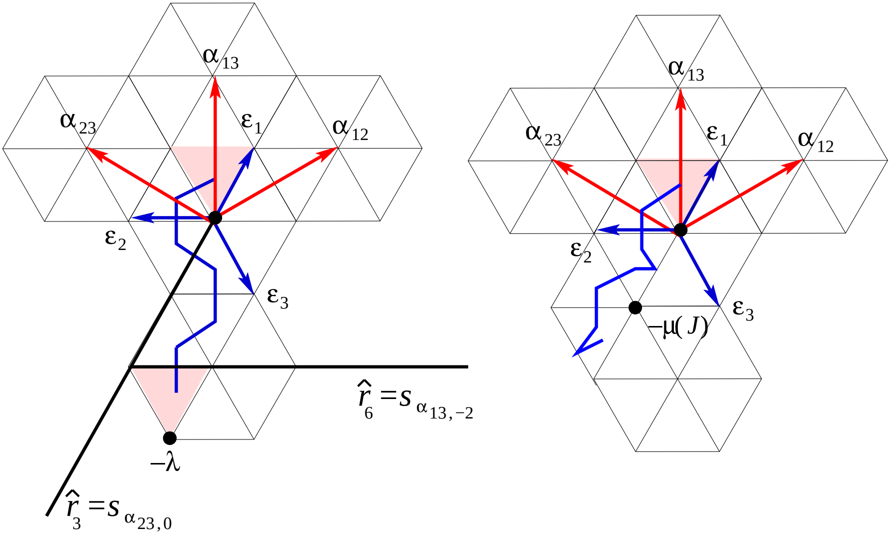
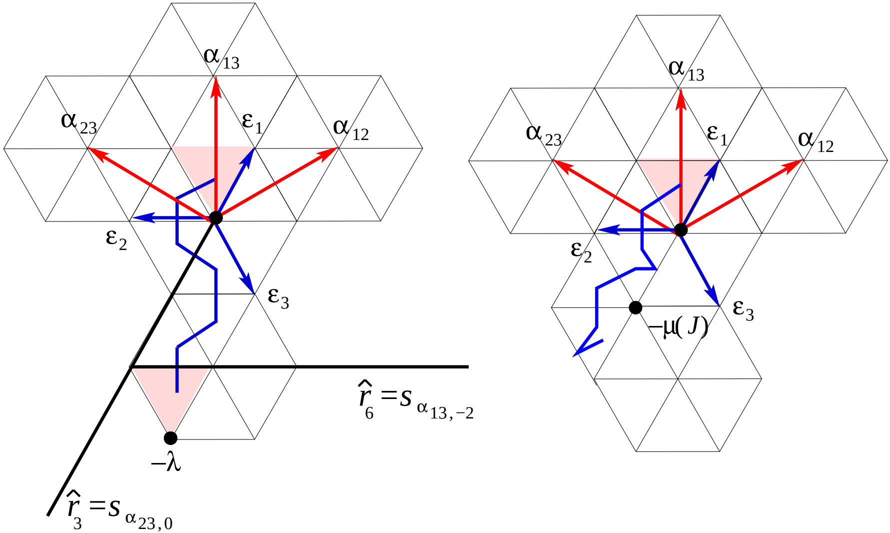


$J = \{3, 6\}$, chain: $Id = 123 \lessdot t_{23} = 132 \lessdot t_{23}t_{13} = 231$.

Type A_2, $\lambda = 3\varepsilon_1 + \varepsilon_2$, $\Gamma = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{13}, \alpha_{12}, \alpha_{13})$.



$J = \{3, 6\}$, chain: $Id = 123 \lessdot t_{23} = 132 \lessdot t_{23}t_{13} = 231$.

Folded path (alcove walk): $\Gamma(J) = (\alpha_{12}, \alpha_{13}, \underline{\alpha_{23}}, \alpha_{12}, \alpha_{13}, \underline{\alpha_{12}})$.

Construction of λ-chains

Method 2. Constructing ω_k-chains explicitly.

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Let $\lambda = \sum_k c_k \omega_k$, and concatenate ω_k-chains.

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Type A_4, ω_2. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

1
2

3
4
5

$$\Gamma_2 = \{ \qquad \} .$$

Construction of λ-chains

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Example

Type A_4, ω_2. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

1
2

3
4
5

$$\Gamma_2 = \{ \{(2, 3) \quad \} .$$

Construction of λ-chains

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Example

Type A_4, ω_2. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

1
2

3
4
5

$$\Gamma_2 = \{ \{(2,3), (2,4) \quad \} .$$

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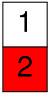

$$\Gamma_2 = \{ \{(2,3), (2,4), (2,5) \qquad \} .$$

Construction of λ-chains

Method 2. Constructing ω_k-chains explicitly.

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Type A_4, ω_2. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

1
2

3
4
5

$$\Gamma_2 = \{ \{(2,3), (2,4), (2,5), (1,3) \qquad \} .$$

Construction of λ-chains

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Example

Type A_4, ω_2. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i, j)$.

1
2

3
4
5

$$\Gamma_2 = \{ \{(2,3),(2,4),(2,5),(1,3),(1,4) \quad \} .$$

Construction of λ-chains

Method 2. Constructing ω_k-chains explicitly.

Let $\lambda = \sum_k c_k \omega_k$, and concatenate ω_k-chains.

Example

Type A_4, ω_2. The roots: $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i,j)$.

1
2

3
4
5

$$\Gamma_2 = \{ \{(2,3),(2,4),(2,5),(1,3),(1,4),(1,5)\} \,.$$

Construction of λ-chains

Method 3. The lexicographic λ-chain.

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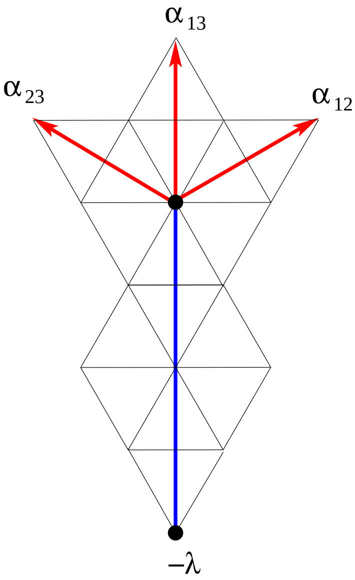

Construction of λ-chains

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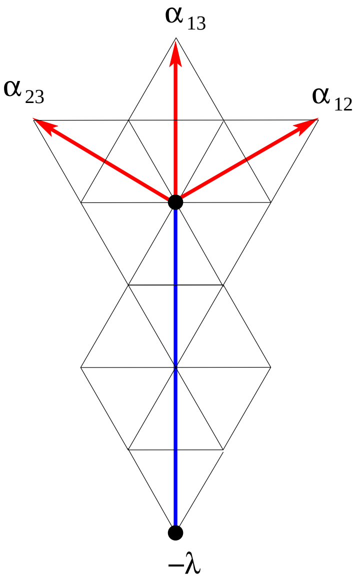
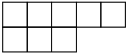
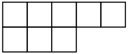


From the alcove model to tableaux

Type A_2: $\lambda = (5, 3, 0) =$  $= 3\omega_2 + 2\omega_1$.

From the alcove model to tableaux

Type A_2: $\lambda = (5,3,0) =$  $= 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i,j)$.

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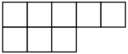
Type A_2: $\lambda = (5,3,0) =$  $= 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i,j)$.

ω_2-chain: 

$\Gamma_2 = ((2,3),(1,3))$.

From the alcove model to tableaux

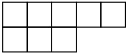
Type A_2: $\lambda = (5,3,0) = \begin{array}{|c|c|c|c|c|}\hline \; & \; & \; & \; & \; \\ \hline \; & \; & \; \\ \cline{1-3}\end{array} = 3\omega_2 + 2\omega_1$.

Recall: roots $\alpha_{ij} = \varepsilon_i - \varepsilon_j = (i,j)$.

ω_2-chain: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|}\hline 3 \\ \hline \end{array}$ $\qquad \Gamma_2 = ((2,3),(1,3))$.

ω_1-chain: $\begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}$ $\qquad \Gamma_1 = ((1,2),(1,3))$.

From the alcove model to tableaux

Type A_2: $\lambda = (5,3,0) =$  $= 3\omega_2 + 2\omega_1$.

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ω_2-chain: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$ $\begin{array}{|c|}\hline 3 \\ \hline \end{array}$ $\quad \Gamma_2 = ((2,3),(1,3))$.

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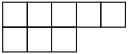
λ-chain: $\Gamma = \Gamma_2\Gamma_2\Gamma_2\Gamma_1\Gamma_1 =$

From the alcove model to tableaux

Type A_2: $\lambda = (5,3,0) =$  $= 3\omega_2 + 2\omega_1$.

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ω_2-chain: 

$\Gamma_2 = ((2,3),(1,3))$.

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$((2,3),(1,3) \mid (2,3),(1,3) \mid (2,3),(1,3) \mid (1,2),(1,3) \mid (1,2),(1,3))$.

Admissible sequence *J* in $\mathcal{A}(\lambda)$: $J = \{3, 6, 9\}$.

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$$((2,3),(1,3)\,|\,\underline{(2,3)},(1,3)\,|\,(2,3),\underline{(1,3)}\,|\,(1,2),(1,3)\,|\,\underline{(1,2)},(1,3))$$

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J corresponds to the following saturated chain in Bruhat order:

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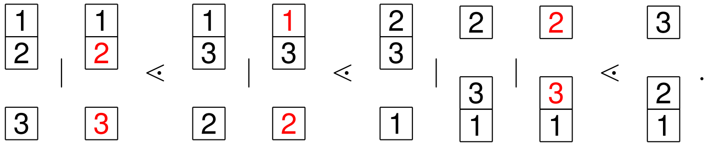

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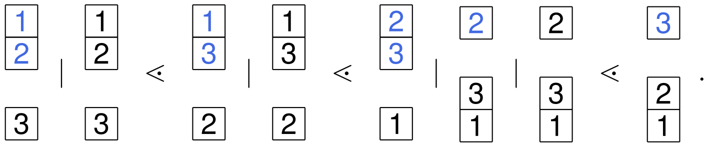


The filling map $\quad \mathrm{fill} \, : \, \mathcal{A}(\lambda) \to SSYT(\lambda)\,.$

Admissible sequence J in $\mathcal{A}(\lambda)$: $J = \{3, 6, 9\}$.

$$((2,3),(1,3)\,|\,\underline{(2,3)},(1,3)\,|\,(2,3),\underline{(1,3)}\,|\,(1,2),(1,3)\,|\,\underline{(1,2)},(1,3))$$

J corresponds to the following saturated chain in Bruhat order:

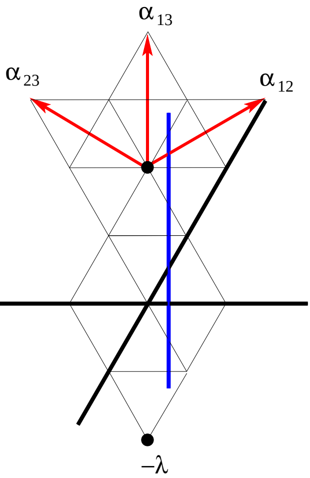
$$\begin{array}{c}\boxed{1}\\\boxed{2}\\ \\\boxed{3}\end{array} \;\Big|\; \begin{array}{c}\boxed{1}\\\boxed{2}\\ \\\boxed{3}\end{array} \;\lessdot\; \begin{array}{c}\boxed{1}\\\boxed{3}\\ \\\boxed{2}\end{array} \;\Big|\; \begin{array}{c}\boxed{1}\\\boxed{3}\\ \\\boxed{2}\end{array} \;\lessdot\; \begin{array}{c}\boxed{2}\\\boxed{3}\\ \\\boxed{1}\end{array} \;\Big|\; \begin{array}{c}\boxed{2}\\ \\\boxed{3}\\\boxed{1}\end{array} \;\Big|\; \begin{array}{c}\boxed{2}\\ \\\boxed{3}\\\boxed{1}\end{array} \;\lessdot\; \begin{array}{c}\boxed{3}\\ \\\boxed{2}\\\boxed{1}\end{array}\;.$$

The filling map $\quad \mathrm{fill} \,:\, \mathcal{A}(\lambda) \to SSYT(\lambda)\,.$

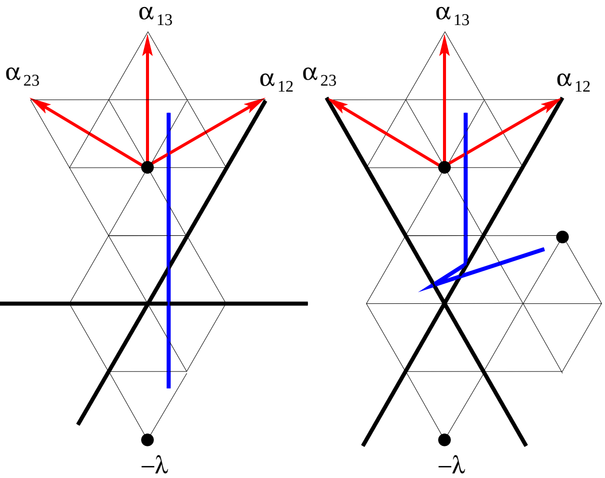
$$\mathrm{fill}(J) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3\\\hline 2 & 3 & 3 & \multicolumn{2}{c}{}\\\cline{1-3}\end{array}\;.$$

From the alcove model to LS-paths

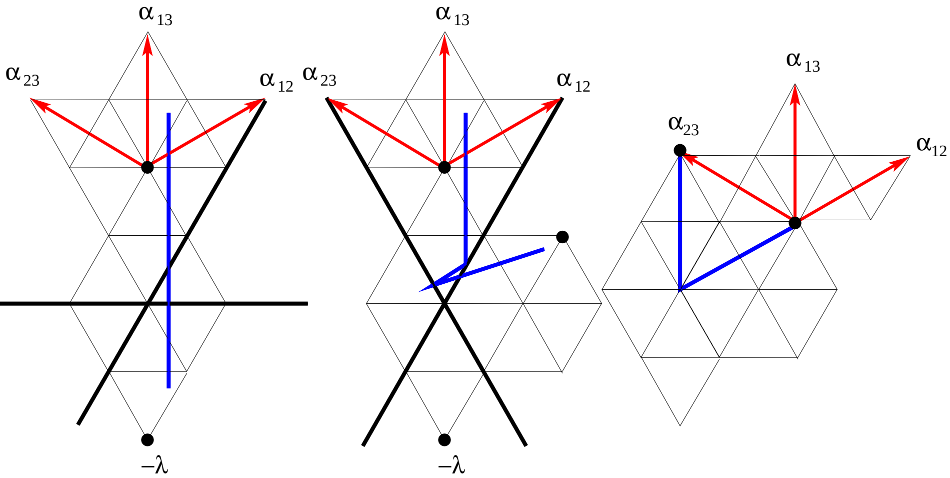
From the alcove model to LS-paths



From the alcove model to LS-paths



From the alcove model to LS-paths



Crystal operators on SSYT (in type A)

Example

$$\lambda = (5, 3, 0)\,, \qquad b = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array}$$

Crystal operators on SSYT (in type *A*)

Example

$$\lambda = (5,3,0)\,, \qquad b = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & \color{red}{2} & 3 \\ \hline \color{red}{2} & 3 & 3 \\ \cline{1-3}\end{array}$$

Action of f_2 on b (changes an entry 2 to 3):

Crystal operators on SSYT (in type A)

Example

$$\lambda = (5,3,0)\,, \qquad b = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array}$$

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \mathrm{word}(b)$ 21313223

Crystal operators on SSYT (in type A)

Example

$$\lambda = (5,3,0)\,, \qquad b = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array}$$

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223

Crystal operators on SSYT (in type A)

Example

$\lambda = (5, 3, 0)\,,\quad b =$

1	1	2	2	3
2	3	3		

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 233223

Crystal operators on SSYT (in type A)

Example

$$\lambda = (5,3,0)\,, \qquad b = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array}$$

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 232323

Crystal operators on SSYT (in type A)

Example

$\lambda = (5, 3, 0)$, $b =$

1	1	2	2	3
2	3	3		

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 233223

Crystal operators on SSYT (in type A)

Example

$$\lambda = (5,3,0)\,, \qquad b = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array}$$

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 2~~3322~~3

Crystal operators on SSYT (in type *A*)

Example

$$\lambda = (5, 3, 0)\,, \qquad b = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3} \end{array}$$

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 233223
- Rightmost $2 \mapsto 3$

Crystal operators on SSYT (in type A)

Example

$\lambda = (5, 3, 0)$, $b =$

1	1	2	2	3
2	3	3		

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 233223
- Rightmost $2 \mapsto 3$

$f_2(b) =$

1	1	2	2	3
3	3	3		

Crystal operators on SSYT (in type A)

Example

$\lambda = (5, 3, 0)$, $b =$

1	1	2	2	3
2	3	3		

Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 233223
- Rightmost $2 \mapsto 3$

$f_2(b) =$

1	1	2	2	3
3	3	3		

Note: f_i is defined by similar procedure on $i, i+1$.

Crystal operators on SSYT (in type A)

Example

$$\lambda = (5,3,0)\,, \qquad b = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 3 & 3 \\ \cline{1-3}\end{array}$$

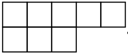
Action of f_2 on b (changes an entry 2 to 3):

- $b \to \text{word}(b)$ 21313223
- Obtain 2-signature 233223
- Cancel 32 pairs 233223
- Rightmost $2 \mapsto 3$

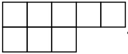
$$f_2(b) = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 \\ \hline 3 & 3 & 3 \\ \cline{1-3}\end{array}$$

Note: f_i is defined by similar procedure on $i, i+1$. There exists a similar procedure for tableaux of type $B - D$.

Crystal operators in the alcove model

Type A_2 , $\lambda = (5, 3, 0) =$  .

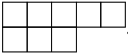
Crystal operators in the alcove model

Type A_2 , $\lambda = (5, 3, 0) =$ .

λ-chain $\Gamma = \Gamma_2\Gamma_2\Gamma_2\Gamma_1\Gamma_1 =$

$((2,3),(1,3) \mid (2,3),(1,3) \mid (2,3),(1,3) \mid (1,2),(1,3) \mid (1,2),(1,3))\,.$

Crystal operators in the alcove model

Type A_2 , $\lambda = (5, 3, 0) =$ .

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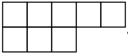
$((2,3),(1,3) \mid (2,3),(1,3) \mid (2,3),(1,3) \mid (1,2),(1,3) \mid (1,2),(1,3))\,.$

Let $J = \{3, 6, 9\}$ in $\mathcal{A}(\lambda)$.

Crystal operators in the alcove model

Type A_2 , $\lambda = (5, 3, 0) =$ .

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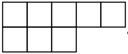
Let $J = \{3, 6, 9\}$ in $\mathcal{A}(\lambda)$.

Step 1: Construct the "folded chain" $\Gamma(J)$.

Crystal operators in the alcove model

Type A_2, $\lambda = (5, 3, 0) =$ .

λ-chain $\Gamma = \Gamma_2 \Gamma_2 \Gamma_2 \Gamma_1 \Gamma_1 =$

$((2,3),(1,3) \mid (2,3),(1,3) \mid (2,3),(1,3) \mid (1,2),(1,3) \mid (1,2),(1,3))$.

Let $J = \{3, 6, 9\}$ in $\mathcal{A}(\lambda)$.

Step 1: Construct the "folded chain" $\Gamma(J)$.

$((2,3),(1,3) \mid \underline{(2,3)},(1,2) \mid (3,2),\underline{(1,2)} \mid (2,3),(2,1) \mid \underline{(2,3)},(3,1))$

Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

$J = \{3, 6, 9\}$.

$$\Gamma(J) = ((2,3),(1,3) \mid \underline{(2,3)},(1,2) \mid (3,2),\underline{(1,2)} \mid (2,3),(2,1) \mid \underline{(2,3)},(3,1))$$

- For f_2 only look at $(2,3)$, $\underline{(2,3)}$, and $(3,2)$ in $\Gamma(J)$.

Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

$J = \{3, 6, 9\}$.

$$\Gamma(J) = ((2,3), (1,3) \mid \underline{(2,3)}, (1,2) \mid (3,2), \underline{(1,2)} \mid (2,3), (2,1) \mid \underline{(2,3)}, (3,1))$$

- For f_2 only look at $(2,3)$, $\underline{(2,3)}$, and $(3,2)$ in $\Gamma(J)$.

Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

$J = \{3, 6, 9\}$.

$$\Gamma(J) = ((2,3), (1,3) \mid \underline{(2,3)}, (1,2) \mid (3,2), \underline{(1,2)} \mid (2,3), (2,1) \mid \underline{(2,3)}, (3,1))$$

- For f_2 only look at $(2,3)$, $\underline{(2,3)}$, and $(3,2)$ in $\Gamma(J)$.
- Cancel pairs $(3,2)$, $(2,3)$ *like before*.

Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

$J = \{3, 6, 9\}$.

$\Gamma(J) = ((2,3),(1,3) \mid \underline{(2,3)},(1,2) \mid (3,2),\underline{(1,2)} \mid (2,3),(2,1) \mid \underline{(2,3)},(3,1))$

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Crystal operators in the alcove model (cont.)

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- Consider rightmost $(2,3)$ *like before*.

Crystal operators in the alcove model (cont.)

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Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

$J = \{3, 6, 9\}$.

$\Gamma(J) = ((2,3), (1,3) \mid \underline{(2,3)}, (1,2) \mid (3,2), \underline{(1,2)} \mid (2,3), (2,1) \mid \underline{(2,3)}, (3,1))$

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- Add corresponding position to J, and remove from J the position corresponding to $\underline{(2,3)}$ to its right (if any).

Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

$J = \{3, 6, 9\}$.

$$\Gamma(J) = (\underline{(2,3)}, (1,3) \mid (2,3), (1,2) \mid (3,2), \underline{(1,2)} \mid (2,3), (2,1) \mid \underline{(2,3)}, (3,1))$$

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- Cancel pairs $(3,2)$, $(2,3)$ *like before*.
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Crystal operators in the alcove model (cont.)

Step 2. Signature rule.

$J = \{3, 6, 9\}$. $f_2(J) = \{1, 6, 9\}$.

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Crystal operators in the alcove model (cont.)

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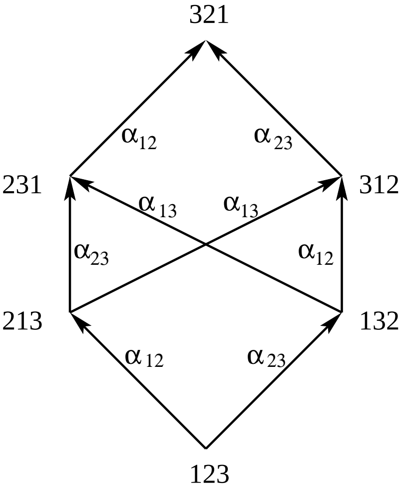
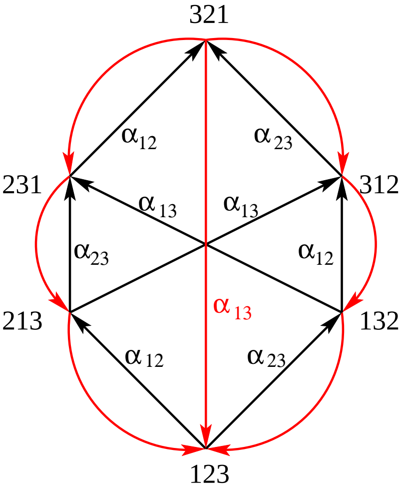
- Add corresponding position to J, and remove from J the position corresponding to $\underline{(2,3)}$ to its right (if any).

Note: In arbitrary type, f_i is defined based on the simple roots α_i in $\Gamma(J)$.

Bruhat graph for S_3:



Quantum Bruhat graph for S_3:



Realizing $\bigotimes B(\omega_i)$ as $\mathcal{A}(\lambda)_q$

Example in type A_2. Consider

$$B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1) \quad \text{and} \quad \lambda = \omega_1 + \omega_2 + \omega_2 + \omega_1 = (4, 2, 0).$$

Realizing $\bigotimes B(\omega_i)$ as $\mathcal{A}(\lambda)_q$

Example in type A_2. Consider

$B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_2)\otimes B(\omega_1)$ and $\lambda = \omega_1+\omega_2+\omega_2+\omega_1 = (4,2,0)$.

A λ-chain as a concatenation of ω_1-, ω_2-, ω_2-, and ω_1-chains:

$$\Gamma = (\ (1,2),\ (1,3) \mid (2,3),\ (1,3) \mid (2,3),\ (1,3) \mid (1,2),\ (1,3)\ ).$$

Realizing $\bigotimes B(\omega_i)$ as $\mathcal{A}(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$\left(\ \underline{(1,2)},\ \underline{(1,3)}\ \middle|\ \underline{(2,3)},\ (1,3)\ \middle|\ (2,3),\ \underline{(1,3)}\ \middle|\ \underline{(1,2)},\ \underline{(1,3)}\ \right).$$

Realizing $\bigotimes B(\omega_i)$ as $\mathcal{A}(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

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Claim: J is in $\mathcal{A}(\lambda)_q$.

Realizing $\bigotimes B(\omega_i)$ as $\mathcal{A}(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$(\ \underline{(1,2)},\ \underline{(1,3)}\ \mid\ \underline{(2,3)},\ (1,3)\ \mid\ (2,3),\ \underline{(1,3)}\ \mid\ \underline{(1,2)},\ \underline{(1,3)}\ ).$$

Claim: J is in $\mathcal{A}(\lambda)_q$. Indeed, the corresponding path in the quantum Bruhat graph is

$$\begin{array}{ccccc} \boxed{1} & & \boxed{2} & & \boxed{3} \\ & < & & < & \\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} & \to & \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} & \to & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \end{array} \Bigg| \begin{array}{ccc} \begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array} & < & \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} \\ \boxed{2} & \to & \boxed{1} \end{array} \Bigg| \begin{array}{ccc} \begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array} & > & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \\ \boxed{1} & \to & \boxed{3} \end{array} \Bigg| \begin{array}{ccccc} \boxed{1} & & \boxed{2} & & \boxed{3} \\ & < & & < & \\ \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} & \to & \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} & \to & \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \end{array}.$$

Realizing $\bigotimes B(\omega_i)$ as $\mathcal{A}(\lambda)_q$, cont.

Example. Let $J = \{1, 2, 3, 6, 7, 8\}$.

$$(\ \underline{(1,2)},\ \underline{(1,3)}\ |\ \underline{(2,3)},\ (1,3)\ |\ (2,3),\ \underline{(1,3)}\ |\ \underline{(1,2)},\ \underline{(1,3)}\ ).$$

Claim: J is in $\mathcal{A}(\lambda)_q$. Indeed, the corresponding path in the quantum Bruhat graph is

$$\begin{array}{c}\boxed{1}\\ \\ \boxed{2}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \rightarrow\end{array} \begin{array}{c}\boxed{2}\\ \\ \boxed{1}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \rightarrow\end{array} \begin{array}{c}\boxed{3}\\ \\ \boxed{1}\\ \boxed{2}\end{array} \ \Big|\ \begin{array}{c}\boxed{3}\\ \boxed{1}\\ \\ \boxed{2}\end{array} \begin{array}{c}<\\ \rightarrow\end{array} \begin{array}{c}\boxed{3}\\ \boxed{2}\\ \\ \boxed{1}\end{array} \ \Big|\ \begin{array}{c}\boxed{3}\\ \boxed{2}\\ \\ \boxed{1}\end{array} \begin{array}{c}>\\ \rightarrow\end{array} \begin{array}{c}\boxed{1}\\ \boxed{2}\\ \\ \boxed{3}\end{array} \ \Big|\ \begin{array}{c}\boxed{1}\\ \\ \boxed{2}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \rightarrow\end{array} \begin{array}{c}\boxed{2}\\ \\ \boxed{1}\\ \boxed{3}\end{array} \begin{array}{c}<\\ \rightarrow\end{array} \begin{array}{c}\boxed{3}\\ \\ \boxed{1}\\ \boxed{2}\end{array}.$$

The corresponding element in $B(\omega_1) \otimes B(\omega_2) \otimes B(\omega_2) \otimes B(\omega_1)$ (column-strict filling), obtained via "fillord:"

$$\boxed{3} \otimes \begin{array}{c}\boxed{2}\\ \boxed{3}\end{array} \otimes \begin{array}{c}\boxed{1}\\ \boxed{2}\end{array} \otimes \boxed{3}\,.$$

The combinatorial R-matrix in type A

Realized by Schützenberger's jeu de taquin (sliding algorithm) on two columns.

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Example. Realize the isomorphism

$$B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_2)\otimes B(\omega_1) \simeq B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_1)\otimes B(\omega_2)\,.$$

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Commute the last two factors as follows:

$$\boxed{3}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\otimes\boxed{3}$$

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Commute the last two factors as follows:

$$\begin{array}{|c|}\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\otimes\begin{array}{|c|}\hline 3\\\hline\end{array}=\begin{array}{|c|}\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & \\\hline\end{array}$$

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$$\begin{array}{|c|}\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\otimes\begin{array}{|c|}\hline 3\\\hline\end{array}=\begin{array}{|c|}\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 1 & 3\\\hline 2 & \\\hline\end{array}\mapsto\begin{array}{|c|}\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 1 & \\\hline 2 & 3\\\hline\end{array}\mapsto$$

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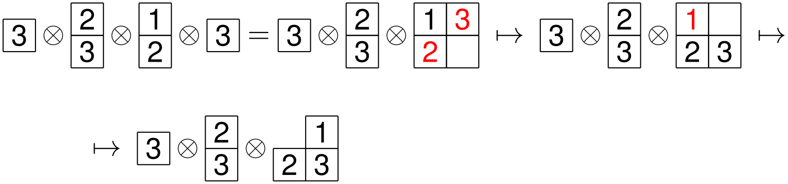

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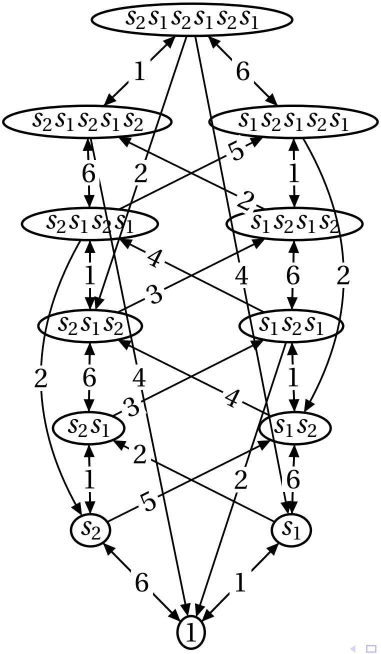
Commute the last two factors as follows:

$$\boxed{3}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\otimes\boxed{3} = \boxed{3}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline 1 & \color{red}{3}\\\hline \color{red}{2} & \\\hline\end{array} \mapsto \boxed{3}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{|c|c|}\hline \color{red}{1} & \\\hline 2 & 3\\\hline\end{array} \mapsto$$

$$\mapsto \boxed{3}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\begin{array}{cc} & \boxed{1}\\ \boxed{2} & \boxed{3}\end{array} = \boxed{3}\otimes\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\otimes\boxed{2}\otimes\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}.$$

Example. Type G_2. $s_2 s_1 s_2 \to s_1$: $1, 2, 5, 6$; $6, 3, 2, 1$.



The quantum Yang–Baxter moves via the running example.

Realize the isomorphism

$$B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_2)\otimes B(\omega_1) \simeq B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_1)\otimes B(\omega_2)\,.$$

The quantum Yang–Baxter moves via the running example.

Realize the isomorphism

$$B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_2)\otimes B(\omega_1)\simeq B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_1)\otimes B(\omega_2)\,.$$

$$(\ \underline{(1,2)},\underline{(1,3)}\mid\underline{(2,3)},(1,3)\mid(2,3),\underline{(1,3)}\mid\underline{(1,2)},\underline{(1,3)}\ )\,,$$

$$(\ (1,2),(1,3)\mid(2,3),(1,3)\mid(1,2),(1,3)\mid(2,3),(1,3)\ )\,.$$

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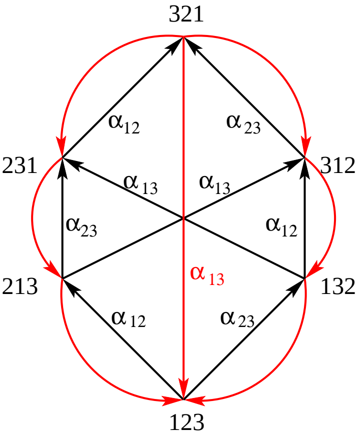
$$(\ \underline{(1,2)},\ \underline{(1,3)}\ |\ \underline{(2,3)},\ (1,3)\ |\ (2,3),\ \underline{(1,3)}\ |\ \underline{(1,2)},\ \underline{(1,3)}\ )\,,$$

$$(\ (1,2),\ (1,3)\ |\ (2,3),\ (1,3)\ |\ (1,2),\ (1,3)\ |\ (2,3),\ (1,3)\ )\,.$$



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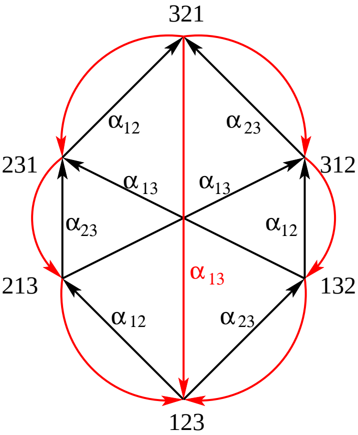
$$(\,\underline{(1,2)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid (2,3), \underline{(1,3)} \mid \underline{(1,2)}, \underline{(1,3)}\,)\,,$$

$$(\,\underline{(1,2)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid (1,2), (1,3) \mid (2,3), \underline{(1,3)}\,)\,.$$



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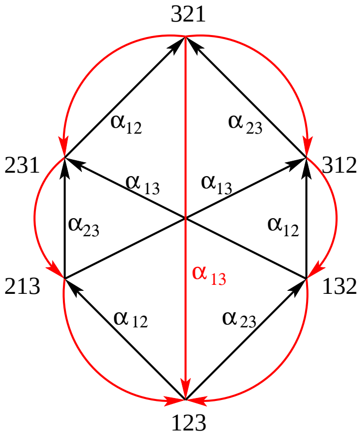
$$(\,\underline{(1,2)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid (2,3), \underline{(1,3)} \mid \underline{(1,2)}, \underline{(1,3)}\,),$$

$$(\,\underline{(1,2)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid \underline{(1,2)}, (1,3) \mid \underline{(2,3)}, \underline{(1,3)}\,).$$



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$$(\, \underline{(1,2)}, \underline{(1,3)} \mid \underline{(2,3)}, (1,3) \mid \underline{(1,2)}, (1,3) \mid \underline{(2,3)}, \underline{(1,3)} \,)\,.$$

The Bruhat chain corresponding to the second case:

$$\begin{array}{c} 1 \\ \\ 2 \\ 3 \end{array} \begin{array}{c} \\ < \\ \to \\ \end{array} \begin{array}{c} 2 \\ \\ 1 \\ 3 \end{array} \begin{array}{c} \\ < \\ \to \\ \end{array} \begin{array}{c} 3 \\ \\ 1 \\ 2 \end{array} \;\Big|\; \begin{array}{c} 3 \\ 1 \\ \\ 2 \end{array} \begin{array}{c} \\ < \\ \to \\ \end{array} \begin{array}{c} 3 \\ 2 \\ \\ 1 \end{array} \;\Big|\; \begin{array}{c} 3 \\ \\ 2 \\ 1 \end{array} \begin{array}{c} \\ > \\ \to \\ \end{array} \begin{array}{c} 2 \\ \\ 3 \\ 1 \end{array} \;\Big|\; \begin{array}{c} 2 \\ 3 \\ \\ 1 \end{array} \begin{array}{c} \\ > \\ \to \\ \end{array} \begin{array}{c} 2 \\ 1 \\ \\ 3 \end{array} \begin{array}{c} \\ < \\ \to \\ \end{array} \begin{array}{c} 3 \\ 1 \\ \\ 2 \end{array}\,.$$

The quantum Yang–Baxter moves via the running example.

Realize the isomorphism

$$B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_2)\otimes B(\omega_1) \simeq B(\omega_1)\otimes B(\omega_2)\otimes B(\omega_1)\otimes B(\omega_2)\,.$$

$$\big(\,\underline{(1,2)},\underline{(1,3)} \mid \underline{(2,3)},(1,3) \mid (2,3),\underline{(1,3)} \mid \underline{(1,2)},\underline{(1,3)}\,\big)\,,$$

$$\big(\,\underline{(1,2)},\underline{(1,3)} \mid \underline{(2,3)},(1,3) \mid \underline{(1,2)},(1,3) \mid \underline{(2,3)},\underline{(1,3)}\,\big)\,.$$

The Bruhat chain corresponding to the second case:

$$\begin{array}{ccccc|ccc|ccc|ccccc}
\boxed{1} & < & \boxed{2} & < & \boxed{3} & \begin{array}{|c|}\hline 3\\ \hline 1\\ \hline\end{array} & < & \begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array} & \boxed{3} & > & \boxed{2} & \begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array} & > & \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} & < & \begin{array}{|c|}\hline 3\\ \hline 1\\ \hline\end{array} \\
\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array} & \to & \begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array} & \to & \begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array} & \boxed{2} & \to & \boxed{1} & \begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array} & \to & \begin{array}{|c|}\hline 3\\ \hline 1\\ \hline\end{array} & \boxed{1} & \to & \boxed{3} & \to & \boxed{2}
\end{array}\,.$$

So we have

$$\boxed{3}\otimes\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\otimes\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\otimes\boxed{3} \;\mapsto\; \boxed{3}\otimes\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\otimes\boxed{2}\otimes\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\,.$$