Computational Challenges in the Method of Integral Representations

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The bulk of my research is in the theory of integral representations of Langlands $L$-functions. This is the art of identifying integrals which

- involve an automorphic form (or some automorphic forms) on an adele group (or some adele groups),
- involve a complex variable, in which they are meromorphic with a functional equation as $s \to 1 - s$
- can be expressed as a product of local integrals corresponding to the places of the ambient global field,

and then trying to compute these local integrals, in the hopes that the answer will come out in terms of Langlands $L$-functions. (It always does.) (My collaborators in this work have been Wee Teck Gan and David Ginzburg, and deserve the credit for the artistry.)
What David and I have mostly been up to lately is trying to push a couple of ideas that have worked in the classical groups into the exceptional groups.

- **restriction** restrict an Eisenstein series on $G(F) \backslash G(\mathbb{A})$ to $H(F) \backslash H(\mathbb{A})$ and integrate against a cusp form or vice versa
- **trilinear forms** integrate the product of three automorphic forms (at least one an Eisenstein series)
- **Fourier coefficients** use a Fourier coefficient attached to a nilpotent orbit to map automorphic forms on $G(F) \backslash G(\mathbb{A})$ to automorphic functions on $C(F) \backslash C(\mathbb{A})$ and pair with forms on $C(F) \backslash C(\mathbb{A})$.

We make educated guesses about where to look based on attaching a dimension to (some?) automorphic representations using these same Fourier coefficients attached to nilpotent orbits (following work of Kawanaka on finite groups of Lie type).
Recently, Ginzburg and I were able to show that a certain integral of the third type, with $G = E_8$ and $C = G_2 \times G_2$ gives a new integral representation for the twisted partial standard $L$ function of a cuspidal representation of $G_2(\mathbb{A})$ (generic or not).

We’ve also done an exhaustive survey of all integrals of any of these three types where $G = F_4$ and our dimension formalism is satisfied.

With some students, I’m working on generalizing a work of Jiang-Rallis which represents $\zeta_K(s)/\zeta_F(s)$, where $K$ is a finite extension of $F$, as a Fourier coefficient of a $G_2$ Eisenstein series. (Jiang-Rallis assume $F$ has three cube roots of 1.)

I’m also looking at integrals of this type (which I call section integrals):

$$\int_{U(F)\backslash U(\mathbb{A})} f_{\chi}(wu) \psi(u) \, du$$

$U$ a unipotent subgroup of a reductive group $G$, $f_{\chi}$ the normalized spherical vector in a principal series representation, $\psi$ a character of $U$, and $w$ a Weyl group element.
To push this program into the big exceptional groups, I’ve been writing computer programs. I look forward to being able to discuss computational approaches and algorithms with others. The “section integrals” have an interesting combinatorics into which I hope to gain some insight. The representation of $\zeta_K/\zeta_F$ seems to be based on a generalization of uniqueness of Whittaker models. The Whittaker model is attached to the regular nilpotent orbit and is unique. We have a model which is attached to the next biggest one, and is not in general unique but seems to be unique for the specific degenerate induced representation to which it is applied here. Interesting things have resulted from considering Whittaker models on metaplectic covers where they are no longer unique. Might similar interesting things result from considering a metaplectic version of this integral? (There is also a baby version on $Sp_4$ for quadratic extensions.)