Homogeneous Representations of Khovanov-Lauda-Rouquier Algebras of type $A_n$

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Dyck Paths

Definition

A **Dyck Path** of semi length \( n \in \mathbb{N} \) is a path starting at \((0, 0)\), ending at \((2n, 0)\), with steps \( \langle 1, 1 \rangle \) and \( \langle 1, -1 \rangle \), never crossing below the \( x \)-axis.

Example \((D_1)\)

Example \((D_2)\)
A Statistic, $k$, on Dyck Paths

**Definition**

For a Dyck path, $D$, define the statistic

$$k = (\text{Sum of Peak Heights}) - (\# \text{ of Peaks})$$

**Example ($D_1$)**

$$k = 2 - 1 = 1$$

**Example ($D_2$)**

$$k = (2 + 3 + 1) - 3 = 3$$
The KLR-Algebra

KLR-Algebra, $R_\alpha$

Rep $R_\alpha$
The KLR-Algebra

KLR-Algebra, $R_\alpha$

$\text{Rep} \, R_\alpha$

Quantum Groups, Crystals
The KLR-Algebra

KLR-Algebra, $R_\alpha$

$\text{Rep } R_\alpha$

Cuspidal Representations

Quantum Groups, Crystals
The KLR-Algebra

Homogeneous Representations \(\supseteq\) Cuspidal Representations

KLR-Algebra, \(R_\alpha\)

\(\text{Rep } R_\alpha\)

Quantum Groups, Crystals
The KLR-Algebra

Set-up:

- Fix a quiver of type $A_n$, $\Gamma : 1 \rightarrow 2 \rightarrow 3 \rightarrow \cdots \rightarrow n$, with vertex set $I = 1, \ldots, n$
- $Q_+ = \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$
- $I^\alpha = \left\{ w = [i_1, \ldots, i_d] \in I^d \middle| \alpha_{i_1} + \cdots + \alpha_{i_d} = \alpha \right\}$ (for $\alpha \in Q$)

Example

For $\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 \in Q_+$ (so $d=6$),

$[1, 2, 2, 3, 3, 4], \quad [2, 1, 3, 2, 4, 3] \in I^\alpha$
The KLR algebra, $R_\alpha$ (associated with quiver $\Gamma$), is the associative $F$-algebra generated by the union of

- idempotents $\{e(w) \mid w \in I^\alpha\}$
- symmetric generators $\{\psi_1, \ldots, \psi_{d-1}\}$
- polynomial generators $\{y_1, \ldots, y_d\}$

subject to relations . . .
The KLR algebra

\[ e(w)e(v) = \delta_{wv}, \quad \sum_{w \in I^\alpha} e(w) = 1 \]

\[ y_k e(w) = e(w)y_k, \]

\[ \psi_k e(w) = e(s_k w)\psi_k, \]

\[ y_k y_\ell = y_\ell y_k, \]

\[ y_k \psi_\ell = \psi_\ell y_k, \text{ (for } k \neq \ell, \ell + 1 \text{)} \]

\[ (y_{k+1}\psi_k - \psi_k y_k)e(w) = \begin{cases} e(w) & \text{if } w_k = w_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \]
The KLR algebra

\begin{align*}
(\psi_k y_{k+1} - y_k \psi_k) e(w) &= \begin{cases} 
  e(w) & \text{if } w_k = w_{k+1}, \\
  0 & \text{otherwise},
\end{cases} \\
\psi_k^2 e(w) &= \begin{cases} 
  0 & \text{if } w_k = w_{k+1}, \\
  (y_k - y_{k+1}) e(w) & \text{if } w_k \to w_{k+1}, \\
  (y_{k+1} - y_k) e(w) & \text{if } w_k \leftarrow w_{k+1}, \\
  e(w) & \text{otherwise},
\end{cases} \\
\psi_k \psi_\ell &= \psi_\ell \psi_k, (\text{for } |k - \ell| > 1) \\
(\psi_{k+1} \psi_k \psi_{k+1} - \psi_k \psi_{k+1} \psi_k) e(w) &= \begin{cases} 
  e(w) & \text{if } w_{k+2} = w_k \to w_{k+1}, \\
  -e(w) & \text{if } w_{k+2} = w_k \leftarrow w_{k+1}, \\
  0 & \text{otherwise},
\end{cases}
\end{align*}
Homogeneous Representations are given by (classes of) words \([i_1, \cdots, i_d] \in I^\alpha\) satisfying the homogeneity condition:

**Homogeneity Condition**

If \(i_r = i_s\) for some \(r < s\), then there exist \(t, u\) with \(r < t < u < s\) such that \(i_r\) is neighbors with both \(i_t\) and \(i_u\).
Homogeneous Representations

Example (Homogeneous)

213243

231243

213423

231243

234123

ψ₂

ψ₄

ψ₄

ψ₂

Homogeneity Condition

If \( i_r = i_s \) for some \( r < s \), then there exist \( t, u \) with \( r < t < u < s \) such that \( i_r \) is neighbors with both \( i_t \) and \( i_u \).

Example (Not Hom.)

132234

ψ₁

312234

A Bijection

Theorem (F, Lee)

There is a bijection between the irreducible homogeneous representations of a KLR algebra of type $A_n$ and the Dyck paths of semi-length $n + 1$. Further, if a representation is given by a component whose words have $k$ letters, the corresponding Dyck path has $(\text{Sum of Peak Heights}) - (\# \text{ of Peaks}) = k$. 

There is a bijection between the irreducible homogeneous representations of a KLR algebra of type $A_n$ and the Dyck paths of semi-length $n + 1$.

Further, if a representation is given by a component whose words have $k$ letters, the corresponding Dyck path has $(\text{Sum of Peak Heights}) - (\# \text{ of Peaks}) = k$. 
Example \((k = 6 = (3 + 3 + 3) - 3)\)

\[
\begin{align*}
234123 & \quad \Leftrightarrow \quad \begin{array}{c}
21 \quad 32 \quad 43
\end{array} \\
\vdots & \\
(21)(32)(43) & \quad \Leftrightarrow \quad (21) \quad (32) \quad (43)
\end{align*}
\]

Example \((k = 6 = (4 + 4) - 2)\)

\[
\begin{align*}
324312 & \quad \Leftrightarrow \quad \begin{array}{c}
21 \quad 32 \quad 43
\end{array} \\
\vdots & \\
(321)(432) & \quad \Leftrightarrow \quad (321) \quad (432)
\end{align*}
\]