Multiple Dirichlet Series

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2013-02-18 Mon

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2. The d^{th} Fourier coefficient is a Dirichlet series with
   - functional equations
   - analytic continuation
3. These properties came from the analytic properties of the Eisenstein series

This talk

- Focus on $n = 2$: Fourier coeffs now involve quadratic Dirichlet L-functions:

$$E(z, s) = * + \sum_{d \neq 0} L(s', \chi_d) e^{2\pi i x d} W_s(dy)$$

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- Properties can be established independent of theory of Eisenstein series (mostly)
- "Axiomatic development" of the theory of WMDS

Transition to *local* parts

- Emphasis on construction of local parts. Two methods:
  - Functional Equations (Chinta/Gunnells)
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- Emphasis on construction of local parts. Two methods:
  - Functional Equations (Chinta/Gunnells)
  - Crystal basis description (Brubaker, Bump, Friedberg)
- Two methods are (more or less) equivalent when they both apply, but it's not so easy to see

A_2 quadratic double Dirichlet series

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Where do these functional equations come from?

Background on quadratic Dirichlet L-functions

Define

$$L(s, \chi_{d_0}) = \sum_{n=1}^{\infty} \frac{\chi_{d_0}(n)}{n^s} = \prod_p \left(1 - \frac{\chi_{d_0}(p)}{p^s}\right)^{-1}$$

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- Euler product
- Analytic continuation
  - For d_0 a **fundamental discriminant**, $L(s, \chi_{d_0})$ is entire unless $d_0 = 1$
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$$L(s, \chi_{d_0}) = *|d|^{1/2-s} L(1-s, \chi_{d_0})$$

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But we need to define for all d. How to do this without messing up the earlier properties?

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- functional equation: $P(s, d) = (d/d_0)^{1/2-s} P(1-s, d)$.
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If this last condition is satisfied, then the double Dirichlet series will have a functional equation $(s,w) \mapsto (1-s, s+w-1/2)$

Passage to p-parts

Two types of p-parts for $L(s,d) = L(s,\chi_{d_0}) \cdot P(s,d)$:

- if $(p, d_0) = 1$:

$$\boxed{(1-\chi_{d_0}(p)p^{-s})^{-1} \ times \ P_p(p^{-s}, d)}$$

- if $p | d_0$,

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We will next describe the construction of the p-part as in Chinta-Gunnells

A generating series

Two variable generating function

$$F(x,y) = \sum_{k\geq 0}(1-x)^{-1}P_p(x,p^{2k})y^{2k} + \sum_{k\geq 0}P_p(x,p^{2k+1})y^{2k+1}$$

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We need F to satisfy

1. $(1-x)\left[F(x,y)+F(x,-y)\right]$ and $\frac{1}{y}\left[F(x,y)-F(x,-y)\right]$ are invariant under $(x,y) \mapsto \left(\frac{1}{px}, xy\sqrt{p}\right)$.
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$$\implies F(x,y) = \frac{1-xy}{(1-x)(1-y)(1-px^2y^2)}$$

Definition of a group action

Let f be the monomial $x^a y^b z^c$. Define

$$(\sigma_1 f)(x,y,z) = \begin{cases} f(y,x,z) & \text{if } a-b \text{ even} \\ f(y,x,z)\frac{x_2 - tx_1}{x_1 - tx_2} & \text{if } a-b \text{ odd} \end{cases}$$

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Extend to polynomials by linearity and then to rational functions. This will define an action of S_3 on rational functions.

Construction of p-parts

For $a \geq b \geq c$ define the polynomials

$$N_{a,b,c}(x, y, z) = \frac{\sum_{w \in W} (-1)^{\mathtt{length}(w)} w(x^{a+2} y^{b+1} z^c)}{\Delta(x)} \cdot D(x)$$

where

$$D(x) = \prod_{i<j} (x_i^2 - t^2 x_j^2) \text{ and } \Delta(x) = \prod_{i<j} (x_i^2 - x_j^2).$$

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These polynomials arise in the construction of the A_2 multiple Dirichlet series. They also arise as Whittaker functions on the double cover of GL_3.

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Define the **weight** of I to be

$$(a_{00} + a_{01} + a_{02} - a_{11} - a_{12}, a_{11} + a_{12} - a_{22}, a_{22})$$

and "(right sum) — (up-and-right sum)" coordinates

$$r_{11} = (a_{11} - a_{01}) + (a_{12} - a_{02}), r_{12} = a_{12} - a_{02}, r_{22} = a_{22} - a_{12}.$$

GT-formula (cont.)

We say

- I is **nonstrict** if the rows of I are not strictly decreasing
- I is **right-leaning** at (i,j) if $a_{i,j} = a_{i-1,j}$
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$$G(I) = \begin{cases} 0 & \text{if } I \text{ is nonstrict} \\ G(I;1,1)G(I;1,2)G(I;2,2) & \text{otherwise} \end{cases}$$

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where

	left-leaning	right-leaning	non-leaning
r_{ij} even	$-t^2$	1	$1 - t^2$
r_{ij} odd	t	1	0

GT-formula (cont.)

Finally we define

$$M_{a,b,c}(x,y,z) = \sum_{I} G(I) x^{wt(I)}$$

where the sum is over all GT-patterns with top row $a+2, b+1, c$.

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(Follows from work of McNamara, Chinta-Offen, Chinta-Gunnells. But more direct proof??)

Other problems

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3. Function fields
   - Multiple Dirichlet series are rational functions

Multiple Dirichlet series arising in other settings

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- Are they related?
- Can techniques introduced/refined in the study of WMDS be used in different settings

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1. Point evaluations of higher rank Eisenstein series
   - Representation numbers of quadratic forms
   - Count integral points on flag varieties
   - Predict or give evidence for automorphic liftings

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2. Zeta functions of prehomogeneous vector spaces
   - Well-developed theory for reductive groups
   - Rich arithmetic theory evident in spaces coming from parabolic subgroups of reductive spaces