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Study $H_T^*(G/B)$

Example: Type C_2, $G = Sp_4$

$$W_0 = \left\{ \begin{array}{cc} & 1 \\ s_1 & s_2 \\ s_1s_2 & s_2s_1 \\ s_1s_2s_1 & s_2s_1s_2 \\ & s_1s_2s_1s_2 \end{array} \right\}$$ with $s_i^2 = 1$ and $s_1s_2s_1s_2 = s_2s_1s_2s_1$

acts on $\mathbb{C}[y_1, y_2]$ by

$s_1 y_1 = y_2,\quad s_2 y_1 = y_1,$

$s_1 y_2 = y_1,\quad s_2 y_2 = -y_2.$

So

$$\mathbb{C}[y_1, y_2]^{W_0} = \mathbb{C}[y_1^2 + y_2^2,\ y_1^2 y_2^2]$$

$$R^+ = \left\{ \begin{array}{l} y_{\alpha_1} = y_1 - y_2 \\ y_{\alpha_2} = 2y_2 \\ y_{\alpha_1+\alpha_2} = y_1 + y_2 \\ y_{2\alpha_1+\alpha_2} = 2y_1 + y_2 \end{array} \right\}$$ and $$R^- = \left\{ \begin{array}{l} y_{-\alpha_1} = y_2 - y_1 \\ y_{\alpha_2} = -2y_2 \\ y_{-\alpha_1-\alpha_2} = -y_1 - y_2 \\ y_{-2\alpha_1-\alpha_2} = -2y_1 \end{array} \right\}$$

The Borel model for $H_T^*(G/B)$

$$H_T^*(G/B) = \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{\langle f(x_1, x_2) - f(y_1, y_2) \mid f \in \mathbb{C}[x_1, x_2]^{W_0} \rangle}$$

$$= \frac{\mathbb{C}[x_1, x_2, y_1, y_2]}{\langle x_1^2 + x_2^2 = y_1^2 + y_2^2,\ x_1^2 x_2^2 = y_1^2 y_2^2 \rangle}$$

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The moment graph model

$$H_T^*(G/B) \xrightarrow{\Phi} \bigoplus_{w \in W_0} \mathbb{C}[y_1, y_2]$$

$$f(x_1, x_2) \longmapsto (wf(y_1, y_2))_{w \in W_0}$$

$$g(y_1, y_2) \longmapsto (g(y_1, y_2))_{w \in W_0}$$

is an injective ring homomorphism

So

$$y_1 = \begin{array}{ccc} & y_1 & \\ y_1 & & y_1 \\ y_1 & & y_1 \\ y_1 & & y_1 \\ & y_1 & \end{array} \qquad y_2 = \begin{array}{ccc} & y_2 & \\ y_2 & & y_2 \\ y_2 & & y_2 \\ y_2 & & y_2 \\ & y_2 & \end{array} \qquad x_1 = \begin{array}{ccc} & y_1 & \\ y_2 & & y_1 \\ -y_2 & & y_2 \\ -y_1 & & -y_2 \\ & -y_1 & \end{array} \qquad x_2 = \begin{array}{ccc} & y_2 & \\ y_1 & & -y_2 \\ y_1 & & -y_1 \\ y_2 & & -y_1 \\ & -y_2 & \end{array}$$

$$x_{-\alpha_1} = x_2 - x_1 = \begin{array}{ccc} & y_2 - y_1 & \\ y_1 - y_2 & & -y_2 - y_1 \\ y_1 + y_2 & & -y_1 - y_2 \\ y_2 + y_1 & & -y_1 + y_2 \\ & -y_2 + y_1 & \end{array} \qquad x_{\alpha_2} = -2x_2 = \begin{array}{ccc} & -2y_2 & \\ -2y_1 & & 2y_2 \\ -2y_1 & & 2y_1 \\ -2y_2 & & 2y_1 \\ & 2y_2 & \end{array}$$

BGG-operators and Bott-Samelson classes

$$A_i = (1 + t_{s_i}) \frac{1}{x_{-\alpha_i}} \quad \text{where} \quad t_{s_i}(f_w)_{w \in W_0} = (f_{s_i z})_{z \in W_0}$$

The Bott-Samelson classes are

$$[Z_{pt}] = \begin{cases} y_{R^-}, & \text{if } w = 1 \\ 0, & \text{if } w \neq 1 \end{cases} \quad \text{where} \quad y_{R^-} = \prod_{\alpha \in R^+} y_{-\alpha}$$

and, for a sequence $s_{i_1}\cdots s_{i_\ell}$ (not necessarily reduced)

$$[Z_{i_1\cdots i_\ell}] = A_{i_1}\cdots A_{i_\ell}[Z_{pt}].$$

For example,

$$[Z_{121}] = A_1[Z_{21}] = (1+t_{s_1})\frac{1}{X_{-\alpha_1}}[Z_{21}]$$

$$= (1+t_{s_1})\begin{pmatrix} & \frac{1}{y_2-y_1} & \\ \frac{1}{y_1-y_2} & & \frac{1}{-y_2-y_1} \\ \frac{1}{y_1+y_2} & & \frac{1}{-y_1-y_2} \\ & \frac{1}{-y_2+y_1} & \end{pmatrix}\begin{pmatrix} & (-y_1-y_2)(-2y_1) & \\ (-y_1-y_2)(-2y_2) & & (-y_1-y_2)(-2y_1) \\ 0 & & (-y_1-y_2)(-2y_2) \\ 0 & & 0 \\ & 0 & \end{pmatrix}$$

$$= (1+t_{s_1})\begin{pmatrix} & \frac{-y_1-y_2}{y_2-y_1}(-2y_1) & \\ \frac{-y_1-y_2}{y_1-y_2}(-2y_2) & & -2y_1 \\ 0 & & -2y_2 \\ 0 & & 0 \\ & 0 & \end{pmatrix}$$

$$= \begin{pmatrix} & \frac{-y_1-y_2}{y_2-y_1}(-2y_1) & \\ \frac{-y_1-y_2}{y_1-y_2}(-2y_2) & & -2y_1 \\ 0 & & -2y_2 \\ 0 & & 0 \\ & 0 & \end{pmatrix} + \begin{pmatrix} & \frac{-y_1-y_2}{y_1-y_2}(-2y_2) & \\ \frac{-y_1-y_2}{y_2-y_1}(-2y_1) & & 0 \\ -2y_1 & & 0 \\ -2y_2 & & 0 \\ & 0 & \end{pmatrix} = \begin{pmatrix} & (-2)(-y_1-y_2) & \\ (-2)(y_1-y_2) & & -2y_1 \\ -2y_1 & & -2y_2 \\ -2y_2 & & 0 \\ & 0 & \end{pmatrix}$$

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Schubert classes

Theorem (Goresky-Kottwitz-MacPherson)

$$\mathrm{im}\,\Phi = \{ (f_w)_{w\in W_0} \mid f_w - f_{ws_\alpha} \in y_{-\alpha}\,\mathbb{C}[y_1,y_2] \}$$

The Schubert classes are $[X_w]$, $w \in W_0$, determined by

(a) $[X_w] \in \mathrm{im}\,\Phi$

(b) $[X_w]_w = \prod_{\substack{\alpha\in R^+ \\ w\alpha \notin R^-}} y_\alpha$ and $[X_w]_v = 0$ unless $v \le w$

(c) $\deg([X_w]_v) = \mathrm{Card}(R^+) - \ell(w)$, for all $v \in W_0$.

i.e.

$$[X_{s_1s_2s_1}] = \begin{array}{ccc} & a & \\ b & & c \\ d & & e \\ f & & 0 \\ & 0 & \end{array}$$

with $f = -2y_2$

$f - d \in (y_2 - y_1)\,\mathbb{C}[y_1,y_2]$

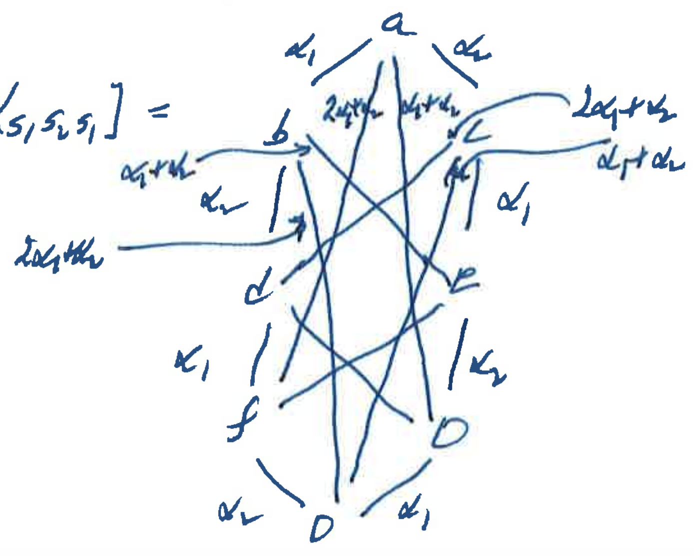
$d - b \in (-2y_2)\,\mathbb{C}[y_1,y_2]$

$b - a \in (y_2 - y_1)\,\mathbb{C}[y_1,y_2]$

$f - a \in (-2y_1)\,\mathbb{C}[y_1,y_2]$

$\vdots$

or $[X_{s_1s_2s_1}] =$



Theorem (BGG = Bernstein-Gelfand-Gelfand)

If $w = s_{i_1} \cdots s_{i_\ell}$ is reduced then

$$[Z_{i_1 \cdots i_\ell}] = [X_w].$$

Schubert products

$$[X_{s_1 s_2 s_1}]^2 = \begin{pmatrix} & 4(-y_1-y_2)^2 & \\ 4(-y_1-y_2)^2 & & 4y_1^2 \\ 4y_1^2 & & 4y_2^2 \\ 4y_2^2 & & 0 \\ & 0 & \end{pmatrix}$$

$$= (-2y_2) \begin{pmatrix} & (-2)(y_1-y_2) & \\ -2(-y_1-y_2) & & -2y_1 \\ -2y_1 & & -2y_2 \\ -2y_2 & & 0 \\ & 0 & \end{pmatrix} + \begin{pmatrix} & 4(-y_1-y_2)(y_1) & \\ 4(-y_1-y_2)(-y_2) & & 4y_1^2-4y_1y_2 \\ 4y_1^2-4y_1y_2 & & 0 \\ 0 & & 0 \\ & 0 & \end{pmatrix}$$

$$= (-2y_2)[X_{s_1 s_2 s_1}] + 2[X_{s_1 s_2}].$$

Some Bott-Samelson classes

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$$[Z_{pt}] = \begin{pmatrix} \multicolumn{2}{c}{(y_2-y_1)(-2y_2)(-y_1-y_2)(-2y_1)} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \multicolumn{2}{c}{0} \end{pmatrix}$$

$$[Z_1] = \begin{pmatrix} \multicolumn{2}{c}{(-y_1-y_2)(-2y_1)(-2y_2)} \\ (-y_1-y_2)(-2y_1)(-2y_2) & 0 \\ 0 & 0 \\ 0 & 0 \\ \multicolumn{2}{c}{0} \end{pmatrix} \qquad [Z_2] = \begin{pmatrix} \multicolumn{2}{c}{(-y_1-y_2)(y_2-y_1)(-2y_1)} \\ 0 & (-y_1-y_2)(y_2-y_1)(-2y_1) \\ 0 & 0 \\ 0 & 0 \\ \multicolumn{2}{c}{0} \end{pmatrix}$$

$$[Z_{12}] = \begin{pmatrix} \multicolumn{2}{c}{(-y_1-y_2)(-2y_1)} \\ (-y_1-y_2)(-2y_1) & (y_2-y_1)(-2y_1) \\ (y_2-y_1)(-2y_1) & 0 \\ 0 & 0 \\ \multicolumn{2}{c}{0} \end{pmatrix} \qquad [Z_{21}] = \begin{pmatrix} \multicolumn{2}{c}{(-y_1-y_2)(-2y_1)} \\ (-y_1-y_2)(-2y_2) & (-y_1-y_2)(-2y_1) \\ 0 & (-y_1-y_2)(-2y_2) \\ 0 & 0 \\ \multicolumn{2}{c}{0} \end{pmatrix}$$

$$[Z_{121}] = \begin{pmatrix} \multicolumn{2}{c}{-2(y_1+y_2)} \\ -2(y_1+y_2) & -2y_1 \\ -2y_1 & -2y_2 \\ -2y_2 & 0 \\ \multicolumn{2}{c}{0} \end{pmatrix} \qquad [Z_{212}] = \begin{pmatrix} \multicolumn{2}{c}{-2y_1} \\ -y_1-y_2 & -2y_1 \\ y_2-y_1 & -y_1-y_2 \\ 0 & y_2-y_1 \\ \multicolumn{2}{c}{0} \end{pmatrix}$$

$$[Z_{1212}] = \begin{pmatrix} \multicolumn{2}{c}{1} \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ \multicolumn{2}{c}{1} \end{pmatrix}$$