Triple Shifted Sums of Automorphic $L$-functions

Thomas Hulse
Brown University

ICERM Semester Program
Automorphic Forms, Combinatorial Representation Theory and Multiple Dirichlet Series
Providence, RI

January 29, 2013
Let \( f(z) \) and \( g(z) \) be even weight \( k > 0 \) holomorphic cusp forms on \( \Gamma_0(N) \backslash \mathbb{H} \) with respective Fourier expansions

\[
\begin{align*}
f(z) &= \sum_{m=1}^{\infty} a(m) e^{2\pi i m z} = \sum_{m=1}^{\infty} A(m) m^{\frac{k-1}{2}} e^{2\pi i m z}, \\
g(z) &= \sum_{m=1}^{\infty} b(m) e^{2\pi i m z} = \sum_{m=1}^{\infty} B(m) m^{\frac{k-1}{2}} e^{2\pi i m z}.
\end{align*}
\]
Let $f(z)$ and $g(z)$ be even weight $k > 0$ holomorphic cusp forms on $\Gamma_0(N) \backslash \mathbb{H}$ with respective Fourier expansions

$$f(z) = \sum_{m=1}^{\infty} a(m) e^{2\pi imz} = \sum_{m=1}^{\infty} A(m) m^{\frac{k-1}{2}} e^{2\pi imz},$$

$$g(z) = \sum_{m=1}^{\infty} b(m) e^{2\pi imz} = \sum_{m=1}^{\infty} B(m) m^{\frac{k-1}{2}} e^{2\pi imz}.$$
In 1965, Selberg\textsuperscript{[3]} constructed and gave meromorphic continuations of shifted convolution sums of the form

$$\sum_{m=1}^{\infty} \frac{a(m)b(m + h)}{(2m + h)^s},$$

by replacing the real-analytic Eisenstein Series in the Rankin-Selberg Convolution with a real-analytic Poincaré Series and then untiling.
In 1965, Selberg\textsuperscript{[3]} constructed and gave meromorphic continuations of shifted convolution sums of the form

\[
\sum_{m=1}^{\infty} \frac{a(m)b(m+h)}{(2m+h)^s},
\]

by replacing the real-analytic Eisenstein Series in the Rankin-Selberg Convolution with a real-analytic Poincaré Series and then untiling.

Such shifted convolution sums have been used to produce subconvexity bounds of moments of $L$-functions by providing asymptotic estimates of “off-diagonal” terms.
In 1965, Selberg\cite{3} constructed and gave meromorphic continuations of shifted convolution sums of the form
\[
\sum_{m=1}^{\infty} \frac{a(m)b(m+h)}{(2m+h)^s},
\]
by replacing the real-analytic Eisenstein Series in the Rankin-Selberg Convolution with a real-analytic Poincaré Series and then untiling.

Such shifted convolution sums have been used to produce subconvexity bounds of moments of $L$-functions by providing asymptotic estimates of “off-diagonal” terms.

By means of an approximation of a non-square integrable Poincaré series devised by Hoffstein, he and I were able to continue a variant of Selberg’s shifted sum with uncoupled denominator\cite{2}.
In 1965, Selberg\cite{3} constructed and gave meromorphic continuations of shifted convolution sums of the form

\begin{align*}
\text{Selberg: } & \sum_{m=1}^{\infty} \frac{a(m)b(m+h)}{(2m+h)^s}, \\
\text{Hoffstein \& H: } & \sum_{m=1}^{\infty} \frac{a(m+h)b(m)}{m^s}
\end{align*}

by replacing the real-analytic Eisenstein Series in the Rankin-Selberg Convolution with a real-analytic Poincaré Series and then untiling.

Such shifted convolution sums have been used to produce subconvexity bounds of moments of $L$-functions by providing asymptotic estimates of “off-diagonal” terms.

By means of an approximation of a non-square integrable Poincaré series devised by Hoffstein, he and I were able to continue a variant of Selberg’s shifted sum with uncoupled denominator\cite{2}.
Similarly, in the case where $N = 1$, this modified Poincaré series can be used to give a meromorphic continuation of the shifted convolution sum

$$\sum_{m=1}^{\infty} \frac{a(m + h)\lambda_\ell(m)}{m^{s + \frac{k-1}{2}}}$$

where the $\lambda_\ell$s are the Fourier coefficients of a weight-zero Maass form with eigenvalue $\frac{1}{2} + it_\ell$. 

Combining this construction with the spectral expansion of Selberg’s shifted convolution sum and employing Bochner’s Theorem on the analytic continuation of functions in several variables, we are able to construct meromorphic continuations of the multivariable functions $T_{\pm}(s_1, s_2, s_3) = \sum_{m,h,n \geq 1} a(m-h)b(m)c(h \pm n) m^{s_1}h^{s_2}n^{s_3}$ to all $(s_1, s_2, s_3) \in \mathbb{C}^3$.
Similarly, in the case where $N = 1$, this modified Poincaré series can be used to give a meromorphic continuation of the shifted convolution sum

$$\sum_{m=1}^{\infty} \frac{a(m + h)\lambda_\ell(m)}{m^{s+\frac{k-1}{2}}},$$

where the $\lambda_\ell$s are the Fourier coefficients of a weight-zero Maass form with eigenvalue $\frac{1}{2} + it_\ell$.

Combining this construction with the spectral expansion of Selberg’s shifted convolution sum and employing Bochner’s Theorem on the analytic continuation of functions in several variables,[1] we are able to construct meromorphic continuations of the multivariable functions

$$T_\pm(s_1, s_2, s_3) = \sum_{m,h,n \geq 1} \frac{a(m - h)b(m)c(h \pm n)}{m^{s_1}h^{s_2}n^{s_3}},$$

to all $(s_1, s_2, s_3) \in \mathbb{C}^3$. 
By taking inverse Mellin Transforms of $T_{\pm}$, we are able to derive the non-trivial estimates

\[
\sum_{m,h,n \geq 1}^{\infty} \frac{a(m - h)b(m)c(h \pm n)}{m^k h^{\frac{k}{2}}} e^{-\left(\frac{m+h+n}{x}\right)}
\]

\[
= \sum_{m,h,n \geq 1}^{\infty} \frac{A(m - h)B(m)C(h \pm n)(1 - \frac{h}{m})^{\frac{k}{2}} (1 \pm \frac{n}{h})^{\frac{k}{2}}}{\sqrt{(m - h)(m)(h \pm n)}} e^{-\left(\frac{m+h+n}{x}\right)}
\]

\[
= O_f(1).
\]
By taking inverse Mellin Transforms of $T_{\pm}$, we are able to derive the non-trivial estimates

$$\sum_{m,h,n \geq 1} \frac{a(m-h)b(m)c(h \pm n)}{m^k h^{k/2}} e^{-\frac{(m+h+n)}{x}}$$

$$= \sum_{m,h,n \geq 1} \frac{A(m-h)B(m)C(h \pm n)(1 - \frac{h}{m})^{k/2} (1 \pm \frac{n}{h})^{k/2} e^{-\frac{(m+h+n)}{x}}}{\sqrt{(m-h)(m)(h \pm n)}}$$

$$= \mathcal{O}_f(1).$$

It is expected that certain binomial and integral expansions can remove the unwanted coupling terms.
By taking inverse Mellin Transforms of $T_{\pm}$, we are able to derive the non-trivial estimates

$$
\sum_{m,h,n \geq 1} \frac{a(m-h)b(m)c(h \pm n)}{m^k h^{\frac{k}{2}}} e^{-\frac{m+h+n}{x}}
$$

$$
= \sum_{m,h,n \geq 1} \frac{A(m-h)B(m)C(h \pm n)(1 - \frac{h}{m})^\frac{k}{2}(1 + \frac{n}{h})^\frac{k}{2}}{\sqrt{(m-h)(m)(h \pm n)}} e^{-\frac{m+h+n}{x}}
$$

$$
= O_f(1).
$$

It is expected that certain binomial and integral expansions can remove the unwanted coupling terms.

Since these objects correspond to the “off-diagonal” terms of third moments of $L$-functions, the ultimate goal of my research is to get a formula for the asymptotics of higher moments and use these to produce subconvexity estimates.
S. Bochner.
A theorem on analytic continuation of functions in several variables. 

J. Hoffstein and T. Hulse.
Multiple dirichlet series and shifted convolutions (in preparation). 

A. Selberg.
On the estimation of Fourier coefficients of modular forms. 