Degenerate Whittaker functionals for real reductive groups

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Whittaker functionals

- For a real reductive group G have equivalence of categories

$$\pi \mapsto \pi^{HC} : \mathcal{M}(G) \longrightarrow \mathcal{HC}(G)$$

- $\mathcal{HC}(G) = \mathcal{HC}(\mathfrak{g}, K)$: category of finite length $(\mathfrak{g}, K)$-modules
- $\mathcal{M}(G)$: smooth admissible Fréchet reps. of moderate growth
- Assume G **quasisplit:** fix Borel $B = HN$, Cartan $H = TA$

$$\text{Define} \quad \mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}], \mathfrak{v} = \mathfrak{n}/\mathfrak{n}', \Psi = \mathfrak{v}^* \subset \mathfrak{n}^*$$

- $\Psi \longleftrightarrow$ Lie algebra characters of $\mathfrak{n} \longleftrightarrow$ group characters of N
- $\Psi_0 = \{\psi \in \Psi : \psi(x) \in i\mathbb{R} \text{ for } x \in \mathfrak{n}_0\} \longleftrightarrow$ unitary characters of N
- For $\psi \in \Psi$, $\pi \in \mathcal{M}$ and $\sigma \in \mathcal{HC}$ define

$$Wh^*_\psi(\pi) := \mathrm{Hom}^{cts}_N(\pi, \psi), \Psi_0(\pi) := \{\psi \in \Psi_0 : Wh^*_\psi(\pi) \neq 0\}$$

$$Wh'_\psi(\sigma) := \mathrm{Hom}_\mathfrak{n}(\sigma, \psi), \Psi(\sigma) := \{\psi \in \Psi : Wh'_\psi(\sigma) \neq 0\}$$

Kostant's theorem

- Let $\mathcal{N} \subset \mathfrak{g}^*$ be the nullcone, write $\mathcal{N}_\theta = \mathcal{N} \cap \mathfrak{k}^\perp$, $\mathcal{N}_0 = \mathcal{N} \cap \mathfrak{g}_0^*$.
- For $\pi \in \mathcal{M}$ can its define annihilator variety, associated variety and wavefront set: $\mathrm{An}\mathcal{V}(\pi) \subset \mathcal{N}$, $\mathrm{As}\mathcal{V}\left(\pi^{HC}\right) \subset \mathcal{N}_\theta$, $\mathrm{WF}(\pi) \subset i\mathcal{N}_0$.
- We say π is *large* if the Gelfand-Kirillov dimension of π^{HC} is dim $\mathfrak{n}$, or equivalently if $\mathrm{An}\mathcal{V}$ (or $\mathrm{As}\mathcal{V}$ or WF) has max dim,
- We say π or σ is *generic* if $\exists\ \psi \in \Psi(\cdot)$ such that $H_\mathbb{C} \cdot \psi$ is open in Ψ.

Theorem (Kostant)

π or σ is large iff it is generic.

- Several authors (Matumoto, Yamashita, ...) consider *generalized* Whittaker functionals $\sim$ generic characters for *smaller* nilradicals
- We consider *degenerate* functionals $\sim$ *arbitrary* characters of $\mathfrak{n}$.

Main results

- Let $pr_{\mathfrak{n}^*} : \mathfrak{g}^* \to \mathfrak{n}^*$ denote the natural projection (restriction to $\mathfrak{n}$).

Theorem (A)

For $\sigma \in \mathcal{HC}$ we have $\Psi(\sigma) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}(\sigma)) \cap \Psi$.

- The finite group $F = Norm_{G_{\mathbb{C}}}(G) / (Z_{G_{\mathbb{C}}} \cdot G)$ acts on $\mathcal{M}(G)$.
- For $\pi \in \mathcal{M}(G)$, define $\tilde{\pi} = \bigoplus \{\pi^a : a \in F\}$.

Theorem (B)

Let $\pi \in \mathcal{M}$ and $\sigma = \pi^{HC}$, then

1. $\Psi_0 \cap \Psi(\sigma) = \Psi_0(\widetilde{\pi})$
2. $\Psi_0(\pi) \subset \mathrm{WF}(\pi) \cap \Psi \subset \Psi_0(\widetilde{\pi})$

Main results

Fact

If $G = GL_n(\mathbb{R})$ or if G is a complex group then $\widetilde{\pi} = \pi$ hence $\Psi_0(\pi) = \mathrm{WF}(\pi) \cap \Psi$.

Theorem (C)

The sets $\Psi_0(\pi)$ and $\mathrm{WF}(\pi)$ determine one another if

1. $G = GL_n(\mathbb{R})$ *or* $GL_n(\mathbb{C})$ *or* $SL_n(\mathbb{C})$ *and* $\pi \in \mathcal{M}(G)$
2. $G = Sp_{2n}(\mathbb{C})$ *or* $O_n(\mathbb{C})$ *or* $SO_n(\mathbb{C})$ *and* π *is irreducible*

- Key observation for the second statement:

Theorem (D)

Let $\mathcal{O}$ be a nilpotent orbit for a complex classical Lie algebra then $\mathcal{O}$ is uniquely determined by $\overline{\mathcal{O}} \cap \Psi$.

Proof of Theorem A

- Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be the Borel subalgebra of $\mathfrak{g}$, let V be a $\mathfrak{b}$-module.
- Define the $\mathfrak{n}$-adic completion and Jacquet module as follows:
- $\widehat{V} = \widehat{V}_{\mathfrak{n}} = \varprojlim V/\mathfrak{n}^i V, \quad J(V) = J_{\mathfrak{b}}(V) = \left(\widehat{V}_{\mathfrak{n}}\right)^{\mathfrak{h}\text{-finite}}$
- We also define $\mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}]$ and $CV = H_0(\mathfrak{n}', V) = V/\mathfrak{n}' V$.
- Let $(\sigma, M) \in \mathcal{HC}(G)$
- (Nakayama) $\Psi(\sigma) = \mathrm{Supp}_{\mathfrak{v}}(CM) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}(CM)$
- (Joseph+Gabber) $\mathrm{An}\mathcal{V}_{\mathfrak{v}}(CM) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}\left(\widehat{CM}\right) = \mathrm{An}\mathcal{V}_{\mathfrak{v}}(J(CM))$
- (Easy) $J(CM) \approx C(JM)$ as $\mathfrak{b}$-modules.
- (Bernstein+Joseph)
  $\mathrm{An}\mathcal{V}_{\mathfrak{v}}(J(CM)) = \mathrm{As}\mathcal{V}_{\mathfrak{v}}(C(JM)) \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(JM) \cap \Psi$.
- (Ginzburg+ENV) $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(JM) \supset \mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) \cap \Psi$.
- (Casselman-Osborne+Gabber) $\mathrm{As}\mathcal{V}_{\mathfrak{n}}(M) = pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(M))$.
- Thus $\Psi(\sigma) \supset pr_{\mathfrak{n}^*}(\mathrm{As}\mathcal{V}_{\mathfrak{g}}(\sigma)) \cap \Psi$; other inclusion is easy.

Proof of Theorem D

- For $GL(n, \mathbb{R})$ and $SL(n, \mathbb{C}) \sim$ Jordan form
- Orbits for $Sp_{2n}(\mathbb{C})$ or $O_n(\mathbb{C}) \sim$ partitions satisfying certain conditions
- An orbit meets Ψ iff it has at most one part ≥ 2 with odd multiplicity
- For each partition λ and each k there is a partition $\mu \leq \lambda$, which meets Ψ and satisfies $\mu_1 + \cdots + \mu_k = \lambda_1 + \cdots + \lambda_k$
- Result for $SO_n(\mathbb{C})$ requires slight additional argument.

Counterexamples for exceptional groups

Fact

Theorem D is false for every exceptional group. The following is a complete list of sets of nilpotent coadjoint orbits whose closures have the same intersection with Ψ, with special orbits underlined.

- For $G = G_2$: $\underline{G_2(a_1)}$ and $\widetilde{A_1}$
- For $G = F_4$:
  1. $\underline{F_4(a_1)}$ and $\underline{F_4(a_2)}$
  2. $\underline{F_4(a_3)}$ and $C_3(a_1)$
- For $G = E_6$:
  1. $\underline{E_6(a_1)}$ and $\underline{D_5}$
  2. $\underline{D_4(a_1)}$ and $A_3 + A_1$
- For $G = E_7$:
  1. $\underline{E_7(a_1)}$ and $\underline{E_7(a_2)}$
  2. $\underline{E_7(a_3)}$ and D_6
  3. $\underline{E_6(a_1)}$ and $\underline{E_7(a_4)}$.

Counterexamples for exceptional groups

- For $G = E_8$:
  1. $\underline{E_8(a_1)}$, $\underline{E_8(a_2)}$, and $\underline{E_8(a_3)}$
  2. $\underline{E_8(a_4)}$, $\underline{E_8(b_4)}$ and $\underline{E_8(a_5)}$
  3. $\underline{E_7(a_1)}$, $\underline{E_8(b_5)}$ and $E_7(a_2)$
  4. $\underline{E_8(a_6)}$ and $\underline{D_7(a_1)}$
  5. $\underline{E_6(a_1)}$ and $\underline{E_7(a_4)}$
  6. $\underline{E_8(a_7)}$, $E_7(a_5)$, $E_6(a_3) + A_1$, $E_7(a_5)$ and $D_6(a_2)$.