Generalized Schubert Calculus

ICERM

04.03.2013

$G \supseteq B \supseteq T$ gives $W_0 = N(T)/T$,

reductive algebraic group; Borel subgroup; maximal torus

$\mathfrak{h}_{\mathbb{Z}}^* = \mathrm{Hom}(T, \mathbb{C}^\times)$,

$\mathfrak{h}_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{C}, T)$, and

$R^- = \{-\alpha \in \mathfrak{h}_{\mathbb{Z}}^*$ appearing in the T-module $\mathfrak{g}/\mathfrak{b} = T(G/B)\}$

Four cohomologies, Four rings

"Ordinary" cohomology

$H_T^*(pt) = S(\mathfrak{h}_{\mathbb{Z}}^*) = \mathbb{C}[x_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$ with $x_{\lambda+\mu} = x_\lambda + x_\mu$

K-theory

$K_T(pt) = \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{C}[e^\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]$ with $e^{\lambda+\mu} = e^\lambda e^\mu$

or, if $x_\lambda = 1 - e^\lambda$ then $x_{\lambda+\mu} = x_\lambda + x_\mu$

Elliptic cohomology

$$E\ell\ell_T(pt) = Th[\mathfrak{h}_{\mathbb{Z}}^*] = \mathrm{span}\left\{ \theta_{\lambda + m\Lambda_0} \,\middle|\, \begin{matrix} m \in \mathbb{Z}_{\geq 0} \\ \lambda \in \mathfrak{h}_{\mathbb{Z}}^* \bmod m\mathfrak{h}_{\mathbb{Z}} \end{matrix} \right\}$$

Cobordism

$\Omega_T(pt) = \mathbb{L}[\mathfrak{h}_{\mathbb{Z}}^*] = \mathbb{L}[[x_\lambda \mid \lambda \in \mathfrak{h}_{\mathbb{Z}}^*]]$ with $x_{\lambda+\mu} = x_\lambda \oplus x_\mu$.

where $\mathbb{L}$ is the ring generated by $a_{11}, a_{12}, a_{21}, a_{31}, \ldots$

$$x \oplus y = x + y + a_{11}xy + a_{12}xy^2 + a_{21}x^2y + \ldots$$

and a_{ij} satisfy relations forced by

$$x \oplus y = y \oplus x, \quad x \oplus (y \oplus z) = (x \oplus y) \oplus z, \quad x \oplus (\ominus x) = 0$$

and $x \oplus 0 = x$

The Borel model

$$H_T^*(G/B) = \frac{\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]}{\langle f(x)-f(y) \mid f(x)\in\mathbb{C}[x_1,\ldots,x_n]^{W_0}\rangle} = S(\mathfrak{h}_{\mathbb{Z}}^*)\otimes_{S(\mathfrak{h}_{\mathbb{Z}}^*)^{W_0}} S(\mathfrak{h}_{\mathbb{Z}}^*)$$

$$K_T(G/B) = \frac{\mathbb{C}[X_1^{\pm1},\ldots,X_n^{\pm1},Y_1^{\pm1},\ldots,Y_n^{\pm1}]}{\langle f(X)-f(Y) \mid f(X)\in\mathbb{C}[X_1^{\pm1},\ldots,X_n^{\pm1}]^{W_0}\rangle} = \mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*]\otimes_{\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*]^{W_0}}\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*]$$

$$Ell_T(G/B) \text{ "=" } Th[\mathfrak{h}_{\mathbb{Z}}^*]\otimes_{Th[\mathfrak{h}_{\mathbb{Z}}^*]^{W_0}} Th[\mathfrak{h}_{\mathbb{Z}}^*]$$

$$\Omega_T(G/B) = \frac{\mathbb{L}[[x_\lambda, y_\mu \mid \lambda,\mu\in\mathfrak{h}_{\mathbb{Z}}^*]]}{\langle f(x)-f(y) \mid f(x)\in\mathbb{L}[[x_\lambda]]^{W_0}\rangle} = \Omega_T(pt)\otimes_{\Omega_G(pt)}\Omega_T(pt)$$

<u>Remark</u> These are four versions of the Chevalley-Shephard-Todd theorem:

(a) $S(\mathfrak{h}_{\mathbb{Z}}^*)$ is a free $S(\mathfrak{h}_{\mathbb{Z}}^*)^{W_0}$-module,

(b) $\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*]$ is a free $\mathbb{C}[\mathfrak{h}_{\mathbb{Z}}^*]^{W_0}$-module,

(c) $Th[\mathfrak{h}_{\mathbb{Z}}^*]$ is a free $Th[\mathfrak{h}_{\mathbb{Z}}^*]^{W_0}$-module,

(d) $\Omega_T(pt)$ is a free $\Omega_G(pt)$-module and

$$\Omega_G(pt) = \Omega_T(pt)^{W_0}.$$

The moment graph model

The T-fixed points $\{wB \mid w \in W_0\}$ in G/B provide

$$\begin{aligned} \iota_w : pt &\to G/B \\ * &\mapsto wB \end{aligned} \quad \text{and} \quad \Phi : \Omega_T(G/B) \xrightarrow{\oplus_w \iota_w^*} \bigoplus_{w \in W_0} \Omega_T(pt)$$

i.e. the following map is an injective ring homomorphism

$$\frac{\mathbb{L}[[x_\lambda, y_\mu]]}{\langle f(x) - f(y) \mid f \in \mathbb{L}[[x_\lambda]]^{W_0} \rangle} \xrightarrow{\Phi} \bigoplus_{w \in W_0} \mathbb{L}[[y_\mu]]$$

$$\begin{aligned} f(x) &\longmapsto (wf(y))_{w \in W_0} \\ g(y) &\longmapsto (g(y))_{w \in W_0} \end{aligned}$$

Parabolics, pullbacks and pushforwards

A parabolic subgroup of G is $P_J \supseteq B$ such that G/P_J is a projective variety.

$$\begin{aligned} \pi_J : G/B &\to G/P_J \\ gB &\mapsto gP_J \end{aligned} \quad \text{gives} \quad \begin{aligned} \pi_J^* &: \Omega_T(G/P_J) \to \Omega_T(G/B) \\ (\pi_J)_! &: \Omega_T(G/B) \to \Omega_T(G/P_J) \end{aligned}$$

which are

$$\Omega_T(pt)^{W_J} \otimes_{\Omega_G(pt)} \Omega_T(pt) \xrightarrow{\pi_J^*} \Omega_T(pt) \otimes_{\Omega_G(pt)} \Omega_T(pt)$$

and

$$(\pi_J)_! = \Big(\sum_{w \in W_J} t_w\Big) \frac{1}{\prod_{-\alpha \in R_J^-} x_{-\alpha}}$$

where W_J is the subgroup of W_0 corresponding to $P_J \subseteq G$, R_J^- are the negative roots in $T(G/P_J)$ and t_w acts as w on the left factor of $\Omega_T(pt) \otimes \Omega_T(pt)$

ICERM
04.03.2013 (4)

BGG operators A_i

If P_i is a minimal parabolic $P_i \neq B$ then

$$A_i = (\pi_i)_! = (1 + t_{s_i}) \frac{1}{x_{-\alpha_i}}$$

See papers of the Ottawa group and collabs (Zainoulline et al) for much information on the calculus of these in cobordism.

Bott-Samelson classes $[Z_{i_1 \cdots i_\ell}]$

Let $s_{i_1} \cdots s_{i_\ell}$ be a sequence (not necessarily reduced).

$$[Z_{i_1 \cdots i_\ell}] = A_{i_1} \cdots A_{i_\ell} [Z_{pt}]$$

where
$$[Z_{pt}]_w = \begin{cases} \prod_{-\alpha \in R^-} y_{-\alpha}, & \text{if } w = 1, \\ 0, & \text{if } w \neq 1. \end{cases}$$

Schubert classes

I think that there exist unique $[X_w]$, $w \in W_0$, characterized by

(a) $[X_w] \in \operatorname{im} \Phi$,

(b) $[X_w]_w = \prod_{\substack{-\alpha \in R^- \\ w\alpha \notin R^-}} y_{-\alpha}$ and $[X_w]_v = 0$ unless $v \leq w$,

(c) If $\lambda \in \mathfrak{h}_{\mathbb{Z}}^*$ is dominant (i.e. in $\sum_{i=1}^n \mathbb{Z}_{\geq 0} \omega_i$) then

$$x_{-\lambda} [X_w] = \sum_{v \in W_0} c_{\lambda v}^w [X_w] \quad \text{with } c_{\lambda v}^w \in \mathbb{L}_{\geq 0}[[y_{-\omega_1}, \ldots, y_{-\omega_n}]]$$

where $\mathbb{L}_{\geq 0} = \mathbb{Z}_{\geq 0}[a_{11}, a_{12}, a_{21}, \ldots]$.

Remarks

(1) If $w = s_{i_1} \cdots s_{i_\ell}$ is reduced then

$$[Z_{i_1 \cdots i_\ell}] = [X_w] \text{ in } \underbrace{H_T^*(G/B)}_{\text{all } a_{ij}=0} \text{ and } \underbrace{K_T(G/B)}_{\substack{\text{all } a_{ij}=0 \\ \text{except } a_{11}}}$$

but $[Z_{121}] \neq [X_{s_1 s_2 s_1}]$ in $\Omega_T(G/B)$ for type A_2

(2) I would guess that

$$[X_u][X_v] = \sum_{w \in W_0} c_{uv}^w [X_w] \text{ with } c_{uv}^w \in \mathbb{L}_{\geq 0}[[y_{-\alpha_1}, \dots, y_{-\alpha_n}]]$$

From the tables at the end of Calmès-Petrov-Zainoulline we should not expect such positivity when multiplying Bott-Samelson classes.

(3) I would guess that

$$[X_{s_i w_0}] = x_{-\omega_i} \oplus y_{w_0 \omega_i} \quad \text{in} \quad \Omega_T(G/B)$$

generalizing the formulas

$$x_{-\omega_i} + y_{w_0 \omega_i} = [X_{s_i w_0}] \quad \text{in} \quad H_T^*(G/B) \text{ and}$$

$$1 - X^{-\omega_i} Y^{w_0 \omega_i} = [X_{s_i w_0}] \quad \text{in} \quad K_T(G/B).$$