Representations of the affine BMW algebra

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ICERM March 7, 2013

Joint with Kevin Walker

. . . via a *mysterious* topological construction (from TQFTs)

What is the algebra and combinatorics behind it?

Recall a partition λ with $|\lambda| = n$ indexes an irreducible representation of $\mathcal{S}_n$.

Example

$\lambda =$  indexes an irrep of $\mathcal{S}_5$.

Hecke algebra

or λ indexes an irrep $M(\emptyset, \lambda, 5)$ of the finite Hecke algebra H_5^{fin} of type A.

H_n^{fin} is a q-deformation of $\mathbb{C}[\mathcal{S}_n]$ with generators
T_i in place of $(i, i{+}1) = s_i \in \mathcal{S}_n$.

basis of the module $M(\emptyset, \lambda, n) \longleftrightarrow \mathsf{SYT}(\lambda)$

1	2	3
4	5	

1	2	4
3	5	

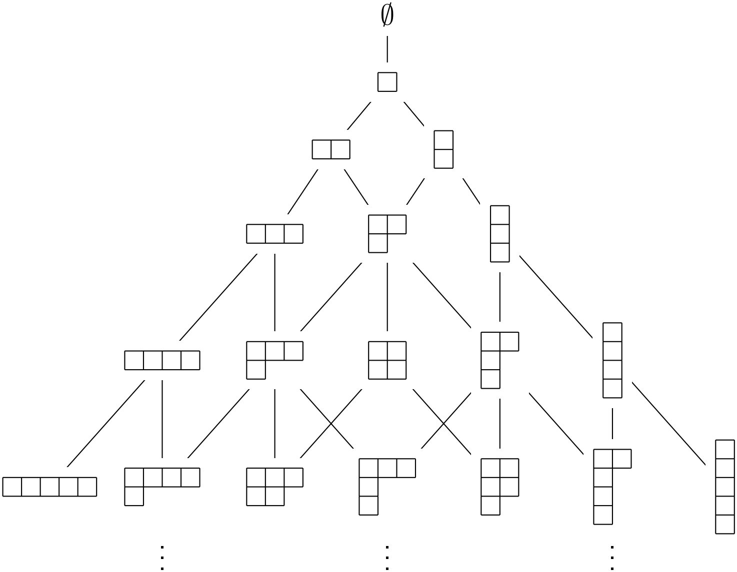
1	2	5
3	4	

1	3	4
2	5	

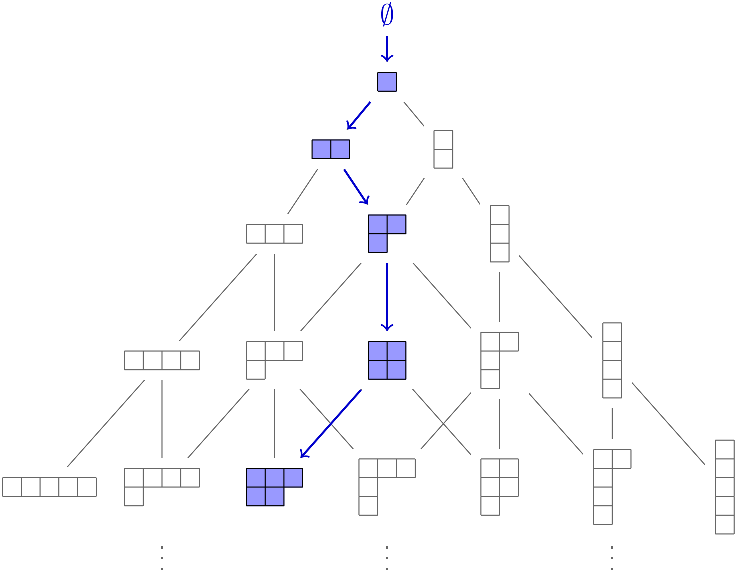
1	3	5
2	4	

A standard Young tableau of shape λ ($\mathcal{T} \in \mathsf{SYT}(\lambda)$) corresponds to a path of length $n = |\lambda|$ from $\emptyset$ to λ in Young's lattice of partitions.

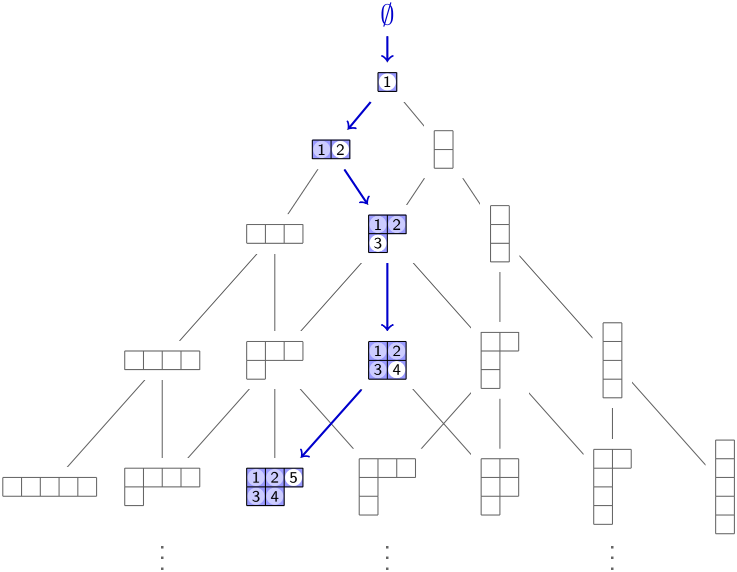
Young's lattice of partitions (is a crystal graph)



a path $\emptyset$ to λ



a path $\emptyset$ to $\lambda \longleftrightarrow \mathcal{T} \in STY(\lambda)$



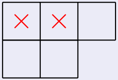
A standard Young tableau of shape λ ($\mathcal{T} \in \mathrm{SYT}(\lambda)$) corresponds to a path of length $n = |\lambda|$ from $\emptyset$ to λ in Young's lattice of partitions.

Why does this index a basis?   Induction/Restriction (among other reasons)

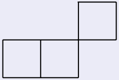
Skew shapes

Example

$\mu = (2) \subseteq \lambda = (3,2)$

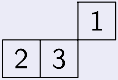


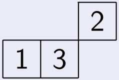
$\lambda/\mu =$

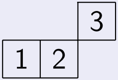


Example (SYT)

$\mathsf{SYT}(\lambda/\mu)$

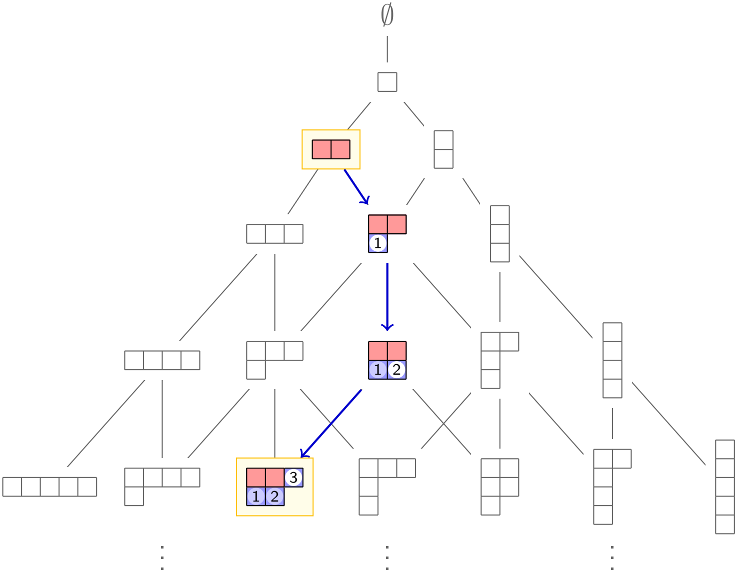






$\mathcal{T} \in \mathsf{SYT}(\lambda/\mu)$ corresponds to a path of length $n = |\lambda/\mu|$ from μ to λ in Young's lattice of partitions.

a path μ to λ



What is the representation theory behind this?

SYT(λ/μ) index a basis of an irrep $M(\mu, \lambda, n)$ of H_n^{aff}.

H_n^{aff} is the (extended) affine Hecke algebra of type A.

H_n^{aff} is a q-deformation of $\mathbb{C}[\mathcal{S}_n \ltimes \mathbb{Z}^n]$ with generators
T_i in place of $(i, i+1) = s_i \in \mathcal{S}_n$,
X_i in place of $(0, 0, \ldots, 1, \ldots 0) \in \mathbb{Z}^n$,

As vector spaces, $H_n^{\mathrm{aff}} \simeq H_n^{\mathrm{fin}} \otimes \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$.

$M(\mu, \lambda, n)$

Question

Are these **all** the irreps of H_n^{aff}?

NO

Why

- Because boxes are in $\mathbb{Z}^2$ (not $\mathbb{C}^2$) we only get representations in $\mathrm{Rep}_q^{\mathrm{aff}}$, the full subcategory on which $\{X_i\}$ take eigenvalues in $\{q^k \mid k \in \mathbb{Z}\}$. (like integral weights)
- The configuration of boxes for skew shapes yield the X-semisimple (aka *calibrated*) representations. (see A. Ram)

Definition

M is X-ssl if its restriction to $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is semisimple, i.e., if

$$M = \bigoplus_{\beta \in \mathbb{Z}^n} M[\beta]$$

weight spaces

Let $\beta \in \mathbb{Z}^n$. The β-weight space of M is

$$M[\beta] = \{v \in M \mid X_i v = q^{\beta_i} v, 1 \leq i \leq n\}.$$

Fact

M is X-ssl $\implies$ $\dim M[\beta] = 1$ or 0.

Determines a weight basis; weights encode SYT

X-ssl $M \in \mathrm{Rep}_q^{\mathrm{aff}}$

$M[\beta] \ni v_{\mathcal{T}}$, $\mathcal{T} \in \mathrm{SYT}(\lambda/\mu)$

β_i describes which diagonal $\boxed{i}$ is on

The combinatorics of sppt $M = \{\beta \in \mathbb{Z}^n \mid M[\beta] \neq 0\}$ is that of $\mathrm{SYT}(\lambda/\mu)$, i.e. of n-step paths μ to λ.

Action of H_n^{aff} generators on basis $\{v_{\mathcal{T}} \mid \mathcal{T} \in \mathrm{SYT}(\lambda/\mu)\}$

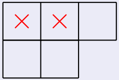
$$X_i v_{\mathcal{T}} = q^{\mathrm{diagonal}\,\boxed{i}} v_{\mathcal{T}}$$

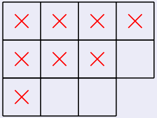
$$T_i v_{\mathcal{T}} \in \mathrm{span}\{v_{\mathcal{T}}, v_{s_i\mathcal{T}}\}$$

If $s_i\mathcal{T} \notin \mathrm{SYT}(\lambda/\mu)$, set $v_{s_i\mathcal{T}} = 0$. (connection of SYT and weights; the algebra dictates the combinatorics)

Facts

- a (directed) n-step path on Young's lattice $\longleftrightarrow$ some $\mathcal{T} \in \mathrm{SYT}(\lambda/\mu)$ $\longleftrightarrow$ some weight β
- There is a unique X-ssl $M \in \mathrm{Rep}_q^{\mathrm{aff}}$ with $\beta \in \operatorname{sppt} M$ ($M[\beta] \neq 0$). i.e., $\operatorname{sppt} M \cap \operatorname{sppt} N = \emptyset$.
- These are *all* the allowable weights across all X-ssl modules.
- Given X-ssl irrep $M \in \mathrm{Rep}_q^{\mathrm{aff}}$, (μ, λ) is unique up to diagonal shift.





finite vs affine

H_n^{fin}-modules have $\mu = \emptyset$, $n = |\lambda|$.

H_n^{fin} is a quotient of H_n^{aff} via

$$H_n^{\mathrm{aff}} \twoheadrightarrow H_n^{\mathrm{fin}}$$
$$T_i \mapsto T_i$$
$$X_1 \mapsto 1 = q^0$$

For SYT, means $\boxed{1}$ can only be on 0^{th} diagonal

What is this construction?

We started with two irreps: $M(\emptyset, \mu, m)$ of H_m^{fin}, $M(\emptyset, \lambda, \ell)$ of H_ℓ^{fin} and produced an X-ssl irrep $M(\mu, \lambda, n)$ of H_n^{aff}, $n = \ell - m$ (or 0 if $\mu \not\subseteq \lambda$).

$$\mathrm{fin} \times \mathrm{fin} \to \mathrm{aff}$$

up-down

What if we now allow n-step paths to go up and down?

Then we capture the combinatorics of weights of X-ssl irreps of the affine BMW algebra B_n^{aff}. (See Leduc-Ram, Orellana-Ram)

B_n^{fin} is a deformation of the Brauer algebra, with generators T_i, E_i, $1 \leq i < n$.
E_{n-1} creates a link between B_n^{fin} and B_{n-2}^{fin}.
The irreps of B_n^{fin} correspond to λ with $n - |\lambda| \equiv 0 \mod 2$.

B_n^{aff} has generators T_i, E_i, X_i.

H_n^{aff} is a quotient of B_n^{aff} $(B_n^{\mathrm{aff}} \twoheadrightarrow H_n^{\mathrm{aff}})$ via

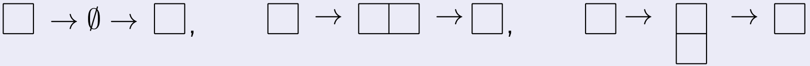
$$T_i \mapsto T_i$$
$$X_i \mapsto X_i$$
$$E_i \mapsto 0$$

n-step paths μ to $\lambda \longleftrightarrow$ basis of the B_n^{aff}-module $M(\mu, \lambda, n)$

Now μ, λ are arbitrary (we drop the requirement $\mu \subseteq \lambda$)
n is fixed, up to *parity*, independent of $|\mu|, |\lambda|$.

Example

$M((1),(1),2)$ has basis indexed by

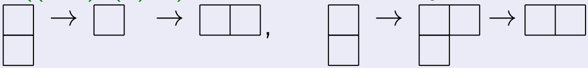


Example

$M((1),(1),3) = 0$

Example

$M((1,1),(2),2)$ has basis indexed by



What is this construction?

We started with two irreps: $M(\emptyset, \mu, m)$ of B_m^{fin}, $M(\emptyset, \lambda, \ell)$ of B_ℓ^{fin}
and produced an X-ssl irrep $M(\mu, \lambda, n)$ of B_n^{aff}
(or 0 if $|\mu| + |\lambda| + n \not\equiv 0 \mod 2$)

In fact, we produced a whole FAMILY (vary n) of irreps—this is really ONE irrep of the affine BMW category $\mathcal{B}^{\mathrm{aff}}$.

We have a functor

$$\mathrm{Rep}(\mathcal{B}^{\mathrm{fin}}) \times \mathrm{Rep}(\mathcal{B}^{\mathrm{fin}}) \to \mathrm{Rep}(\mathcal{B}^{\mathrm{aff}}).$$

It comes from some bi-module. Where does that bi-module come from?

Topology, TQFTs

This construction actually comes from topology (3-manifolds, $(3+1)$-dim TQFTs (actually $(3+\epsilon)$-dim TQFT), . . .)

TQFT = topological quantum field theory

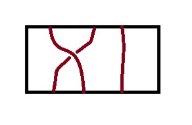
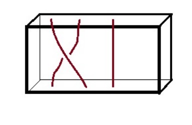
a TQFT is a machine for turning topology into algebra.

The BMW TQFT

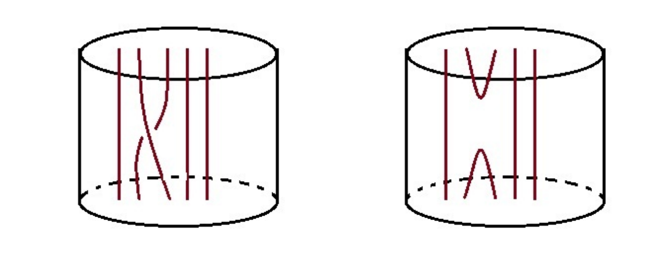
- assigns to a 3-manifold M a vector space $V(M)$ (the BMW skein module). We actually get a family of vector spaces $V(M; c)$ depending on boundary conditions c on M.
- assigns a linear category $\mathcal{B}(Y)$ to a surface Y.
- If Y is contained in the boundary M, then the various vector spaces $V(M; c)$ afford a representation of $\mathcal{B}(Y)$.
- If a pair of surfaces $Y_1 \cup Y_2$ is contained in the boundary of M, then the various $V(M; c)$ constitute a $(\mathcal{B}(Y_1), \mathcal{B}(Y_2))$-bimodule.
- Gluing 3-manifolds along a surface Y corresponds to taking tensor product over $\mathcal{B}(Y)$

$\mathcal{B}^{\mathrm{fin}}$

We often depict $T_1 \in B_3^{\mathrm{fin}}$ (or H_3^{fin}) by

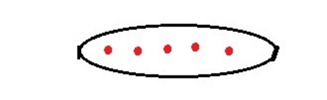
 or .

But here we'll draw (say T_2 and $E_2 \in B_5^{\mathrm{fin}}$) as



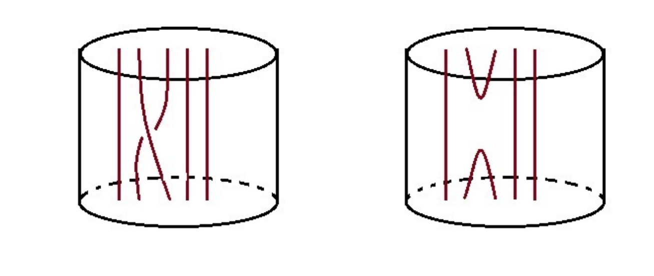
$Y = D^2$ disk

Category $\mathcal{B}(Y) = \mathcal{B}^{\mathrm{fin}}$



Pick n framed (oriented) points in Y.
This collection **c** is an object of $\mathcal{B}(Y)$.

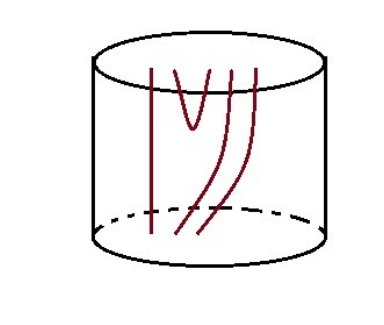
End(**c**) $= B_n^{\mathrm{fin}}$ (resp. H_n^{fin})



$Y = D^2$ disk

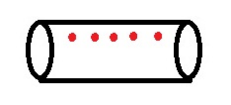
Category $\mathcal{B}(Y) = \mathcal{B}^{\text{fin}}$

We also have morphisms $Mor(\mathbf{c}, \mathbf{d})$.



$Y = S^1 \times [0,1]$ annulus

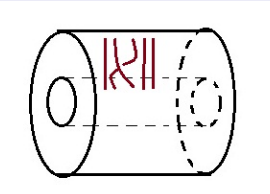
Category $\mathcal{B}(Y) = \mathcal{B}^{\mathrm{aff}}$

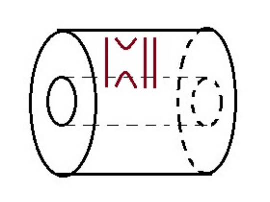


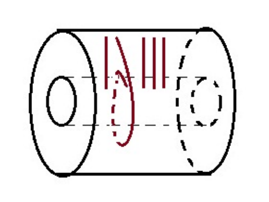
Pick n framed (oriented) points in Y.
This collection **c** is an object of $\mathcal{B}(Y)$.

$\mathrm{End}(\mathbf{c}) = B_n^{\mathrm{aff}}$ (resp. H_n^{aff})

Draw (say T_2 and E_2 and $X_2 \in B_5^{\mathrm{aff}}$) as

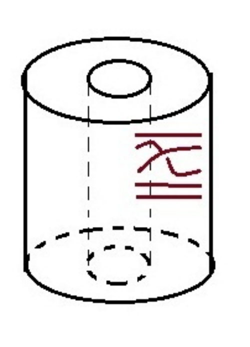


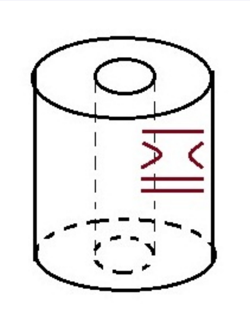


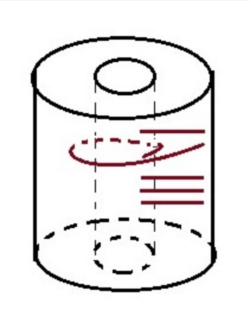


$\mathcal{B}^{\mathrm{aff}}$ corresponds to $Y = S^1 \times [0,1]$ annulus

Or we turn the picture sideways and draw (say T_2 and E_2 and $X_2 \in B_5^{\mathrm{aff}}$) as

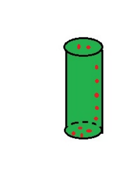




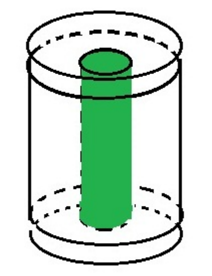


“hat-box” (or 1-handle) construction

bi-module $V(M)$ for $M = D^2 \times [0,1]$



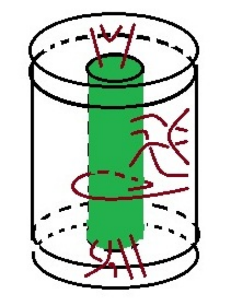
The boundary of the *solid* cylinder is TWO disks and an annulus.
When we draw the thickened boundary, this is where our morphisms from $\mathcal{B}^{\text{fin}} \times \mathcal{B}^{\text{fin}}$ and $\mathcal{B}^{\text{aff}}$ can act



"hat-box" (or 1-handle) construction

bi-module

The $(\mathrm{Rep}(\mathcal{B}^{\mathrm{fin}}) \times \mathrm{Rep}(\mathcal{B}^{\mathrm{fin}}), \mathrm{Rep}(\mathcal{B}^{\mathrm{aff}}))$-bimodule structure:



Recap: we produce $M(\mu, \lambda, n)$ by imposing the finite irreps μ, λ (or idempotents) on the top/bottom of the hatbox, and then let $\mathcal{B}^{\mathrm{aff}}$ act on the remaining cylindrical boundary.

We can apply this "machine" to other 3-manifolds . . .

Other constructions, directions

other 3-manifolds M

other ways of slicing up the boundary (to yield a bi-module)

gluing and tensor product

non X-ssl representations?? (topology and category have rigidity)