q-deformed Whittaker functions and the local Langlands correspondence

Sergey OBLEZIN , ITEP

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ICERM (Providence)

References

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Whittaker functions

The Gauss (Bruhat) decomposition of $G = G(F)$:

$$G^0 = U_- \cdot A \cdot U_+ \,, \qquad A = \left\{ e^{x_1}, \ldots, e^{x_N} \right\}.$$

Character of $B_- = U_- A$ with $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$:

$$\chi_{\underline{\lambda}} : \quad B_- \longrightarrow \mathbb{C}^* \,, \qquad \chi_{\underline{\lambda}}(ua) = \prod_{i=1}^{N} e^{(\lambda_i + \rho_i)\, x_i} \,.$$

The principal series representation $(\pi_{\underline{\lambda}}, \mathcal{V}_{\underline{\lambda}})$ of G and of $\mathcal{U}(\mathfrak{g})$:

$$\mathrm{Ind}_{B_-}^{G} \chi_{\underline{\lambda}} = \left\{ f \in \mathrm{Fun}(G) \,\middle|\, f(bg) = \chi_{\underline{\lambda}}(b)\, f(g) \,, \quad b \in B_- \right\}$$

The Whittaker function $\Psi_{\underline{\lambda}}(g)$ is a smooth function on $X = N_- \backslash G$ analytic in $\underline{\lambda}$ given by

$$\Psi_{\underline{\lambda}}(g) = e^{\rho(g)} \left\langle \psi_L \,, \pi_{\underline{\lambda}}(e^{-H(g)})\, \psi_R \right\rangle , \qquad (1)$$

$\psi_L, \psi_R \in \mathcal{V}_{\underline{\lambda}}$ are defined by character $\psi_0 : F \to \mathbb{C}^*$:

$$\psi : U \longrightarrow \mathbb{C} \,, \qquad \psi(u) = \prod_{\text{simple roots}} \psi_0\big(u_{\alpha_i}\big) \,.$$

Spherical Whittaker functions

The Iwasawa decomposition of $G = G(F)$:

$$G \;=\; K\cdot A\cdot U_+\,.$$

The spherical Whittaker function $\Psi_{\underline{\lambda}}(z)$ is a smooth function on $\mathfrak{H}=K\backslash G$ analytic in $\underline{\lambda}$ given by

$$\Psi_{\underline{\lambda}}(g)\;=\;e^{\rho(g)}\bigl\langle\psi_K\,,\,\pi_{\underline{\lambda}}(e^{-H(g)})\,\psi_R\bigr\rangle\,, \tag{2}$$

with the spherical vector $\psi_K\in\mathcal{V}_{\underline{\lambda}}$.

Quantum Toda lattice

In the real case, $G = G(\mathbb{R})$, generators $C_r\,, r = 1, \ldots, N$ of the center $\mathcal{ZU}(\mathfrak{g})$ define quantum Toda Hamiltonians:

$$\mathcal{H}_r \cdot \Psi_{\underline{\lambda}}(\underline{x}) \ := \ e^{-\rho(x)} \big\langle \psi_K\,,\, \pi_{\underline{\lambda}}(C_r\, e^{-H(\underline{x})})\, \psi_R \big\rangle\,. \tag{3}$$

The $G(\mathbb{R})$-Whittaker function is an eigenfunction:

$$\mathcal{H}_r \cdot \Psi_{\underline{\lambda}}(\underline{x}) \ = \ \sigma_r(\underline{\lambda})\, \Psi_{\underline{\lambda}}(\underline{x})\,, \tag{4}$$

$\sigma_r(\underline{\lambda})$ are r-symmetric functions in $\underline{\lambda} = (\lambda_1, \ldots, \lambda_N)$.

Example

In the case $G = GL(2;\,\mathbb{R})$

$$\mathcal{H}_1 \ = \ -\hbar\Big(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\Big)\,, \qquad \mathcal{H}_2 \ = \ -\hbar^2\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big) \ + \ e^{x_1 - x_2}\,,$$

Example: the $GL(2;\,\mathbb{R})$-Whittaker functions

$$\Psi^{\mathbb{R}}_{\lambda_1,\,\lambda_2}(e^{x_1},\,e^{x_2}) \,=\, \int_{\mathbb{R}} dT\; e^{\frac{\imath}{\hbar}\lambda_2(x_1+x_2-T)+\frac{\imath}{\hbar}\lambda_1 T-\frac{1}{\hbar}\left(e^{x_1-T}+e^{T-x_2}\right)} \tag{5}$$

$$=\, e^{\frac{\lambda_1+\lambda_2}{2}}\, e^{\frac{x_1+x_2}{2}}\; K_{\frac{\lambda_1-\lambda_2}{\hbar}}\big(\frac{2}{\hbar} e^{\frac{x_1-x_2}{2}}\big)\,.$$

The Mellin-Barnes integral representation:

$$\Psi^{\mathbb{R}}_{\lambda_1,\,\lambda_2}(e^{x_1},\,e^{x_2}) \,=\, \int\limits_{\mathbb{R}-\imath\epsilon} d\gamma\;\, e^{\frac{\imath}{\hbar}x_2(\lambda_1+\lambda_2-\gamma)+\frac{\imath}{\hbar}x_1\gamma}\; \prod_{i=1}^{2} \hbar^{\frac{\lambda_i-\gamma}{\hbar}}\,\Gamma\Big(\frac{\lambda_i-\gamma}{\hbar}\Big) \tag{6}$$

Both integral representations can be generalized to $GL(N;\,\mathbb{R})$ by induction over the rank N, using the Baxter $\mathcal{Q}$-operator formalism, [GLO].

Baxter operators for spherical Whittaker functions, [GLO]

One-parameter family of K-biinvariant functions $\mathcal{Q}_s$ in the Hecke algebra:

$$\big(\mathcal{Q}_s * \Psi_{\underline{\lambda}}\big)(g) \,=\, \int_G dh\; Q_s(gh^{-1})\,\Psi_{\underline{\lambda}}(h) \,=\, L_p(s;\,V)\,\Psi_{\underline{\lambda}}(g)\,.$$

Theorem

The L-function is the spherical transform of the Baxter operator kernel

$$L_p(s;\,V\otimes\delta^{-1/2}) \,=\, \int_A da\; Q_s(a^{-1})\,\varphi_{\underline{\lambda}}(a)\,,$$

with spherical function given by

$$\varphi_{\lambda}(g) \,=\, \int dk\; e^{\langle H(kg),\,\underline{\lambda}\rangle}\,.$$

Example

In the case $G = GL(1;\,F)$*:*

$$L_p(s;\,V) \,=\, \frac{1}{1-p^{\lambda-s}}\,, \qquad\qquad L_\infty(s;\,V) \,=\, \hbar^{\frac{\lambda-s}{\hbar}}\,\Gamma\Big(\frac{\lambda-s}{\hbar}\Big)\,.$$

Explicit formulas: non-Archimedean case

Let $\xi_{\underline{\lambda}} : \mathcal{H}(G,\, K) \to \mathbb{C}$ and $\sigma_{\underline{\lambda}} \subset GL(N;\, \mathbb{C})$ is the (semisimple) conjugacy class, attached to $\xi_{\underline{\lambda}}$ via Satake correspondence. **The class-one** $GL(N;\, F)$-**Whittaker function** associated with $\xi_{\underline{\lambda}}$:

1. $\Psi_{\underline{\lambda}}(ug) \,=\, \psi(u)\,\Psi_{\underline{\lambda}}(ug)\,;$
2. $\int\limits_G dh\; \Psi_{\underline{\lambda}}(gh)\,\phi(h^{-1}) \,=\, \xi_{\underline{\lambda}}(\phi)\,\Psi_{\underline{\lambda}}(g)$ for any $\phi \in \mathcal{H}(G,\, K)\,;$
3. $\Psi_{\underline{\lambda}}(1) \,=\, 1\,.$

The Langlands-Shintani (LS) formula

The class-one $GL(N;\, \mathbb{Q}_p)$-Whittaker function reads

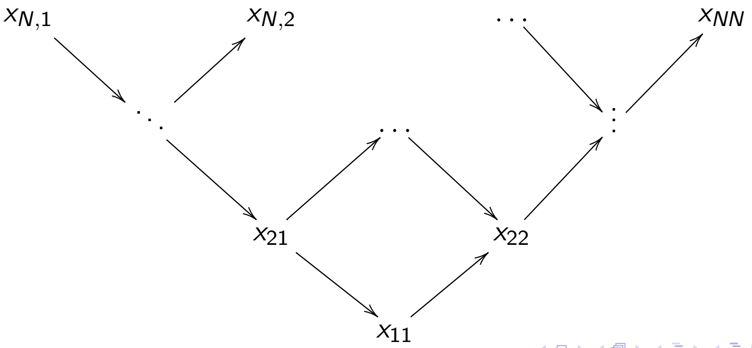
$$\Psi^{\mathbb{Q}_p}_{\underline{\lambda}}(p^{\mathbf{n}}) \;=\; \begin{cases} p^{-\varrho(\mathbf{n})}\,\mathrm{ch}\,_{V_{\mathbf{n}}}\begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_N} \end{pmatrix}, & \mathbf{n} = (n_1 \geq \ldots \geq n_N) \\ 0\,, & \mathbf{n} \quad \textit{non-dominant} \end{cases} \qquad (7)$$

Explicit formulas: Archimedean case

The Givental stationary phase integral formula:

$$\Psi^{\mathbb{R}}_{\underline{\lambda}}(\mathbf{x}) = \int\limits_{C} \prod_{k\leq n<N} dx_{nk} \;\; e^{\hbar^{-1}\mathcal{F}_{\underline{\lambda}}(x_{nk})}\,, \qquad C \sim \mathbb{R}^{\frac{N(N-1)}{2}} \subset \mathbb{C}^{\frac{N(N-1)}{2}}\,, \tag{8}$$

$$\mathcal{F}_{\underline{\lambda}}(x_{nk}) = \sum_{n=1}^{N} \imath\lambda_n \Big(\sum_{k=1}^{n} x_{n,k} - \sum_{i=1}^{n-1} x_{n-1,\,i} \Big) - \sum_{\text{arrows}} e^{\text{target}(a) - \text{source}(a)}$$



(9)

Baxter operators for Macdonald polynomials, [GLO]

$$\check{\mathcal{Q}}_x^{q,t}\cdot f(\Lambda)\,=\,\sum_{\mu\in\mathbb{Z}^N}\check{Q}_x(\mu,\Lambda)\,f(\mu)\,,\qquad \check{Q}_x(\mu,\Lambda)\,=\,x^{|\mu|-|\Lambda|}\varphi_{\mu/\Lambda}\,,\quad (10)$$

$$\varphi_{\mu/\Lambda}\,=\,\prod_{\substack{i,j=1\\ i\leq j}}^{N}\frac{\Gamma_{q,tq^{-1}}\big(t^{j-i}q^{\mu_i-\mu_j+1}\big)}{\Gamma_{q,tq^{-1}}\big(t^{j-i}q^{\mu_i-\Lambda_j+1}\big)}\,\frac{\Gamma_{q,tq^{-1}}\big(t^{j-i}q^{\Lambda_i-\Lambda_{j+1}+1}\big)}{\Gamma_{q,tq^{-1}}\big(t^{j-i}q^{\Lambda_i-\mu_{j+1}+1}\big)}\,,$$

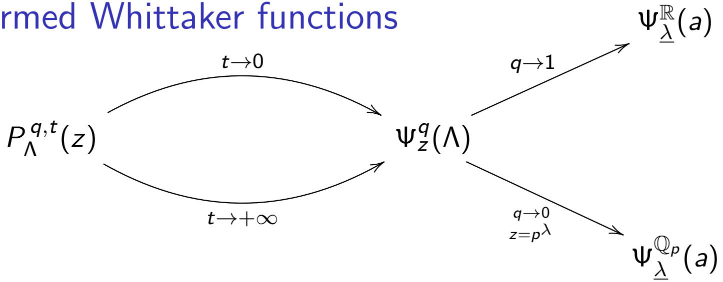
$$\Gamma_{q,t}(z)\,=\,\prod_{n\geq 0}\frac{1-tzq^n}{1-zq^n}\,,\qquad \Gamma_{q,\,t}(z)\times\Gamma_{q,\,t^{-1}}(qz^{-1})\,=\,t^{1/2}\frac{\theta_1\big((tz)^{1/2};\,q\big)}{\theta_1\big(z^{1/2};\,q\big)}\,.$$

Theorem

The Macdonald polynomials are eigenfunctions under the action of (10):

$$\check{\mathcal{Q}}_x^{q,t}\cdot P_\Lambda(z)\,=\,L_x^{\vee}(z)\,P_\Lambda(z)\,,\qquad L_x^{\vee}(z)\,=\,\prod_{i=1}^{N}\Gamma_{q,t}(xz_i)\,.\quad (11)$$

q-deformed Whittaker functions



$$H_r \cdot \Psi^q_z(\Lambda) \;=\; e_r(x)\,\Psi^q_z(\Lambda)\,. \tag{12}$$

$$H_r \;=\; \sum_{I_r} \prod_{k=1}^{r} \bigl(1 - q^{\Lambda_{i_k} - \Lambda_{i_{k+1}} + 1}\bigr)^{1-\delta_{i_{k+1}-i_k,\,1}}\; T_{I_r}\,, \qquad\qquad T_{I_r} \;=\; \prod_{i \in I_r} T_{q,\,q^{\Lambda_i}}\,,$$

Example

In the case $GL(2;\,F)$:

$$H_1 \;=\; \bigl(1 - q^{\Lambda_1 - \Lambda_2 + 1}\bigr)\,T_1 \;+\; T_2\,, \qquad\qquad H_2 \;=\; T_1\,T_2$$

Explicit formulas: q-analog of the LS formula, [GLO]

$$\Psi^q_z(\underline{p}_N) = \sum_{GZ} \prod_{n=1}^{N} z^{|\underline{p}_n| - |\underline{p}_{n-1}|} \frac{\prod_{i=1}^{n-1} (p_{n,i} - p_{n,\,i+1})_q!}{\prod_{i=1}^{n} (p_{n,i} - p_{n-1,\,i})_q!\,(p_{n-1,i} - p_{n,\,i+1})_q!} \qquad (13)$$

$$(m)_q! := (1-q)\cdot \ldots \cdot (1-q^m)\ ;$$

when $\underline{p}_N = (p_{N,1} \geq \ldots \geq p_{NN})$, and $\Psi^q_z(\underline{p}_N) = 0$ otherwise.
Summation is over the Gelfand-Zetlin (GZ) patterns:

$$\begin{array}{ccccccc} p_{N,1} & & p_{N,2} & & \ldots & & p_{NN} \\ & \ddots & & \ldots & & & \\ & & p_{21} & & p_{22} & & \\ & & & p_{11} & & & \end{array} \qquad \begin{array}{c} p_{n+1,\,k} \geq p_{nk} \geq p_{n+1,\,k+1}\,, \\ 1 \leq k \leq n < N \end{array}$$

"$\mathcal{U}_q(\mathfrak{gl}_N)$-Whittaker function" is a character of Demazure module of $\widehat{\mathfrak{gl}}_N$:

$$\Psi^q_{\underline{\lambda}}(\underline{p}) = \begin{cases} \Delta_q(\underline{\lambda})^{-1}\,\mathrm{ch}\,V_{w(\underline{p}')}\,, & \underline{p} = (p_1 \geq \ldots \geq p_N) \\ 0\,, & \underline{p} \quad \text{non-dominant} \end{cases} \qquad (14)$$

Archimedean limit $q \to 1$, [GLO]

$$q = e^{-\varepsilon}, \qquad m_\varepsilon = -\left[\varepsilon^{-1} \log \varepsilon\right]$$

Lemma

Let $f_\alpha(y;\,\varepsilon) := \left(\varepsilon^{-1} y + \alpha m_\varepsilon\right)_q!\,,$ *then as* $\varepsilon \to +0$

$$f_\alpha(y;\,\varepsilon) \sim \begin{cases} e^{A(\varepsilon) + e^{-y} + O(\varepsilon)}, & \alpha = 1 \\ e^{A(\varepsilon) + O(\varepsilon^{\alpha-1})}, & \alpha > 1 \end{cases}, \qquad A(\varepsilon) = -\frac{\pi^2}{6} - \frac{1}{2} \ln \frac{\varepsilon}{2\pi}\,.$$

Theorem

Set

$$p_{n,k} = (n + 1 - 2k) m_\varepsilon + \frac{x_{n,k}}{\varepsilon}\,, \qquad z_n = e^{\imath\,\varepsilon \Lambda_n}\,, \quad 1 \le n \le k \le N\,,$$

then for **the general partition** $\underline{p}_N$*:*

$$\Psi^{\mathrm{Givental}}_{\underline{\lambda}}(\mathbf{x}_N) = \lim_{\epsilon \to +0} \left[\varepsilon^{\frac{N(N-1)}{2}}\, e^{\frac{(N-1)(N+2)}{2} A(\varepsilon)}\ \Psi^q_z(\underline{p}_N)\right]. \tag{15}$$

Example: $\underline{p}_N = (\underbrace{n, \ldots, n}_{m}, 0, \ldots, 0)$

$$\lim_{\epsilon \to +0} \Big[\varepsilon^{m(N-m)} \, e^{[m(N-m)+1]A(\varepsilon)} \; \Psi^q_z(n^m, \, 0^{N-m}) \Big] = \int\limits_{C_m} \prod_{k,i} dx_{nk} \; e^{\mathcal{F}_{\underline{\lambda}}(x_{nk})} \,,$$

$$\mathcal{F}_{\underline{\lambda}}(x_{k,i}) \;=\; F_m(\underline{\lambda}) \;-\; \sum_{\text{arrows}} e^{\text{target}(a) - \text{source}(a)}$$

$$\begin{array}{ccccccc}
x & & & & & & \\
\downarrow & & & & & & \\
x_{N-m,\,1} & \longrightarrow & \cdots & \longrightarrow & x_{N-1,\,m} & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & \longrightarrow & \ddots & \longrightarrow & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
x_{11} & \longrightarrow & \cdots & \longrightarrow & x_{m,\,m} & \longrightarrow & 0
\end{array}$$

Parabolic $GL(N;\mathbb{R})$-Whittaker functions, [GLO, O]

$$\mathfrak{b}_+ = \mathfrak{h}^J + \mathfrak{n}_+^J\,, \qquad \chi_+^J : \quad \mathfrak{n}_+^J \longrightarrow \mathbb{C}\,,$$

$$\Psi_{\underline{\lambda}}^J(\mathbf{x}) = e^{-\rho(\underline{x})}\langle\psi_L\,,\pi_{\underline{\lambda}}\big(e^{-H^J(\underline{x})}\big)\psi_R^J\rangle\Big|_{\substack{x_i=0\\ i\notin J}}\,, \tag{16}$$

$$\mathbf{x} = (x_1,\ldots,x_r)\,, \qquad J = \big(J_1\leq\ldots\leq J_r\leq N\big)\,.$$

Theorem

Type J-parabolic $GL(N;\mathbb{R})$-Whittaker function (14) possesses the Mellin-Barnes integral representation:

$$\Psi_{\underline{\lambda}}^J(\mathbf{x}) = \int_{\mathcal{S}}\prod_{n,k}d\gamma_{nk}\prod_{n=1}^r e^{\frac{x_n}{\hbar}\Big(\sum\limits_{i=1}^{J_n}\gamma_{J_n,\,i}-\sum\limits_{j=1}^{J_{n+1}}\gamma_{J_{n+1},\,j}\Big)}\frac{\prod\limits_{i=1}^{J_n}\prod\limits_{j=1}^{J_{n+1}}\Gamma\Big(\frac{\gamma_{J_n,\,i}-\gamma_{J_{n+1},\,j}}{\hbar}\Big)}{\prod\limits_{\substack{i,k=1\\ i\neq k}}^{J_n}\Gamma\Big(\frac{\gamma_{J_n,\,i}-\gamma_{J_n,\,k}}{\hbar}\Big)} \tag{17}$$

Parabolic $GL(N;\mathbb{R})$-Whittaker functions as equivariant volumes of $\mathcal{M}_{\rm hol}(D\to G/P_J)$, [GLO]

Let $J=(1;\,N)$ and consider $\mathcal{QM}_d(\mathbb{P}^1\to\mathbb{P}^{N-1})\simeq\mathbb{P}^{N(d+1)-1}$, which is acted by the group $G=S^1\times U(N)$.

The G-equivariant symplectic form on $\mathcal{QM}_d(\mathbb{P}^1\to\mathbb{P}^{N-1})$ is

$$\omega_G=\omega_{\rm K\ddot{a}hler}+\hbar H_{S^1}+\lambda_1H_{U(1)}+\ldots+\lambda_NH_{U(1)}\,,\qquad U(1)^N\subset U(N)\,,$$

The G-equivariant volume of $\mathcal{QM}_d(\mathbb{P}^1\to\mathbb{P}^{N-1})$:

$$Z_d(x|\lambda,\,\hbar)=\langle e^{\omega_G},\,[\mathcal{QM}_d]\rangle_G=\oint_C d\gamma\,e^{\frac{x}{\hbar}\gamma}\prod_{i=1}^N\prod_{n=0}^d\frac{1}{\gamma-\lambda_i-n\hbar}\qquad(18)$$

Type A sigma-model provides a regularization of limit $d\to+\infty$

$$\Psi^{(1;\,N)}_{\lambda_1,\ldots,\lambda_N}(x)=\lim_{d\to+\infty}Z_d(x|\lambda,\hbar)=\int\limits_{\mathbb{R}-\imath\varepsilon}d\gamma\;e^{\frac{x}{\hbar}\gamma}\prod_{i=1}^N\Gamma\Big(\frac{\gamma-\lambda_i}{\hbar}\Big)\qquad(19)$$

$$=\Big\langle e^{\Omega_G},\,\big[\widetilde{\mathcal{L}_+\mathbb{P}^{N-1}}\big]\Big\rangle_G\,,\qquad\big[\widetilde{\mathcal{L}_+\mathbb{P}^{N-1}}\big]=\lim_{d\to+\infty}[\mathcal{QM}_d]\,.$$

Archimedean analog of the LS formula, [GLO, O]

Conjecture

The J-parabolic $G_{\mathbb{R}}$-Whittaker function $\Psi^J_{\underline{\lambda}}(\mathbf{x})$ can be identified with the $S^1 \times G_K$-equivariant volume of

$$\mathcal{M}_{\mathrm{hol}}\big(D \longrightarrow \mathrm{Fl}^\vee_J\big)\,,$$

and possesses the stationary phase integral representation

$$\Psi^J_{\underline{\lambda}}(\mathbf{x}) = \int\limits_{C_J} \prod_{n,k} dx_{nk}\; e^{\mathcal{F}^J_{\underline{\lambda}}(x_{nk})}\,, \qquad C_J \sim \mathbb{R}^{\dim \mathrm{Fl}_J} \subset \mathrm{Fl}^\vee_J(\mathbb{C})\,,$$

$\mathcal{F}^J_{\underline{\lambda}}(x_{nk})$ is a superpotential in type B sigma-model.

1. Representation theoretic proof: for minimal P_J in type A, [O];
2. TQFT proof for $\mathrm{Fl}_J = \mathbb{P}^{N-1}$ via Mirror Symmetry between the two topological sigma-models in disk D, [GLO].