Atkin and Swinnerton-Dyer Congruences on Noncongruence Modular Forms

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Noncongruence subgroups

- Bass-Lazard-Serre: All finite index subgroups of $SL_n(\mathbb{Z})$ for $n \geq 3$ are congruence subgroups.
- $SL_2(\mathbb{Z})$ contains far more noncongruence subgroups than congruence subgroups.
- Let Γ be a finite index subgroup of $SL_2(\mathbb{Z})$. The orbit space $\Gamma\backslash\mathfrak{H}^*$ is a Riemann surface, called the modular curve X_Γ for Γ. It has a model defined over a number field.
- The modular curves for congruence subgroups are defined over $\mathbb{Q}$ or cyclotomic fields $\mathbb{Q}(\zeta_N)$.
- Belyi: Every smooth projective irreducible curve defined over a number field is isomorphic to a modular curve X_Γ (for infinitely many finite-index subgroups Γ of $SL_2(\mathbb{Z})$).

Modular forms for congruence subgroups

Let $g = \sum_{n\geq 1} a_n(g)q^n$, where $q = e^{2\pi iz}$, be a normalized ($a_1(g) = 1$) newform of weight $k \geq 2$ level N and character χ.

I. Hecke theory

- It is an eigenfunction of the Hecke operators T_p with eigenvalue $a_p(g)$ for all primes $p \nmid N$, i.e., for all $n \geq 1$,

$$a_{np}(g) - a_p(g)a_n(g) + \chi(p)p^{k-1}a_{n/p}(g) = 0.$$

- The space of weight k cusp forms for a congruence subgroup contains a basis of forms with algebraically integral Fourier coefficients. An algebraic cusp form has bounded denominators.

II. Galois representations

- (Eichler-Shimura, Deligne) There exists a compatible family of degree two l-adic rep'ns $\rho_{g,l}$ of $G_{\mathbb{Q}} := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ such that at primes $p \nmid lN$, the char. poly.

$$H_p(T) = T^2 - A_pT + B_p = T^2 - a_p(g)T + \chi(p)p^{k-1}$$

of $\rho_{g,l}(\mathrm{Frob}_p)$ is indep. of l, and

$$a_{np}(g) - A_p\, a_n(g) + B_p\, a_{n/p}(g) = 0$$

for $n \geq 1$ and primes $p \nmid lN$.

- Ramanujan-Petersson conjecture holds for newforms. That is, $|a_p(g)| \leq 2p^{(k-1)/2}$ for all primes $p \nmid N$.

Modular forms for noncongruence subgroups

Γ : a noncongruence subgroup of $SL_2(\mathbb{Z})$ with finite index

$S_k(\Gamma)$: space of cusp forms of weight $k \geq 2$ for Γ of dim d

A cusp form has an expansion in powers of $q^{1/\mu}$.

Assume the modular curve X_Γ is defined over $\mathbb{Q}$ and the cusp at infinity is $\mathbb{Q}$-rational.

Atkin and Swinnerton-Dyer: there exists a positive integer M such that $S_k(\Gamma)$ has a basis consisting of forms with coeffs. integral outside M (called M-integral) :

$$f(z) = \sum_{n \geq 1} a_n(f) q^{n/\mu}.$$

No efficient Hecke operators on noncongruence forms

- Let Γ^c be the smallest congruence subgroup containing Γ. Naturally, $S_k(\Gamma^c) \subset S_k(\Gamma)$.
- $Tr_{\Gamma}^{\Gamma^c} : S_k(\Gamma) \to S_k(\Gamma^c)$ such that $S_k(\Gamma) = S_k(\Gamma^c) \oplus ker(Tr_{\Gamma}^{\Gamma^c})$.
- $ker(Tr_{\Gamma}^{\Gamma^c})$ consists of genuinely noncongruence forms in $S_k(\Gamma)$.

Conjecture (Atkin). The Hecke operators on $S_k(\Gamma)$ for $p \nmid M$ defined using double cosets $\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma$ as for congruence forms is zero on genuinely noncongruence forms in $S_k(\Gamma)$.

This was proved by Serre, Berger.

So the progress has been led by computational data.

Atkin and Swinnerton-Dyer congruences for elliptic curves

Let E be an elliptic curve defined over $\mathbb{Q}$ with conductor M. By Belyi, $E \simeq X_\Gamma$ for a finite index subgroup Γ of $SL_2(\mathbb{Z})$.
Ex. $E : x^3 + y^3 = z^3$, Γ is an index-9 noncong. subgp of $\Gamma(2)$.

Atkin and Swinnerton-Dyer: The normalized holomorphic differential 1-form $f\frac{dq}{q} = \sum_{n\geq 1} a_n q^n \frac{dq}{q}$ on E satisfies the congruence relation

$$a_{np} - [p + 1 - \#E(\mathbb{F}_p)]a_n + pa_{n/p} \equiv 0 \mod p^{1+\mathrm{ord}_p n} \quad (1)$$

for all primes $p \nmid M$ and all $n \geq 1$.

Sketch of a proof using formal group laws

- The formal power series $\ell(x) := \sum_{n\geq 1} a_n x^n$ is the formal log of a formal group law G, which is isomorphic to the group law on the elliptic curve E in a neighborhood of the identity element.
- The Hasse-Weil L-function of E is

  $$L(s,E) = \prod_{p\nmid M} \frac{1}{1 - b_p p^{-s} + p^{1-2s}} \prod_{p\mid M} \frac{1}{1 - b_p p^{-s}} = \sum_{n\geq 1} b_n n^{-s},$$

  in which $b_p = p + 1 - \#E(\mathbb{F}_p)$ for $p \nmid M$.
- Honda: $\tilde{\ell}(x) := \sum_{n\geq 1} b_n x^n$ is the formal log of a formal group law which is strictly isomorphic to G.

- Hence for $p \nmid M$, the sequences $\{a_n\}$ and $\{b_n\}$ satisfy the same congruence relation of ASD type:

$$c_{np} - A_p c_n + B_{p,1} c_{n/p} + B_{p,2} c_{n/p^2} + \cdots \equiv 0 \mod p^{1+\mathrm{ord}_p n}.$$

- The b_n's satisfy the three term recursion

$$b_{np} - b_p b_n + p b_{n/p} = 0 \quad \text{for all } p \nmid M \text{ and } n \geq 1.$$

So $A_p = b_p$, $B_{p,1} = p$, and other $B_{p,i} = 0$. This proves (1).

- For $p \nmid M$, $H_p(T) = T^2 - b_p T + p$ is the characteristic poly. of the $Frob_p$ acting on the Tate module $T_l(E)$ for $l \neq p$.

- Taniyama-Shimura modularity theorem: $g = \sum_{n \geq 1} b_n q^n$ is a normalized congruence newform. Note that $f \in S_2(\Gamma)$. Thus (1) gives congruence relations between f and g.

ASD congruences in general

Back to general case where X_Γ has a model over $\mathbb{Q}$, and the d-dim'l space $S_k(\Gamma)$ has a basis of M-integral forms.

ASD congruences (1971): for each prime $p \nmid M$, $S_k(\Gamma, \mathbb{Z}_p)$ has a p-adic basis $\{h_j\}_{1\le j\le d}$ such that the Fourier coefficients of h_j satisfy a three-term congruence relation

$$a_{np}(h_j) - A_p(j)a_n(h_j) + B_p(j)a_{n/p}(h_j) \equiv 0 \mod p^{(k-1)(1+\mathrm{ord}_p n)}$$

for all $n \ge 1$. Here

- $A_p(j)$ is an algebraic integer with $|A_p(j)| \le 2p^{(k-1)/2}$, and
- $B_p(j)$ is equal to p^{k-1} times a root of unity.

This is proved to hold for $k = 2$ and $d = 1$ by ASD.

The basis varies with p in general.

Galois representations attached to $S_k(\Gamma)$ and Scholl congruences

Theorem[Scholl] *Suppose that the modular curve X_Γ has a model over $\mathbb{Q}$ such that the cusp at infinity is $\mathbb{Q}$-rational. Attached to $S_k(\Gamma)$ is a compatible family of 2d-dim'l l-adic rep'ns ρ_l of $G_\mathbb{Q}$ unramified outside lM such that for primes $p > k+1$ not dividing Ml, the following hold.*

(i) The char. polynomial

$$H_p(T) = T^{2d} + C_1(p)T^{2d-1} + \cdots + C_{2d-1}(p)T + C_{2d}(p)$$

of $\rho_l(\mathrm{Frob}_p)$ lies in $\mathbb{Z}[T]$, is indep. of l, and its roots are algebraic integers with complex absolute value $p^{(k-1)/2}$;

(ii) For any form f in $S_k(\Gamma)$ integral outside M, its Fourier coeffs satisfy the $(2d+1)$-term congruence relation

$$\begin{aligned} & a_{np^d}(f) + C_1(p)a_{np^{d-1}}(f) + \cdots + \\ & + C_{2d-1}(p)a_{n/p^{d-1}}(f) + C_{2d}(p)a_{n/p^d}(f) \\ & \equiv 0 \mod p^{(k-1)(1+\mathrm{ord}_p n)} \end{aligned}$$

for $n \geq 1$.

Theorem *If $S_k(\Gamma)$ is 1-dimensional, then*

- *the ASD congruences hold for almost all p;*
- *the degree two l-adic Scholl rep'ns of $G_{\mathbb{Q}}$ are modular.*

The 2nd statement follows from Kahre-Wintenberger's work on Serre's conjecture on modular rep'ns, and various modularity lifting theorems.

Application: Characterizing noncongruence modular forms

The following conjecture, supported by all known examples, gives a simple characterization for noncongruence forms. If true, it has wide applications.

Conjecture. A modular form in $S_k(\Gamma)$ with algebraic Fourier coefficients has bounded denominators if and only if it is a congruence modular form, i.e., lies in $S_k(\Gamma^c)$.

Theorem[L-Long] *The conjecture holds when* $S_k(\Gamma)$ *is* 1*-dim'l, containing a basis with Fourier coefficients in* $\mathbb{Q}$*.*

The proof uses ASD congruences, modularity of the Scholl representations, and the Selberg bound on Fourier coefficients of a wt k cusp form f: $a_n(f) = O(n^{k/2-1/5})$.

From Scholl congruences to ASD congruences

Ideally one hopes to factor

$$H_p(T) = \prod_{1 \le j \le d} (T^2 - A_p(j)T + B_p(j))$$

and find a p-adic basis $\{h_j\}_{1 \le j \le d}$, depending on p, for $S_k(\Gamma, \mathbb{Z}_p)$ such that each h_j satisfies the three-term ASD congruence relations given by $A_p(j)$ and $B_p(j)$.

For a congruence subgroup Γ, this is achieved by using Hecke operators to further break the l-adic space and $S_k(\Gamma)$ into pieces. For a noncongruence Γ, no such tools are available.

Theorem[Scholl] *If $\rho_l(Frob_p)$ is diagonalizable and ordinary (i.e. half of the eigenvalues are p-adic units), then ASD congruences at p hold.*

Examples of ASD congruences

The group $\Gamma^1(5)$ has genus 0, no elliptic elements, and 4 cusps. The wt 3 Eisenstein series

$$E_1(z) = 1 - 2q^{1/5} - 6q^{2/5} + 7q^{3/5} + 26q^{4/5} + \cdots ,$$
$$E_2(z) = q^{1/5} - 7q^{2/5} + 19q^{3/5} - 23q^{4/5} + \cdots .$$

have simple zeros at all cusps except ∞ and -2, resp., and non-vanishing elsewhere. So $X_{\Gamma^1(5)}$ is defined over $\mathbb{Q}$ with $t = \frac{E_2}{E_1}$ as a Hauptmodul, and $t_n = \sqrt[n]{t}$ is a Hauptmodul of a smooth irred. modular curve X_{Γ_n} over $\mathbb{Q}$.

Let $\rho_{n,l}$ be the l-adic Scholl representation attached to $S_3(\Gamma_n)$.

Ex 1. When $n = 2$, $S_3(\Gamma_2) =< E_1 t_2 >$ is 1-dim'l, ASD congruences hold for odd p, and $\rho_{2,l}$ are isom. to $\rho_{\eta(4z)^6,l}$.

Ex 2. (L-Long-Yang)

(1) When $n = 3$, the space $S_3(\Gamma_3) =< E_1 t_3, E_1 t_3^2 >$ has a basis

$$f_\pm(z) = q^{1/15} \pm iq^{2/15} - \frac{11}{3}q^{4/15} \mp i\frac{16}{3}q^{5/15} - \\ -\frac{4}{9}q^{7/15} \pm i\frac{71}{9}q^{8/15} + \frac{932}{81}q^{10/15} + \cdots .$$

(2) (Modularity) There are two cuspidal newforms of weight 3 level 27 and quadratic character χ_{-3} given by

$$g_\pm(z) = q \mp 3iq^2 - 5q^4 \pm 3iq^5 + 5q^7 \pm 3iq^8 + \\ +9q^{10} \pm 15iq^{11} - 10q^{13} \mp 15iq^{14} - \cdots$$

such that $\rho_{3,l} = \rho_{g_+,l} \oplus \rho_{g_-,l}$ over $\mathbb{Q}_l(\sqrt{-1})$.

(3) $f_\pm$ satisfy the 3-term ASD congruences with $A_p = a_p(g_\pm)$ and $B_p = \chi_{-3}(p)p^2$ for all primes $p \geq 5$.

Ex 3. (Atkin, Li, Long) $S_3(\Gamma_4) = < h_1, h_2, h_3 >$, where $h_i = E_1 t_4^i$ and $h_2 = E_1 t_2$ satisfies ASD congruences for all odd p.

The space $< h_1, h_3 >$ has a basis satisfying the ASD congruence depending on the residue of odd $p \mod 8$:

1. If $p \equiv 1 \mod 8$, then both h_1 and h_3 satisfy ASD with $A_p = \mathrm{sgn}(p) a_1(p)$ and $B_p = p^2$, where $\mathrm{sgn}(p) = \pm 1 \equiv 2^{(p-1)/4} \mod p$;
2. If $p \equiv 5 \mod 8$, then h_1 (resp. h_3) satisfies ASD with $A_p = 4i a_5(p)$ (resp. $-4i a_5(p)$) and $B_p = -p^2$;
3. If $p \equiv 3 \mod 8$, then $h_1 \pm h_3$ satisfy ASD with $A_p = \pm 2\sqrt{-2} a_3(p)$ and $B_p = -p^2$;
4. If $p \equiv 7 \mod 8$, then $h_1 \pm i h_3$ satisfy ASD with $A_p = \mp 8\sqrt{-2} a_7(p)$ and $B_p = -p^2$.

Here $a_1(p), a_3(p), a_5(p), a_7(p)$ are the Fourier coefficients of the wt 3 congruence forms f_1, f_3, f_5, f_7 given below:

$$f_1(z) = \frac{\eta(2z)^{12}}{\eta(z)\eta(4z)^5} = q^{1/8}(1 + q - 10q^2 + \cdots) = \sum_{n\geq 1} a_1(n)q^{n/8},$$

$$f_3(z) = \eta(z)^5\eta(4z) = q^{3/8}(1 - 5q + 5q^2 + \cdots) = \sum_{n\geq 1} a_3(n)q^{n/8},$$

$$f_5(z) = \frac{\eta(2z)^{12}}{\eta(z)^5\eta(4z)} = q^{5/8}(1 + 5q + 8q^2 + \cdots) = \sum_{n\geq 1} a_5(n)q^{n/8},$$

$$f_7(z) = \eta(z)\eta(4z)^5 = q^{7/8}(1 - q - q^2 + \cdots) = \sum_{n\geq 1} a_7(n)q^{n/8}.$$

For the last two examples, both l-adic space and $S_3(\Gamma_n)$ admit quaternion multiplications, which are used to break the spaces.

An example of failure of ASD congruences

Ex 4. (Kibelbek) $X: \ y^2 = x^5 + 1$ is a genus 2 curve defined over $\mathbb{Q}$. By Belyi, $X \simeq X_\Gamma$ for a finite index subgroup Γ of $SL_2(\mathbb{Z})$. Put

$$\omega_1 = \frac{dx}{2y} = f_1 \frac{dq^{1/10}}{q^{1/10}}, \quad \omega_2 = x\frac{dx}{2y} = f_2 \frac{dq^{1/10}}{q^{1/10}}.$$

Then $S_2(\Gamma) =< f_1, f_2 >$, where

$$f_1 = q^{1/10} - \frac{8}{5}q^{6/10} - \frac{108}{5^2}q^{11/10} + \frac{768}{5^3}q^{16/10} + \frac{3374}{5^4}q^{21/10} + \cdots ,$$

$$f_2 = q^{2/10} - \frac{16}{5}q^{7/10} + \frac{48}{5^2}q^{12/10} + \frac{64}{5^3}q^{17/10} + \frac{724}{5^4}q^{22/10} + \cdots .$$

The l-adic representations attached to $S_2(\Gamma)$ are the dual of the Tate modules on the Jacobian of X_Γ. They are strongly ordinary at $p \equiv \pm 1 \mod 5$, so ASD congruences hold for such p.

For primes $p \equiv \pm 2 \mod 5$, $H_p(T) = T^4 + p^2$ (not ordinary).

$S_2(\Gamma)$ has no nonzero forms satisfying the ASD congruences for $p \equiv \pm 2 \mod 5$.

Scholl congruences

Back to the finite index subgroup Γ of $SL_2(\mathbb{Z})$; $S_k(\Gamma)$ is d-dim'l with an M-integral basis.

- The $2d$-dim'l Scholl rep'ns ρ_l of $G_{\mathbb{Q}}$ are generalizations of Deligne's construction to the noncongruence case.
- For $p \nmid M$, Scholl constructed p-adic de Rham space $DR(\Gamma, k, \mathbb{Z}_p)$ of rank $2d$, which contains $S_k(\Gamma, \mathbb{Z}_p)$.
- Scholl: de Rham cohomology is isomorphic to the crystalline cohomology. Thus the ϕ operator in the crystalline theory is transported to the de Rham space.
- The action of $Frob_p$ on l-adic side ($l \neq p$) and ϕ on the p-adic side have the same characteristic poly.

$$H_p(T) = T^{2d} + C_1(p)T^{2d-1} + \cdots + C_{2d}(p).$$

Scholl congruences for weakly holomorphic modular forms

$M_k^{wk}(\Gamma, R) \supset M_k^{wk-ex}(\Gamma, R) \supset S_k^{wk-ex}(\Gamma, R) \supset D^{k-1}M_{2-k}^{wk}(\Gamma, R)$

- A modular form is *weakly holo.* if it is holo. on $\mathfrak{H}$ and mero. at cusps.
- $f \in M_k^{wk}(\Gamma, R)$ is called *weakly exact* if at each cusp c of Γ the Fourier coefficients $a_n(f, c)$ of f at c is divisible by n^{k-1} in R for each $n < 0$.
- $f \in M_k^{wk-ex}(\Gamma, R)$ is a *cusp* form if it has vanishing constant term at all cusps.
- $D^{k-1} : \sum_n c_n q^{n/\mu} \mapsto \sum_n n^{k-1} c_n q^{n/\mu}$ maps $M_{2-k}^{wk}(\Gamma, R)$ to $S_k^{wk-ex}(\Gamma, R)$.

- Using geometric interpretation of weakly holo. modular forms, Kazalicki-Scholl proved

$$DR(\Gamma, k, \mathbb{Z}_p) = \frac{S_k^{wk-ex}(\Gamma, \mathbb{Z}_p)}{D^{k-1}(M_{2-k}^{wk}(\Gamma, \mathbb{Z}_p))}.$$

$\sum_n a_n q^{n/\mu}$ and $\sum_n b_n q^{n/\mu}$ in S_k^{wk-ex} are equal in $DR(\Gamma, k, \mathbb{Z}_p)$ $\Leftrightarrow a_n \equiv b_n \mod p^{(k-1)\mathrm{ord}_p n}$.

- $\phi : \sum_n a_n q^{n/\mu} \mapsto p^{k-1} \sum_n a_n q^{np/\mu}$.
- Since $H_p(\phi) = 0$ on $DR(\Gamma, k, \mathbb{Z}_p)$, any $f = \sum_n a_n q^{n/\mu}$ in $S_k^{wk-ex}(\Gamma, \mathbb{Z}_p)$ satisfies the congruence

$$\begin{aligned} &a_{np^d}(f) + C_1(p) a_{np^{d-1}}(f) + \cdots + C_{2d}(p) a_{n/p^d}(f) \\ &\equiv 0 \mod p^{(k-1)\mathrm{ord}_p n} \qquad \text{for all } n \geq 1. \end{aligned}$$

- Since $\phi(S_k(\Gamma, \mathbb{Z}_p)) \subset p^{k-1}DR(\Gamma, k, \mathbb{Z}_p)$, for $f \in S_k(\Gamma, \mathbb{Z}_p)$ the above congruence holds mod $p^{(k-1)(1+\text{ord}_p n)}$.

Ex 5. $S_{12}(SL_2(\mathbb{Z}))$ is 1-dim'l spanned by $\Delta(z) = \eta(z)^{24} = \sum_{n\geq 1} \tau(n)q^n$. The char. poly $H_p(T) = T^2 - \tau(p)T + p^{11}$.

$$E_4(z)^6/\Delta(z) - 1464E_4(z)^3 = q^{-1} + \sum_{n=1}^{\infty} a_n q^n$$
$$= q^{-1} - 142236q + 51123200q^2 + 39826861650q^3 + \cdots$$

lies in $DR(SL_2(\mathbb{Z}), k, \mathbb{Z})$. For every prime $p \geq 11$ and integers $n \geq 1$, its coefficients satisfy the congruence

$$a_{np} - \tau(p)a_n + p^{11}a_{n/p} \equiv 0 \pmod{p^{11\text{ord}_p n}}.$$