The Iwahori-Hecke Algebra, the Ramanujan Conjecture, and Expander Graphs

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ICERM
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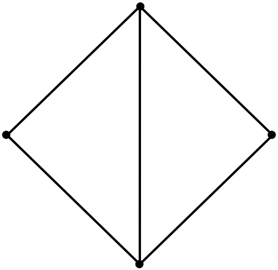

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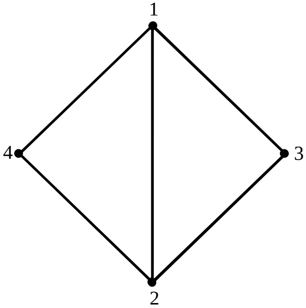

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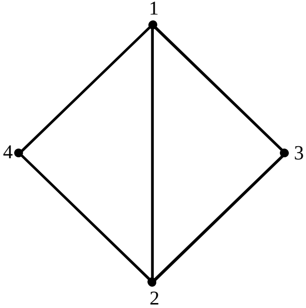

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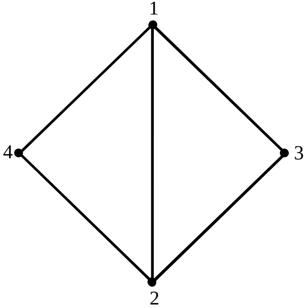

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$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 0 & \text{if } \{v_i, v_j\} \notin E \end{cases}$$

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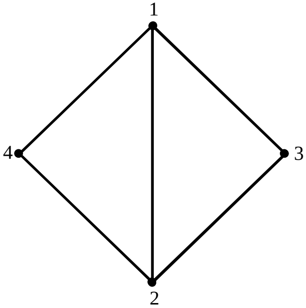

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Primes in Graphs

Graph $X = (V, E)$

- Path C

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  equivalence class of a closed, backtrack-less, tail-less, primitive path C

Zeta Function for Graphs

$$Z_X(u) = \prod_{[C] \text{ prime}} \left(1 - u^{|C|}\right)^{-1}, \quad u \in \mathbb{C}$$

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Theorem (Ihara 1966)

If $X = (V, E)$ is a finite, connected, $(q+1)$-regular multigraph (q odd), then

$$Z_X(u)^{-1} = (1 - u^2)^{r-1} \det(I - uA + qu^2 I),$$

where $r = |E| - |V| + 1$ and I is the $|V| \times |V|$ identity matrix.

Riemann Hypothesis for Graphs

Definition

A finite, connected, $(q+1)$-regular graph X satisfies the Riemann Hypothesis if for $Re(s) \in (0,1)$

$$\left(Z_X(q^{-s})\right)^{-1} = 0 \text{ implies } Re(s) = \frac{1}{2}$$

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- Which graphs satisfy the Riemann Hypothesis?

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- If $(X_{n,k})$ is a k-regular graph on n vertices, the eigenvalues of A are $k \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n-1} \geq -k$

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$$c = \inf \left\{ \frac{|\partial(F)|}{|F|} : \; F \subseteq V, 0 < |F| \leq \frac{n}{2} \right\}$$

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Want large expansion coefficient.

Good expanders/Ramanujan Graphs

Proposition (Cheeger's inequality)

Connected $X_{n,k}$ *is an* (n, k, c)*-expander with* $c \geq \dfrac{k - \lambda(X_{n,k})}{2}$.

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Corollary (to Ihara's Theorem)

X satisfies the Riemann hypothesis $\iff$ *X is a Ramanujan graph*

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- group structure makes estimation of eigenvalues possible

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(Generalize to $GL_n(\mathbb{Q}_p)$: B. 2001.)

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- Ramanujan hypergraphs: Li (Laumot-Rapoport-Stuhler)

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- n_i number of vertices with valency $q_i + 1$ (then, $n_2 \geq n_1$)
- $\mathrm{Spec}(X) = \{\pm\lambda_1, \pm\lambda_2, \ldots \pm \lambda_{n_1}, \underbrace{0, \ldots, 0}_{n_2 - n_1}\}$

  $\lambda_1 = \sqrt{(1+q_1)(1+q_2)} > \lambda_2 \geq \cdots \geq \lambda_{n_1} \geq 0$

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Definition

A (q_1+1, q_2+1)-bigraph X is a *weak Ramanujan bigraph* if $\lambda_{n_1} > 0$, *i.e.*, 0 has multiplicity exactly $n_2 - n_1$ in $\mathrm{Spec}(X)$.

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- Maximal compact subgroups of G: $\{K_1\}, \{K_2\}$.
- $I = K_1 \cap K_2$: Iwahori subgroup (fixes an edge)
- $\mathcal{H}(G, I) = \langle T_1, T_2 \rangle / (T_i^2 = (q_i - 1)T_i + q_i)$

Adjacency and the Iwahori-Hecke algebra

- T_1, T_2 endomorphisms on $\mathbb{C}[E(X)]$,

$$(T_i f)(e) := \sum_{e' \in E_i(e)} f(e') - f(e) \quad (i = 1, 2).$$

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Theorem (Hashimoto)

$$Z_X(u)^{-1} = \det(I - (T_1 T_2)u)$$

$$= (1-u)^{r-1}(1+q_2 u)^{n_2 - n_1} \prod_{j=1}^{n_1} (1 - (\lambda_j^2 - q_1 - q_2)u + q_1 q_2 u^2)$$

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- Iwahori-Hecke algebra $\mathcal{H} = \mathcal{H}(G, I) \cong \mathbb{C}[T_1, T_2]$

Representations of the Iwahori-Hecke Algebra (Hashimoto)

• Finite dim'l irreducible representations φ of $\mathcal{H}$ have dimension 1 or 2 and they are determined by the characteristic polynomial p_φ of $\varphi(T_1 T_2)$.

• Degree two irreducible representations are parameterized by $c \in \mathbb{C}$, $c \neq 0$, $c \neq (q_1 + 1)(q_2 + 1)$.

$$p_\varphi(u) = \det(1 - \varphi(T_1 T_2)u) = 1 - (c - q_1 - q_2)u + q_1 q_2 u^2.$$

• One dimensional irreducible representation have characteristic polynomial

$$p_{\mathrm{St}}(u) = 1 - u; \qquad p_{\mathrm{ds}}(u) = 1 + q_2 u;$$

$$p_{\mathrm{sph}}(u) = 1 - q_1 q_2 u; \qquad p_{\mathrm{nt}}(u) = 1 + q_1 u.$$

Representations (continued)

$$c \in Spec(X)$$

$$\updownarrow$$

$\varphi \leftrightarrow \pi$ sherical unitary irreducible representation of G appearing in $L^2(G/\Gamma)$

mult. of $p_\varphi(u)$ in $Z_X(u)^{-1}$ = mult. of π in $L^2(G/\Gamma)^I =: m(\pi)$

Iwahori-Hecke Algebra

- (Bernstein-Lusztig presentation) $\mathcal{H} = \mathcal{H}_W \otimes \mathcal{A}$

  $\mathcal{H}_W = \mathbb{C}[T]/\langle T^2 = (z^{2\lambda} - 1)T + z^{2\lambda}\rangle$, $\mathcal{A} = \mathbb{C}[\theta]$

  commutation relation:

$$\theta T - T\theta^{-1} = (z^{2\lambda} - 1)\theta + (z^{\lambda+\lambda^*} - z^{\lambda-\lambda^*})$$

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- Iwahori $\leftrightarrow$ Berenstein-Lusztig

  $T = T_1, \quad \theta = \frac{1}{\sqrt{q_1 q_2}} T_1 T_2, \quad z^{2\lambda} = q_1$ and $z^{2\lambda^*} = q_2$

$$p_\varphi(u) = \det(1 - \sqrt{q_1 q_2}\varphi(\theta)u) = 1 - \sqrt{q_1 q_2}\mathrm{Tr}(\varphi(\theta))u + q_1 q_2 u^2$$

Tempered representations

If $m(\pi) > 0$ and $\varphi \notin \{\mathrm{St}, \mathrm{ds}\}$,

$$Tr(\varphi(\Theta)) = \frac{1}{\sqrt{q_1 q_2}} Tr(\varphi(T_1 T_2)) = \frac{c - q_1 - q_2}{\sqrt{q_1 q_2}}$$

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Ramanujan condition $|c - q_1 - q_2| \leq 2\sqrt{q_1 q_2}$

$$\Updownarrow$$

$|Tr(\varphi(\Theta))| \leq 2$ (i.e., φ is tempered)

Ramanujan bigraphs

Ramanujan Type Conjecture: Every nontrivial irreducible unitary $\mathcal{H}(G, I)$-module that appears in the decomposition of $L^2(G/\Gamma)^I = L^2(I\backslash G/\Gamma)$ is tempered.

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Theorem (B.-C.)

$G = SU(3)$.
$\Gamma \leq G$ discrete, co-compact, acts on G without fixed points.
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Corollary

The graph X is Ramanujan if and only if X is weakly Ramanujan.

Constructions

Theorem (Rogawski)

If G is a compact inner form of $U(3)$ arising from a division algebra with an involution of the second kind, there are no non-tempered representations (the Ramanujan-Petersson conjecture is satisfied).