

**The zeta-function of the root system
of type G_2**

Kohji Matsumoto (Nagoya University)

(a joint work with Y.Komori and H.Tsumura)

V : r -dimensional real vector space

\langle , \rangle : inner product defined on V

Δ : finite reduced root system in V

$$\Delta = \Delta_+ \cup \Delta_-$$

$\Psi = \{\alpha_1, \dots, \alpha_r\}$: fundamental system

$\{\lambda_1, \dots, \lambda_r\}$: fundamental weights (with $\langle \alpha_i^\vee, \lambda_j \rangle = \delta_{ij}$)

$$P_{++} = \bigoplus_{i=1}^r \mathbb{N}\lambda_i \quad (\mathbb{N} = \mathbb{Z}_{\geq 1})$$

Define the zeta-function of Δ by

$$\zeta_r(\mathbf{s}, \Delta) = \sum_{\lambda \in P_{++}} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}}$$

where $\mathbf{s} = (s_\alpha)_{\alpha \in \Delta_+} \in \mathbb{C}^n$, $n = |\Delta_+|$.

Examples.

Ex 1. $\Delta = A_1$

$$\Rightarrow \Delta_+ = \{\alpha_1\}, P_{++} = \{m\lambda_1 \mid m \in \mathbb{N}\},$$

$$\Rightarrow \zeta_1(s, A_1) = \sum_{\lambda \in P_{++}} \langle \alpha_1^\vee, \lambda \rangle^{-s} = \sum_{m=1}^{\infty} m^{-s} = \zeta(s)$$

(the Riemann zeta-function)

Ex 2. $\Delta = A_2$

$$\Rightarrow \Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\},$$

$$P_{++} = \{m_1\lambda_1 + m_2\lambda_2 \mid m_1, m_2 \in \mathbb{N}\},$$

Therefore

$$\zeta_2(s, A_2) = \sum_{\lambda \in P_{++}} \prod_{\alpha \in \Delta_+} \frac{1}{\langle \alpha^\vee, \lambda \rangle^{s_\alpha}} = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1^{-s_1} m_2^{-s_2} (m_1 + m_2)^{-s_3}$$

(the Tornheim double sum)

Ex 3. The case $\mathbf{s} = (s, s, \dots, s)$

\mathfrak{g} : a semisimple Lie algebra over \mathbb{C}

$$\Delta = \Delta(\mathfrak{g})$$

Witten zeta-function : $\zeta_W(s, \mathfrak{g}) = \sum_{\varphi} (\dim \varphi)^{-s}$

(φ runs over all finite dim irreducible representation of \mathfrak{g})

By Weyl's dimension formula we have

$$\begin{aligned} & \zeta_W(s, \mathfrak{g}) \\ &= \sum_{m_1=1}^{\infty} \cdots \sum_{m_r=1}^{\infty} \prod_{\alpha \in \Delta_+} \left(\frac{\langle \alpha^\vee, m_1 \lambda_1 + \cdots + m_r \lambda_r \rangle}{\langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle} \right)^{-s} \\ &= \left(\prod_{\alpha \in \Delta_+} \langle \alpha^\vee, \lambda_1 + \cdots + \lambda_r \rangle \right)^s \zeta_r((s, s, \dots, s), \Delta(\mathfrak{g})) \end{aligned}$$

Ex 4. The Euler-Hoffman-Zagier multiple sum

$$\zeta_{EHZ,r}(\mathbf{s}) = \sum_{n_1=1}^{\infty} \cdots \sum_{n_r=1}^{\infty} n_1^{-s_1} (n_1 + n_2)^{-s_2} \cdots (n_1 + \cdots + n_r)^{-s_r}$$

This can be regarded as a zeta-function of the root system of type C_r , which is NOT simply-laced.

$$\Delta_+ = \Delta_{l+} \cup \Delta_{s+} \quad (l: \text{long roots, } s: \text{short roots})$$

$$\mathbf{s}_l = (s_\alpha)_{\alpha \in \Delta_+}, \text{ with } s_\alpha = 0 \text{ for any } \alpha \in \Delta_{s+}$$

$$\text{Then we find } \zeta_r(\mathbf{s}_l, C_r) = \zeta_{EHZ,r}(\mathbf{s})$$

(It is also possible to understand $\zeta_{EHZ,r}(\mathbf{s})$ as a zeta-function of the root system of type A_r ; cf. Komori-M-Tsumura, Math. Z. **268** (2011))

Special values (at positive integer points)

The study of special values is important for both Witten zeta-functions and Euler-Hoffman-Zagier sums

⇒ How are the special values of zeta-functions of root systems?

Recall : Witten's volume formula (connected with the volumes of certain moduli spaces appearing in quantum gauge theory)

$$\zeta_W(2k, \mathfrak{g}) = C_W(2k, \mathfrak{g}) \pi^{2kn}$$

for $k \in \mathbb{N}$, where $n = |\Delta_+|$ and $C_W(2k, \mathfrak{g}) \in \mathbb{Q}$

What is the explicit value of $C_W(2k, \mathfrak{g})$?

1. A_1 case (the Riemann zeta)

$$\zeta(2) = \pi^2/6, \quad \zeta(4) = \pi^4/90, \dots$$

That is, $C_W(2, A_1) = 1/6, \quad C_W(4, A_1) = 1/90, \dots$

2. A_2 case (the Tornheim sum)

$$\zeta_W(s, A_2) = 2^s \zeta_2((s, s, s), \Delta(A_2))$$

and Mordell proved $\zeta_2((2, 2, 2), A_2) = \pi^6/2835$

$$\Rightarrow C_W(2, A_2) = 4/2835$$

Algorithm of computing $C_W(2k, \mathfrak{g})$ for general case:

Szenes (1998), Gunnells-Sczech (2003),

Our approach (Around 2005 –)

$W = W(\Delta)$: The Weyl group of Δ

Define

$$S(\mathbf{s}, \Delta) = \sum_{w \in W} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}\mathbf{s}, \Delta),$$

where $w^{-1}\mathbf{s}$ is defined by $(\sigma_\beta \mathbf{s})_\alpha = s_{\sigma_\beta \alpha}$ for the reflection σ_β with respect to (the hyperplane orthogonal to) β

(If $\sigma_\beta \alpha \in \Delta_-$, then we understand that $s_{\sigma_\beta \alpha} = s_{-\sigma_\beta \alpha}$)

$S(\mathbf{s}, \Delta)$ is the “Weyl group symmetric linear combination” of $\zeta_r(\mathbf{s}, \Delta)$

\Rightarrow can be expressed as a certain multiple integral involving a product of Lerch-type zeta-functions

\Rightarrow When $\mathbf{s} = \mathbf{k} = (k_\alpha)_{\alpha \in \Delta_+}$ ($k_\alpha \in \mathbb{N}$), we have

$$S(\mathbf{k}, \Delta) = (-1)^n \left(\prod_{\alpha \in \Delta_+} \frac{(2\pi\sqrt{-1})^{k_\alpha}}{k_\alpha!} \right) B_{\mathbf{k}}(\Delta)$$

($B_{\mathbf{k}}(\Delta)$: a “root-theoretic” generalization of Bernoulli numbers)

In particular, if $k_\alpha = k_\beta$ whenever $\|\alpha\| = \|\beta\|$, then $w^{-1}\mathbf{k} = \mathbf{k}$, so

$$\begin{aligned} S(2\mathbf{k}, \Delta) &= \sum_{w \in W} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-2k_\alpha} \right) \zeta_r(w^{-1}2\mathbf{k}, \Delta) \\ &= \sum_{w \in W} \zeta_r(2\mathbf{k}, \Delta) = |W| \zeta_r(2\mathbf{k}, \Delta), \quad \text{and hence} \end{aligned}$$

$$\zeta_r(2\mathbf{k}, \Delta) = \frac{(-1)^n}{|W|} \left(\prod_{\alpha \in \Delta_+} \frac{(2\pi\sqrt{-1})^{2k_\alpha}}{(2k_\alpha)!} \right) B_{2\mathbf{k}}(\Delta)$$

$B_{\mathbf{k}}(\Delta)$ can be calculated via its generating function

$$F(\mathbf{t}, \Delta) = \sum_{\mathbf{k}} B_{\mathbf{k}}(\Delta) \prod_{\alpha \in \Delta_+} \frac{t_\alpha^{k_\alpha}}{k_\alpha!} \quad (\mathbf{t} = (t_\alpha)_{\alpha \in \Delta_+})$$

Example:

$$F(t, A_1) = \frac{t}{e^t - 1} \quad (\text{classical case})$$

$$\begin{aligned} & F((t_1, t_2, t_3), A_2) \\ &= \frac{t_1 t_2 t_3 e^{t_1+t_2} (e^{t_3-t_1-t_2} - 1)}{(e^{t_1} - 1)(e^{t_2} - 1)(e^{t_3} - 1)(t_3 - t_1 - t_2)} \end{aligned}$$

Now we consider our main target : type G_2

Type G_2 :

$$\Psi = \{\alpha_1, \alpha_2\}$$

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$$

$$P_{++} = \{m\lambda_1 + n\lambda_2 \mid m, n \in \mathbb{N}\}$$

Therefore

$$\begin{aligned} \zeta_2(\mathbf{s}, G_2) &= \zeta_2((s_1, s_2, s_3, s_4, s_5, s_6), G_2) \\ &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^{-s_1} n^{-s_2} (m+n)^{-s_3} (m+2n)^{-s_4} \\ &\quad \times (m+3n)^{-s_5} (2m+3n)^{-s_6} \end{aligned}$$

Now compute the generating function $F((t_1, t_2, t_3, t_4, t_5, t_6), G_2)$ of the Bernoulli numbers $B_{\mathbf{k}}(G_2)$:

$$\begin{aligned}
F((t_1, t_2, t_3, t_4, t_5, t_6), G_2) &= t_1 t_2 t_3 t_4 t_5 t_6 \\
&\times \left(\frac{1}{(e^{t_1} - 1)(e^{t_2} - 1)} (t_1 + t_2 - t_3)^{-1} (t_1 + 2t_2 - t_4)^{-1} \right. \\
&\quad \left. \times (t_1 + 3t_2 - t_5)^{-1} (2t_1 + 3t_2 - t_6)^{-1} \right. \\
&+ \frac{1}{(e^{t_1} - 1)(e^{t_3} - 1)} (t_1 + t_2 - t_3)^{-1} (t_1 - 2t_3 + t_4)^{-1} \\
&\quad \left. \times (2t_1 - 3t_3 + t_5)^{-1} (t_1 - 3t_3 + t_6)^{-1} \right. \\
&+ \frac{e^{t_1/2+t_4/2} + 1}{2(e^{t_1} - 1)(e^{t_4} - 1)} (t_1/2 + t_2 - t_4/2)^{-1} (t_1/2 - t_3 + t_4/2)^{-1} \\
&\quad \left. \times (t_1/2 - 3t_4/2 + t_5)^{-1} (t_1/2 + 3t_4/2 - t_6)^{-1} \right. \\
&- \frac{e^{2t_1/3+t_5/3} + e^{t_1/3+2t_5/3} + 1}{3(e^{t_1} - 1)(e^{t_5} - 1)} (t_1/3 - t_4 + 2t_5/3)^{-1} \\
&\quad \left. \times (t_1/3 + t_2 - t_5/3)^{-1} (2t_1/3 - t_3 + t_5/3)^{-1} (t_1 + t_5 - t_6)^{-1} \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{e^{t_1/3+t_6/3} + e^{2t_1/3+2t_6/3} + 1}{3(e^{t_1} - 1)(e^{t_6} - 1)} (t_1 + t_5 - t_6)^{-1} \\
& \quad \times (t_1/3 + t_4 - 2t_6/3)^{-1} (2t_1/3 + t_2 - t_6/3)^{-1} (t_1/3 - t_3 + t_6/3)^{-1} \\
& - \frac{e^{t_2}}{(e^{t_2} - 1)(e^{t_3} - 1)} (t_1 + t_2 - t_3)^{-1} (t_2 + t_3 - t_4)^{-1} \\
& \quad \times (2t_2 + t_3 - t_5)^{-1} (t_2 + 2t_3 - t_6)^{-1} \\
& - \frac{e^{t_2}}{(e^{t_2} - 1)(e^{t_4} - 1)} (t_1 + 2t_2 - t_4)^{-1} (t_2 + t_3 - t_4)^{-1} \\
& \quad \times (t_2 + t_4 - t_5)^{-1} (t_2 - 2t_4 + t_6)^{-1} \\
& + \frac{e^{t_2}}{(e^{t_2} - 1)(e^{t_5} - 1)} (t_1 + 3t_2 - t_5)^{-1} (2t_2 + t_3 - t_5)^{-1} \\
& \quad \times (t_2 + t_4 - t_5)^{-1} (3t_2 - 2t_5 + t_6)^{-1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{e^{t_2/2+t_6/2} + e^{t_2}}{2(e^{t_2} - 1)(e^{t_6} - 1)} (t_1 + 3t_2/2 - t_6/2)^{-1} \\
& \quad \times (t_2/2 + t_3 - t_6/2)^{-1} (t_2/2 - t_4 + t_6/2)^{-1} (3t_2/2 - t_5 + t_6/2)^{-1} \\
& - \frac{e^{t_4}}{(e^{t_3} - 1)(e^{t_4} - 1)} (t_2 + t_3 - t_4)^{-1} (t_1 - 2t_3 + t_4)^{-1} \\
& \quad \times (t_3 - 2t_4 + t_5)^{-1} (t_3 + t_4 - t_6)^{-1} \\
& + \frac{e^{t_5}}{2(e^{t_3} - 1)(e^{t_5} - 1)} (t_2 + t_3/2 - t_5/2)^{-1} (t_3/2 - t_4 + t_5/2)^{-1} \\
& \quad \times (t_1 - 3t_3/2 + t_5/2)^{-1} (3t_3/2 + t_5/2 - t_6)^{-1} \\
& + \frac{e^{t_6}}{(e^{t_3} - 1)(e^{t_6} - 1)} (3t_3 + t_5 - 2t_6)^{-1} (t_2 + 2t_3 - t_6)^{-1} \\
& \quad \times (t_3 + t_4 - t_6)^{-1} (t_1 - 3t_3 + t_6)^{-1}
\end{aligned}$$

$$\begin{aligned}
& - \frac{e^{t_5}}{(e^{t_4} - 1)(e^{t_5} - 1)} (t_2 + t_4 - t_5)^{-1} (t_3 - 2t_4 + t_5)^{-1} \\
& \quad \times (t_1 - 3t_4 + 2t_5)^{-1} (3t_4 - t_5 - t_6)^{-1} \\
& - \frac{e^{t_4}}{(e^{t_4} - 1)(e^{t_6} - 1)} (t_1 + 3t_4 - 2t_6)^{-1} (t_3 + t_4 - t_6)^{-1} \\
& \quad \times (t_2 - 2t_4 + t_6)^{-1} (3t_4 - t_5 - t_6)^{-1} \\
& + \frac{e^{t_5} + e^{2t_5/3+2t_6/3} + e^{t_5/3+t_6/3}}{3(e^{t_5} - 1)(e^{t_6} - 1)} (t_1 + t_5 - t_6)^{-1} \\
& \quad \times (t_3 + t_5/3 - 2t_6/3)^{-1} (t_4 - t_5/3 - t_6/3)^{-1} (t_2 - 2t_5/3 + t_6/3)^{-1}
\end{aligned}$$

From this explicit form of the generating function, we can calculate the special values of $\zeta_2(\mathbf{s}, G_2)$

$$\zeta_2((2, 2, 2, 2, 2, 2), G_2) = \frac{23}{297904566960} \pi^{12}$$

$$\zeta_2((4, 4, 4, 4, 4, 4), G_2) = \frac{8165653}{1445838676129559305994400000} \pi^{24}$$

$$\begin{aligned} & \zeta_2((6, 6, 6, 6, 6, 6), G_2) \\ &= \frac{55940539974690617}{131888156302530666544150214880458495963616000000} \pi^{36} \end{aligned}$$

Also (since our condition is $k_\alpha = k_\beta$ for $\|\alpha\| = \|\beta\|$) we have

$$\zeta_2((2, 4, 4, 4, 2, 2), G_2) = \frac{467}{213955059990672000} \pi^{18}$$

How to evaluate the values at “odd” integer points?

Recall :

$$S(\mathbf{k}, G_2) = \sum_{w \in W} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-k_\alpha} \right) \zeta_r(w^{-1}\mathbf{k}, G_2)$$

When k_α is odd, the signature part appears.

For G_2 , it is well-known that $|W| = 12$, and it is easy to see that $|\Delta_+ \cap w\Delta_-|$ is odd for 6 elements of W and is even for the other 6 elements of W .

Therefore, when $\mathbf{k} = (k, k, k, k, k, k)$ for some odd positive integer k , all terms on the right-hand side of the above equation are cancelled, and we obtain NO information.

However, we can show, for example, the following “functional relation” :

$$\begin{aligned}
& \zeta_2((2, s, 1, 1, 1, 1), G_2) + \zeta_2((2, 1, s, 1, 1, 1), G_2) \\
& - \zeta_2((1, 1, 1, s, 2, 1), G_2) + \zeta_2((1, 1, 1, s, 1, 2), G_2) \\
& - \zeta_2((1, 1, s, 1, 1, 2), G_2) + \zeta_2((1, s, 1, 1, 2, 1), G_2) \\
& = \frac{1}{9}\zeta(2)\zeta(s+4) - \frac{109}{648}\zeta(s+6)
\end{aligned}$$

Putting $s = 1$, we obtain

$$\zeta_2((2, 1, 1, 1, 1, 1), G_2) = \frac{1}{18}\zeta(2)\zeta(5) - \frac{109}{1296}\zeta(7)$$

Note : $2 + 1 + 1 + 1 + 1 + 1$ is odd

Similarly,

$$\zeta_2((2, 1, 1, 1, 2, 2), G_2) = -\frac{187}{324}\zeta(2)\zeta(7) + \frac{11149}{11664}\zeta(9)$$

$$\zeta_2((4, 2, 2, 1, 1, 1), G_2) = \frac{1}{18}\zeta(4)\zeta(7) + \frac{595}{648}\zeta(2)\zeta(9) - \frac{73201}{46656}\zeta(11)$$

$$\zeta_2((2, 4, 4, 3, 3, 3), G_2) = \frac{1}{8}\zeta(4)\zeta(15) + \frac{281221}{23328}\zeta(2)\zeta(17) - \frac{11177971}{559872}\zeta(19)$$

Note : $2 + 1 + 1 + 1 + 2 + 2$, $4 + 2 + 2 + 1 + 1 + 1$, $2 + 4 + 4 + 3 + 3 + 3$

are all odd

Underlying reason :

$$I \subset \{1, 2, \dots, r\}, \quad I \neq \emptyset$$

$$\Psi_I = \{\alpha_i \mid i \in I\} \subset \Psi$$

V_I : the linear space spanned by Ψ_I

$$\Delta_I = \Delta \cap V_I$$

$$W^I = \{w \in W \mid \Delta_{I+}^\vee \subset \Delta_+^\vee\}$$

and define

$$S(\mathbf{s}, I, \Delta) = \sum_{w \in W^I} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}\mathbf{s}, \Delta)$$

Then we can show the following theorem:

Theorem : If there exist $w_1 \in W$ and \mathbf{s} satisfying

- $s_\alpha \in \mathbb{Z}$ for $\alpha \in \Delta_+ \cap w_1\Delta_-$,
- $w_1^{-1}\mathbf{s} = \mathbf{s}$,
- $w_1^{-1}W^I = w_1W^I$,

and moreover

$$\left(\prod_{\alpha \in \Delta_+ \cap w_1\Delta_-} (-1)^{s_\alpha} \right) = -1,$$

then we have

$$S(\mathbf{s}, I, \Delta) = \sum_{w \in W^I \setminus w_1W^I} \left(\prod_{\alpha \in \Delta_+ \cap w\Delta_-} (-1)^{-s_\alpha} \right) \zeta_r(w^{-1}\mathbf{s}, \Delta)$$

In the case of G_2 , we can apply this theorem with

$$I = \{2\}, w_1 = -\sigma_{\alpha_1}, \text{ and}$$

$\mathbf{s} = (s_1, s_2, s_2, s_4, s_5, s_5)$ with odd s_4 (necessary for $w_1\mathbf{s} = \mathbf{s}$ and the signature condition)

$$\Rightarrow S(\mathbf{s}, \{2\}, G_2) = \zeta_2(\mathbf{s}, G_2) + (-1)^{s_1}\zeta_2(\mathbf{s}, G_2)$$

$$\Rightarrow \zeta_2(\mathbf{s}, G_2) = (1/2)S(\mathbf{s}, \{2\}, G_2)$$

for $\mathbf{s} = (s_1, s_2, s_2, s_4, s_5, s_5)$ with even s_1 and odd s_4

(therefore $s_1 + s_2 + s_2 + s_4 + s_5 + s_5$ is odd)

Parity result : Some multiple zeta value whose weight and depth are of different parity can be written in terms of multiple zeta values of lower depth

- Euler proved that $\sum_{m \geq 1} \sum_{n \geq 1} m^{-p} (m+n)^{-q}$ can be written as a linear combination of $\zeta(j)$ ($j \geq 2$) with rational coefficients, when $p+q$ is odd

- Similar result for zeta values of type A_2 (Tornheim)
- Similar result for zeta values of type B_2 (Tsumura)

How is the case of G_2 ?

\Rightarrow It seems that we need some modification, because

$$\zeta_2((1, 2, 1, 1, 1, 1), G_2) = \frac{1}{2}\zeta(2)\zeta(5) - \frac{109}{16}\zeta(7) + \frac{81}{8}L(1, \chi_3)L(6, \chi_3)$$

(χ_3 : the primitive Dirichlet character mod 3)

Problem 1 : Can the values $\zeta_2((k_1, k_2, k_3, k_4, k_5, k_6), G_2)$ be expressed in terms of $\zeta(j)$ and $L(j, \chi_3)$ when $k_1 + k_2 + k_3 + k_4 + k_5 + k_6$ is odd?

cf. It is known that they are expressed in terms of double polylogarithms (Zhao, 2010). Okamoto (2012) proved that they can be expressed by $\zeta(j)$, $L(j, \chi_3)$, $S_r(j/l)$ and $C_r(j/l)$ with $l = 4, 12$ and $0 < j < l$, $(j, l) = 1$, where $S_r(x) = \sum_{m \geq 1} \sin(2\pi mx) m^{-r}$ and $C_r(x) = \sum_{m \geq 1} \cos(2\pi mx) m^{-r}$ (Clausen functions)

Problem 2 : When the values $\zeta_2((k_1, k_2, k_3, k_4, k_5, k_6), G_2)$ be expressed only in terms of $\zeta(j)$?