Wellhausen's conjecture - what is it trying to tell us?

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16th April, 2013

Notations:

k - global field containing the n^{th} roots of 1

S - finite set of places including those with $|n|_v \neq 1$

R - ring of S-integers, assumed to be a principal ideal domain

$k_S = \prod_{v \in S} k_v$

$e : k_S \rightarrow \boldsymbol{C}^x$ - non-trivial additive character trivial on R

$\mu_n(k)$ - group of n^{th} roots of 1 in k

$\epsilon : \mu_n(k) \rightarrow \boldsymbol{C}^x$ - an injection

$(-)$ - n^{th} order Jacobi-Legendre symbol

$$g_S(r,\epsilon,c)=\sum_{x \bmod c} \epsilon\left(\frac{x}{c}\right) e(rx/c)$$ - the standard Gauss sum

$$\psi_S(r,\epsilon,\eta,s)=\zeta_S(ns-n+1)\sum_{c\sim\eta} g_S(r,\epsilon,c)\,N(c)^{-s}$$

$a\sim b$ means that $a/b \in k_S^{\times n}$

$$\Psi_S(r,\epsilon,\eta,s)=\prod_{1\leqslant j<n} G(s-j/n)^N\,\psi_S(r,\epsilon,\eta,s)$$

- where $G(s)=(2\pi)^{-s}\,\Gamma(s)$ and $N=[k:\mathbb{Q}]/2$

$$\rho_S(r,\epsilon,\eta)=Res_{s=1+1/n}\Psi_S(r,\epsilon,\eta,s)$$

- one should really regard r and η as components of an idele, then one can remove the dependence on S.

If r_1 and r_2 are such that r_1/r_2 is an n^{th} power in k then

$$\rho_S(r_{1,}\epsilon,\eta)=\rho_S(r_2,\epsilon,\eta)$$

(periodicity theorem). From this follows if π is a prime in R and if r_o is coprime to π then for $0\leq j\leq n\text{-}2$

$$\rho_s(r_o\pi^j,\epsilon,\eta)=N(\pi)^{-(j+1)/n}g_S(r_o,\epsilon^{j+1},\pi)\epsilon(\pi,-\eta)^{j+1}\rho_s(r_o\pi^{n-j-2},\epsilon,\eta\pi^{-j-1})$$

and

$$\rho_s(r_o\pi^{n-1},\epsilon,\eta)=0.$$

There are also two elementary properties that we shall need. Let u be a unit of R. Then we have:

$$\rho_S(ru,\epsilon,\eta)=\epsilon(u,\eta)^{-1}\rho_S(r,\epsilon,\eta)$$

and

$$\rho_S(r,\epsilon,\eta u)=\epsilon(u,\eta)\rho_S(r,\epsilon,\eta)$$

In particular, if R is a principal ideal domain then, to determine the ρ's we need to determine the

$$\rho_S(r_1 r_2^2 \cdots r_{n-2}^{n-2}, \epsilon, \eta)$$

where the $r_1,\ldots,r_{n-2}$ are square-free and mutually coprime. In principle we could also assume $r_j=1$ for $j>(n-2)/2$. The dependence on r_{n-2} follows from the general theory.

For what comes next we need some further notations. We assume k is totally imaginary and let S_∞ be the set of embeddings of k into $\mathbb{C}$ and we shall consider the set $S_{\infty+}$ of non-negative integral combinations

$$\sigma = \sum_{i \in S_\infty} \sigma(i)\, i$$

Let $Z_k(s) = G(s)^{N/2}\zeta_k(s)$ and let T_k be the residue of this function at $s=1$.

Gunther Wellhausen (1996) investigated the case n=6 and made a number of conjectures. There are some general ideas behind this which can be extended to a more general case – where they have been subject to hardly any tests. The following version is a little more explicit that those I have essayed in the past but the underlying motivation is the same.

First assertion: There is algebraic number field of finite degree which contains all the

$$(\rho_S(r, \epsilon, \eta)/T)^n$$

This is surprising in itself.

Second assertion: There are $\sigma_1,\ldots,\sigma_{n-2} \in S_{\infty+}$ so that

$$(\rho_S(r_1 r_2^2 \cdots r_{n-2}^{n-2}, \epsilon, \eta)/T)^n \prod (r_j)^{\sigma_j}$$

has a bounded denominator and a numerator that is is, for any $\varepsilon>0$,

$$O(N(r_1 r_2^2 \cdots r_{n-2}^{n-2}))^{\epsilon}$$

This is also very unexpected. Note that the σ_j-σ_{n-2-j} are given by Stickelberger's theorem.

Suppose that $k=\mathbb{Q}(\mu_n)$. Then one has

$$\sigma_{n-2}=\sum_{j:gcd(j,n)=1}(n-1-\bar{j})(\epsilon^j)$$

Here $\bar{j}$ is the inverse (mod n) of j. Also

(ϵ^j) is the injection determined by $(\epsilon^j)|\mu_n(k)=\epsilon^j$

For the quadratic cases n=3,4 or 6 (the case n=2 is uninteresting) come down to

$$\sigma_{n-2}=(n-2)(\epsilon)$$

Conjecturally for n=4 (Eckhardt-Patterson) and n=6 (Wellhausen) one has generally:

$$\sigma_j=j(\epsilon)$$

These formulæ look simple but they are restricted to these very simple cases. The next complicated cases would be the quartic cyclic case $n=5$ and the biquadratic cases $n=8$ and $n=12$. In these cases one needs something more complicated.

The reciprocity formulæ for the suggest certain forms for the σ_j. One takes the numerator and denominator of

$$N(\pi)^{-(j+1)/n} g_S(r_o, \epsilon^{j+1}, \pi).$$

There is no evidence at all about these cases and it would be very rash to jump to any conclusions.

Apropos tests – the most effective algorithm for computing the ρ's is that developed by Carsten Eckhardt which uses the Mellin transform of the functional equations of the Ψ's. The optimal function decays increasingly slowly, so roughly like:

$$m \rightarrow \exp(-2\pi m^{a})$$

where $a = 2/((n-1)\phi(n))$ which is very small.

Added to this is the large number of η's needed, namely

$$n^{Card(S)}$$

If $n=6$ then S has the infinite place and the divisors of 2 and 3. We therefore have $6^3=216$ η's. Here $a=1/5$. For $n=5$ we have $5^3=125$ η's but $a=1/8$. This just might be feasible.

What is wrong with this conjecture?

The point that causes the most trouble is the the numerator and the estimate given for it:

$$O(N(r_1 r_2^2 \cdots r_{n-2}^{n-2}))^{\epsilon}.$$

This is vague and as such is difficult to test. What one would like would be that this is replaced by a divisor sum. This is what happens when $n=4$ and the numerator is just a power of 2. The experimental evidence leads to no direct conjectures. One also does not know a priori if one should be working with some linear combinations of the ρ's. There is another line of thought which looks as if it might help and which I shall describe anon; unfortunately it also leads nowhere.

Let Φ be an irreducible reduced root system, i.e. on of the standard types. Then one can associate with this a generalized Gauss sum $g_{\Phi,S}(\mathbf{r},\varepsilon,\mathbf{c})$. The theory is not yet complete but one can say a lot about these functions. One could extend our conjecture to the multi-dimensional residues of the associated WMDs. There is, regrettably, no way of testing these ideas at the moment, even in simple rank 2 cases with small values of n.

Even if we understood these functions there seems to be no direct method of using them to understand the original ρ's.   One now knows enough about them that one can estimate what one might be able to use them for.   One thing that is missing is an arithmetic interpretation of the $g_{\Phi,S}(\mathbf{r},\varepsilon,\mathbf{c})$ which would provide a motivation for analytic investigations.

A comparison of the WMDs with the earlier calculations of Proskurin (and Kataev) suggests that there might be a formula for certain forms of degree n-2 in the $\rho_S(r,\varepsilon,\eta)$ (as functions of η). This would then be something like

$$\omega(r)\sum_{d_1 d_2 \cdots d_{n-2}=r} g(1,\bar{\epsilon},d_1)\cdots g(1,\bar{\epsilon},d_{n-2})$$

Here ω would be a simple function, something like $\omega(r_1 r_2^2 \cdots r_{n-2}^{n-2}) = \omega_1(r_1)\cdots\omega_{n-2}(r_{n-2})$ with $\omega_1, \ldots, \omega_{n-2}$ Größencharaktere.

Unfortunately all the variants I have looked at turn out to be inconsistent with the general properties of the ρ – except when $n=4$ (Eckhardt-Patterson conjecture).

I see no way of rescuing something out of all this.

This is not to say that it cannot be done, only that my imagination is not up to finding something that is at least consistent with what is known.

One blemish in the conjecture is that it does not admit a natural analogue in the function field case. There are algorithms for computing the ρ in these cases (Jeff Hoffstein, sjp) but the complexity grows rapidly with n and the ground field. It should be possible to make tables of the ρ and to interpret them. Even for rational function fields the results should be illuminating. (I began this project about 8 years ago but got diverted by other matters – it is now again high on the agenda.)

What we learn from Wellhausen's conjecture is that the important feature is the denominator. We would expect the numerator not to grow rapidly. There are cases which should be feasible, for example $k=\boldsymbol{F}_p((t))$ with $n=6$ and $p=7$ or $n=5$ and $p=11$.

There is no information at all available as to how the geometry of a non-rational curve shows up in the the behaviour of metaplectic forms or of Gauss sums. Although the computational complexity is considerable it should be possible to make progress in the experimental investigation of these objects.

Coming back to the case of characteristic zero - how good can one judge the chances of getting some sort of understanding of the ρ – even at the level of an illuminating conjecture? The version of Wellhausen's conjecture I have given here is not at the level of being what one really is hoping for.

At the moment I am pessimistic, for I do not see a good way forward. But I am not quite so pessimistic, however, that I am prepared to give up trying.