Graph cut, convex relaxation and continuous max-flow problem

Xue-Cheng Tai,
Christian Michelsen Research AS,
Bergen, Norway.
and
University of Bergen, Norway

Collaborations with:
Egil Bae, Yuri Boykov, Jun Liu, Jing Yuan and others

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Interface problems

Interface problems exist everywhere in science and technology. For imaging and vision, it is somehow classical:

- Mumford-Shal model
- GAC model
- Chan-Vese model

How to solve these interface problems?
Max-Flow / Min-Cut

Max-Flow / Min-Cut

Max-flow = min-cut.
Max-Flow / Min-Cut

$(V_s, V_t)$ is a cut, $w_{ij} =$ cost of cutting edge $(i, j)$

Cost of cut $c(V_s, V_t) = \sum_{i \in V_s, j \in V_t} w_{ij}$
Max-Flow / Min-Cut

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Min-cut: find cut of minimum cost,
Max-Flow / Min-Cut

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\[
\text{cost of cut } c(V_s, V_t) = \sum_{i \in V_s, j \in V_t} w_{ij}
\]

Min-cut: find cut of minimum cost,
Max-Flow: Find the maximum amount of flow from \(s\) to \(t\).
Max-Flow / Min-Cut

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Min-cut: find cut of minimum cost,

Max-flow: Find the maximum amount of flow from $s$ to $t$.

Max-flow = min-cut.
A simple 1d signal $I(x)$:

Graph-cut for images: Boykov-Kolmogorov (2001).
Graph-cut for image segmentation

The graph:

\[
\begin{align*}
    w_{s,p} &= |l(p) - c_1|^2, \\
    w_{t,p} &= |l(p) - c_2|^2, \\
    c_1 &= 0, \\n    c_2 &= 1.
\end{align*}
\]

More generally

\[
\begin{align*}
    w_{s,p} &= f_1(p), \\
    w_{t,p} &= f_2(p), \\
    w(p, q) &= \alpha \text{ or } g(p, q) \text{ (edge force)}. \nonumber
\end{align*}
\]
Relation with k-mean ($\alpha = 0$)

- Given $c_1$ and $c_2$.

![Diagram with nodes and connections]

- Use cut (threshold) to get $\Omega_1$ and $\Omega_2$.

- Update $c_i = \int_{\Omega_i} I(x)$, $i = 1, 2$.

- Go to the next iteration.

| 1 | 1 | 2 | 1 | 1 | 1 | 2 |
Relation with k-mean ($\alpha = 0$)

Given $c_1$ and $c_2$.

- use cut (threshold) to get $\Omega_1$ and $\Omega_2$. 

```
1 1 2 1 1 2
```
Relation with k-mean ($\alpha = 0$)

- Given $c_1$ and $c_2$.
- Use cut (threshold) to get $\Omega_1$ and $\Omega_2$.
- Update

$$c_i = \frac{\int_{\Omega_i} l(x)}{\text{Area}(\Omega_i)}, i = 1, 2.$$
Relation with k-mean ($\alpha = 0$)

- Given $c_1$ and $c_2$.
- Use cut (threshold) to get $\Omega_1$ and $\Omega_2$.
- Update
  \[ c_i = \frac{\int_{\Omega_i} I(x)}{\text{Area}(\Omega_i)}, \quad i = 1, 2. \]
- Go to the next iteration.
Regularized Graph-cut: $\alpha \neq 0$

The "virtual graph and the corresponding label function $u(p), p = 1, 2, \ldots$.

Costs:

\[ w_{s,p} = |l(p) - c_1|^2, \; w_{t,p} = |l(p) - c_2|^2, \; w_{p,q} = \alpha. \]

The corresponding minimization problem is: ($N(p)$ – neighbors of $p$)

\[
\min_{u(p) \in \{1,2\}} \sum_{p \in \Omega_1} |l(p) - c_1|^2 + \sum_{p \in \Omega_2} |l(p) - c_2|^2 + \alpha \sum_p \sum_{q \in N(p)} |u(p) - u(q)|.
\]
Discrete vs continuous

Discrete minimization:

$$\min_{u(p) \in \{0,1\}} \sum_{p \in \Omega_1} |l(p) - c_1|^2 + \sum_{p \in \Omega_2} |l(p) - c_2|^2 + \alpha \sum_{p} \sum_{q \in N(p)} |u(p) - u(q)|.$$ 

Continuous minimization:

$$\min_{u(x) \in \{0,1\}} \int_{\Omega_1} |l(x) - c_1|^2 + \int_{\Omega_2} |l(x) - c_2|^2 + \alpha \int_{\Omega} |Du|.$$ 

$$\min_{u(x) \in \{0,1\}} \int_{\Omega} |l(x) - c_1|^2 (1 - u) + \int_{\Omega} |l(x) - c_2|^2 u + \alpha \int_{\Omega} |Du|.$$
Higher dimensional problems

A graph for 2D images:

Figure: Graph used for discrete 2D binary labeling
Two-phase Min-cut – Discretized setting

\[
\min_{u \in \{0,1\}} \sum_{p \in \mathcal{P}} f_1(p)(1 - u(p)) + f_2(p)u(p) + \sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{N}^k_p} g(p, q)|u(p) - u(q)|.
\]

Costs:

\[
w_{s,p} = f_1(p), \quad w_{t,p} = f_2(p), \quad w_{p,q} = g(p, a).
\]

\(^1\mathcal{N}^k_p\) is the k-neighborhood of \(p \in \mathcal{P}\).
Max-Flow / Min-Cut (graph cut)

Max-flow formulation

\[
\begin{align*}
\max_{\mathbf{p}_s, \mathbf{p}_t, \mathbf{q}} & \quad \sum_{v \in V \setminus \{s,t\}} p_s(v) \\
\text{subject to} & \quad |q(v, u)| \leq g(v, u), \quad \forall (v, u) \in V \times V \\
& \quad 0 \leq p_s(v) \leq f_1(v), \quad \forall v \in V \setminus \{s, t\}; \\
& \quad 0 \leq p_t(v) \leq f_2(v), \quad \forall v \in V \setminus \{s, t\}; \\
& \quad \left( \sum_{u \in N(v)} \tilde{q}(v, u) \right) - p_s(v) + p_t(v) = 0, \quad \forall v \in V \setminus \{s, t\}.
\end{align*}
\]

Figure: Graph used for discrete binary labeling
Continuous max-flow formulation

$$\sup_{p_s, p_t, q} \int_{\Omega} p_s(x) \, dx$$

subject to

$$|q(x)| = |q_1(x)| + |q_2(x)| \leq g(x), \quad \forall x \in \Omega;$$

$$p_s(x) \leq f_1(x), \quad \forall x \in \Omega;$$

$$p_t(x) \leq f_2(x), \quad \forall x \in \Omega;$$

$$\text{div} \, q(x) - p_s(x) + p_t(x) = 0, \quad \text{a.e.} \, x \in \Omega.$$
Continuous max-flow formulation (G. Strang (1983))

\[
\sup_{p_s, p_t, q} \int_{\Omega} p_s(x) \, dx
\]

subject to

\[
|q(x)| = \sqrt{q_1^2(x) + q_2^2(x)} \leq g(x), \quad \forall x \in \Omega;
\]

\[
p_s(x) \leq f_1(x), \quad \forall x \in \Omega;
\]

\[
p_t(x) \leq f_2(x), \quad \forall x \in \Omega;
\]

\[
\text{div} \, q(x) - p_s(x) + p_t(x) = 0, \quad \text{a.e. } x \in \Omega.
\]
Lagrange multiplier $u$ for flow conservation condition

$$\text{div } q(x) - p_s(x) + p_t(x) = 0, \quad \text{a.e. } x \in \Omega.$$  

yields primal-dual formulation

$$\sup_{p_s, p_t, q} \inf_u \int_\Omega p_s + u(\text{div } q - p_s + p_t) \, dx$$

s.t. $p_s(x) \leq f_1(x), \quad p_t(x) \leq f_2(x), \quad |q(x)| \leq g(x).$

Optimizing for flows $p_s, p_t, q$ results in:

$$\min_{u \in [0,1]} \int_\Omega f_1(x)(1 - u(x)) + f_2(x)u(x) \, dx + g(x) |\nabla u(x)| \, dx.$$  

This is exactly the same model as in Chan et al (2006).
Three problems

\[
\min_{u(x) \in \{0,1\}} \int_{\Omega} f_1(1 - u) + f_2 u + g(x)|\nabla u| dx.
\]
Three problems

\[
\min_{u(x) \in \{0, 1\}} \int_{\Omega} f_1(1 - u) + f_2 u + g(x) |\nabla u| \, dx.
\]

\[
\min_{u(x) \in [0, 1]} \int_{\Omega} f_1 u + f_2 (1 - u) + g(x) |\nabla u| \, dx.
\]
Three problems

\[
\begin{align*}
\min_{u(x) \in \{0, 1\}} \int_{\Omega} f_1(1 - u) + f_2 u + g(x) |\nabla u| dx. \\
\min_{u(x) \in [0, 1]} \int_{\Omega} f_1 u + f_2 (1 - u) + g(x) |\nabla u| dx.
\end{align*}
\]

\[
\begin{align*}
\max_{p_s, p_t, q} \int_{\Omega} p_s dx \text{ subject to:} \\
p_s(x) \leq f_1(x), \quad p_t(x) \leq f_2(x), \quad |p(x)| \leq g(x), \\
\text{div} p(x) - p_s(x) + p_t(x) = 0.
\end{align*}
\]
Three problems

\[ \min_{u(x) \in \{0,1\}} \int_{\Omega} f_1(1 - u) + f_2 u + g(x)|\nabla u| dx. \]

\[ \min_{u(x) \in [0,1]} \int_{\Omega} f_1 u + f_2(1 - u) + g(x)|\nabla u| dx. \]

\[ \max_{p_s,p_t,q} \int_{\Omega} p_s dx \text{ subject to:} \]
\[ p_s(x) \leq f_1(x), \quad p_t(x) \leq f_2(x), \quad |p(x)| \leq g(x), \]
\[ \text{div} p(x) - p_s(x) + p_t(x) = 0. \]
Three problems

PCLMS or Binary LM (Lie-Lysaker-T.,2005):

\[ \min_{u(x) \in \{0,1\}} \int_{\Omega} f_1(1 - u) + f_2 u + g(x) |\nabla u| \, dx. \]

Convex problem (CEN, (Chan-Esdoiglu-Nikolova,2006))

\[ \min_{u(x) \in [0,1]} \int_{\Omega} f_1(1 - u) + f_2 u + g(x) |\nabla u| \, dx. \]

Graph-cut (Boykov-Kolmogorov,2001)

\[ \max_{p_s,p_t,q} \int_{\Omega} p_s \, dx \text{ subject to:} \]
\[ p_s(x) \leq f_1(x), \ p_t(x) \leq f_2(x), \ |p(x)| \leq g(x), \]
\[ \text{div} \ p(x) - p_s(x) + p_t(x) = 0. \]
The following approached are solving the same problem, but did not know each other:

- max-flow and min-cut.
- CEN 2006 (convex relaxation approach)
- Binary Level set methods and PCLSM (piecewise constant level set method)
- A cut is nothing else, but the Lagrangian multiplier for the flow conservation constraint!!!
Continuous Max-Flow: Remarks

- Min-cut problem is minimizing an energy functional. Not using the decent (gradient) info of the energy.
- Continuous max-flow/min-cut is a convex minimization problem. A lot of choices, can use decent (gradient) info.
Continuous Max-Flow: How to solve it (Only 2-phase case)?

- Min-cut algorithms: Augmented Path. Push-relabel, etc,
- Split-Bregman, Augmented Lagrangian, Primal-Dual approaches: we can use these approach to solve the convex min-cut problem.
Continuous Max-Flow and Min-Cut

Multiplier-Based Maximal-Flow Algorithm
Augmented lagrangian functional (Glowinski & Le Tallec, 1989)

\[ L_c(p_s, p_t, q, \lambda) := \int_\Omega p_s \, dx + \lambda \left( \text{div} \, q - p_s + p_t \right) - \frac{c}{2} \left| \text{div} \, q - p_s + p_t \right|^2 \, dx. \]

minmax subject to:
\[ p_s(x) \leq f_1(x), \quad p_t(x) \leq f_2(x), \quad |q(x)| \leq g(x) \]

ADMM algorithm: For \( k=1, \ldots \) until convergence, solve

\[ q^{k+1} := \arg \max_{\|q\|_\infty \leq \alpha} L_c(p_s^k, p_t^k, q, \lambda^k) \]
\[ p_s^{k+1} := \arg \max_{p_s(x) \leq f_1(x)} L_c(p_s, p_t^k, q^{k+1}, \lambda^k) \]
\[ p_t^{k+1} := \arg \max_{p_t(x) \leq f_2(x)} L_c(p_s^{k+1}, p_t, q^{k+1}, \lambda^k) \]

\[ \lambda^{k+1} = \lambda^k - c \left( \text{div} \, q^{k+1} - p_s^{k+1} + p_t^{k+1} \right) \]
Continuous Max-Flow and Min-Cut

Other algorithms for solving relaxed problem: $p = \nabla u$

- Bresson et. al.
  - fix $\mu^k$ and solve ROF problem
    
    $$\lambda^{k+1} := \arg \min_\lambda \left\{ \alpha \int_\Omega |\nabla \lambda(x)| \, dx + \frac{1}{2\theta} \|\lambda(x) - \mu^k(x)\|^2 \right\}$$

- fix $\lambda^{k+1}$ and solve

  $$\mu^{k+1} := \arg \min_{\mu \in [0,1]} \left\{ \frac{1}{2\theta} \|\mu(x)-\lambda^{k+1}\|^2 + \int_\Omega \mu(x)(f_1(x)-f_2(x)) \, dx \right\}$$

- Goldstein-Osher: Split Bregman / augmented lagrangian
Convergence

Figure: Red line: max-flow algorithm. Blue line: Splitting algorithm (Bresson et al. 2007)
Multiphase problems

Multiphase problem
Related to graph cut, $\alpha$-expansion and $\alpha - \beta$ swap are mostly popular.

Approximations are made and upper bounded has been given.

Each point $x \in \Omega$ is labelled by

$$u(x) = i, \quad i = 1, 2, \ldots n.$$  

- One label function is enough for any $n$ phases.
- More generally

$$u(x) = \ell_i, \quad i = 1, 2, \ldots n.$$
Each point $x \in \Omega$ is labelled by a vector function:

$$u(x) = (u_1(x), u_2(x), \ldots u_d(x)).$$
Multiphase problems – Approach II

Each point \( x \in \Omega \) is labelled by a vector function:

\[
u(x) = (u_1(x), u_2(x), \ldots, u_d(x)).\]

- **Multiphase**: Total number of phases \( n = 2^d \). (Chan-Vese)

  \[
  u_i(x) \in \{0, 1\}. 
  \]
Each point $x \in \Omega$ is labelled by a vector function:

$$u(x) = (u_1(2), u_2(x), \ldots u_d(x)).$$

- **Multiphase**: Total number of phases $n = 2^d$. (Chan-Vese)

  - $u_i(x) \in \{0, 1\}$.  

- **More than binary labels**: Total number of phases $n = B^d$.

  - $u_i(x) \in \{0, 1, 2, \ldots B\}$.  

We need to identify $n$ characteristic functions $\psi_i(x)$, $i = 1, 2 \cdots n$:

$$\psi_i(x) \in \{0, 1\}, \quad \sum_{i=1}^{n} \psi_i(x) = 1.$$ 

Relation between Approach I and III:

$$u(x) = i, \quad i = 1, 2, \cdots n.$$ 

$$u(x) = \sum_{i=1}^{n} i \psi_i(x).$$
Multiphase problems

Multiphase problem (I)

Special graph cut for Chan-Vese approach
Associate two vertices to each grid point \((v_{p,1} \text{ and } v_{p,2})\).

For any cut \((V_s, V_t)\):
- If \(v_{p,i} \in V_s\) then \(\phi^i = 1\) for \(i = 1, 2\).
- If \(v_{p,i} \in V_t\) then \(\phi^i = 0\) for \(i = 1, 2\).

Figure left: graph corresponding to one grid point \(p\).

Figure right: graph corresponding to two grid points \(p\) and \(q\):
- **Red**: Data edges, constituting \(E_{data}(\phi_1, \phi_2)\).
- **Blue**: Regularization edges with weight \(w_{pq}\).
Cuts for the CV-graph (Bae-Tai, EMMCVPR 2009)
Minimization by graph cut

Graph construction

Linear system for finding edge weights

\[
\begin{align*}
A(p) + B(p) &= |c_2 - u_0^p|_\beta \\
C(p) + D(p) &= |c_3 - u_0^p|_\beta \\
A(p) + E(p) + D(p) &= |c_1 - u_0^p|_\beta \\
B(p) + F(p) + C(p) &= |c_4 - u_0^p|_\beta
\end{align*}
\]

such that $E(p), F(p) \geq 0$

For each $p$, $E_p^{\text{data}}(\phi^1_p, \phi^2_p)$ interaction between two binary variables. Linear system has solution iff $E_p^{\text{data}}(\phi^1_p, \phi^2_p)$ is submodular.
Restriction $E(p), F(p) \geq 0$ implies

$$|c_1 - u_p^0|^\beta + |c_4 - u_p^0|^\beta = A(p) + B(p) + C(p) + D(p) + E(p) + F(p) \geq A(p) + B(p) + C(p) + D(p) = |c_2 - u_p^0|^\beta + |c_3 - u_p^0|^\beta.$$ 

Therefore it is sufficient that

$$|c_2 - l|^\beta + |c_3 - l|^\beta \leq |c_1 - l|^\beta + |c_4 - l|^\beta, \quad \forall l \in [0, L],$$

At most three edges are required for a general submodular function of two binary variables (Kolmogorov et. al.)
Global minimizer – Guarantees

Submodularity condition

\[ |c_2 - I|^\beta + |c_3 - I|^\beta \leq |c_1 - I|^\beta + |c_4 - I|^\beta, \quad \forall I \in [0, L], \]

Proposition 1: Let \(0 \leq c_1 < c_2 < c_3 < c_4\). Condition is satisfied for all \(I \in \left[\frac{c_2 - c_1}{2}, \frac{c_4 - c_3}{2}\right]\).

Proposition 2: Let \(0 \leq c_1 < c_2 < c_3 < c_4\). There exists a \(B \in \mathbb{N}\) such that condition is satisfied for any \(\beta \geq B\).
There are infinite many solution for $A, B, C, D, E, F$ for each pixel.

We can guarantee $A > 0, B > 0, C > 0, D > 0$. If one of $E, F$ is negative, there is a modified graph.

Some arts: sort $c_i$ as $c_1 < c_2 < c_3 < c_4$, then choose

$$f_1(p) = |c_2 - u_p^0|^\beta, \quad f_2(p) = |c_3 - u_p^0|^\beta,$$

$$f_3(p) = |c_1 - u_p^0|^\beta, \quad f_4(p) = |c_4 - u_p^0|^\beta.$$
Experiment 1

Figure: Experiment 3: (a) Input image, (b) ground truth, (c) gradient descent, (d) our approach, (e) alpha expansion, (f) alpha-beta swap.
Numerical experiments

Experiment 2

Figure: Experiment 3: (a) Input image, (b) ground truth, (c) gradient descent, (d) our approach, (e) alpha expansion, (f) alpha-beta swap.
Numerical experiments

Experiment 3

- $L^2$ data term ($\beta = 2$)
- Right: Input image.
- Left: Output.
Experiment 4, non-submodular minimization

- $L^1$ data term ($\beta = 1$)
- Right: Input image.
- Left: Set of pixels where residual criterion was not satisfied (empty set).
Experiment 4, non-submodular minimization

- \( L^1 \) data term (\( \beta = 1 \))
- Right: Input image.
- Left: Output (global solution).
Exact convex formulation for the Multiphase Chan-Vese model by Continuous max-flow/min-cuts
Multiphase level set representation of CV model

\[
\min_{\phi^1, \phi^2, \{c_i\}_{i=1}^4} \alpha \int_{\Omega} |\nabla H(\phi^1)| + \alpha \int_{\Omega} |\nabla H(\phi^2)| + E^{data}(\phi^1, \phi^2),
\]

where

\[
E^{data}(\phi^1, \phi^2) = \int_{\Omega} \{H(\phi^1)H(\phi^2)|c_2-u^0|^{\beta} + H(\phi^1)(1-H(\phi^2))|c_1-u^0|^{\beta}
\]
\[
+(1-H(\phi^1))H(\phi^2)|c_4-u^0|^{\beta} + (1-H(\phi^1))(1-H(\phi^2))|c_3-u^0|^{\beta}\} \, dx.
\]

\[
\begin{align*}
\Omega_1 &= \{x \in \Omega \text{ s.t. } \phi^1(x) > 0, \phi^2(x) < 0\} \\
\Omega_2 &= \{x \in \Omega \text{ s.t. } \phi^1(x) > 0, \phi^2(x) > 0\} \\
\Omega_3 &= \{x \in \Omega \text{ s.t. } \phi^1(x) < 0, \phi^2(x) < 0\} \\
\Omega_4 &= \{x \in \Omega \text{ s.t. } \phi^1(x) < 0, \phi^2(x) > 0\}
\end{align*}
\]
Binary formulation of multiphase Chan-Vese model

Wish to obtain global optimization framework for

$$\min_{\phi^1, \phi^2 \in \{0,1\}} \alpha \int_{\Omega} |\nabla \phi^1| dx + \alpha \int_{\Omega} |\nabla \phi^2| dx + E^{data}(\phi^1, \phi^2),$$

with

$$E^{data}(\phi^1, \phi^2) = \int_{\Omega} \left\{ \phi^1 \phi^2 |c_2 - u^0|^\beta + \phi^1 (1 - \phi^2) |c_1 - u^0|^\beta \\
+(1 - \phi^1) \phi^2 |c_4 - u^0|^\beta + (1 - \phi^1)(1 - \phi^2) |c_3 - u^0|^\beta \right\} dx.$$ 

Phase 1: $\phi^1 = 1, \phi^2 = 0$  Phase 2: $\phi^1 = 1, \phi^2 = 1$
Phase 3: $\phi^1 = 0, \phi^2 = 0$  Phase 4: $\phi^1 = 0, \phi^2 = 1$
Binary formulation of multiphase Chan-Vese model

Wish to obtain global optimization framework for

\[
\min_{\phi^1, \phi^2 \in \{0, 1\}} \alpha \int_{\Omega} |\nabla \phi^1| \, dx + \alpha \int_{\Omega} |\nabla \phi^2| \, dx + E^{data}(\phi^1, \phi^2),
\]

with

\[
E^{data}(\phi^1, \phi^2) = \int_{\Omega} \{\phi^1 \phi^2 |c_2 - u^0|^\beta + \phi^1 (1 - \phi^2) |c_1 - u^0|^\beta \\
+ (1 - \phi^1) \phi^2 |c_4 - u^0|^\beta + (1 - \phi^1)(1 - \phi^2) |c_3 - u^0|^\beta \} \, dx.
\]

Phase 1: \( \phi^1 = 1, \phi^2 = 0 \)  
Phase 2: \( \phi^1 = 1, \phi^2 = 1 \)  
Phase 3: \( \phi^1 = 0, \phi^2 = 0 \)  
Phase 4: \( \phi^1 = 0, \phi^2 = 1 \)

Can this non-convex problem be equivalent to a convex model???
Binary formulation of multiphase Chan-Vese model

Wish to obtain global optimization framework for

\[
\min_{\phi^1, \phi^2 \in \{0, 1\}} \alpha \int_\Omega |\nabla \phi^1| \, dx + \alpha \int_\Omega |\nabla \phi^2| \, dx + E^{\text{data}}(\phi^1, \phi^2),
\]

with

\[
E^{\text{data}}(\phi^1, \phi^2) = \int_\Omega \{\phi^1 \phi^2 |c_2 - u^0|^\beta + \phi^1 (1 - \phi^2) |c_1 - u^0|^\beta \\
+ (1 - \phi^1) \phi^2 |c_4 - u^0|^\beta + (1 - \phi^1)(1 - \phi^2) |c_3 - u^0|^\beta \} \, dx.
\]

Phase 1: \( \phi^1 = 1, \phi^2 = 0 \)  Phase 2: \( \phi^1 = 1, \phi^2 = 1 \)
Phase 3: \( \phi^1 = 0, \phi^2 = 0 \)  Phase 4: \( \phi^1 = 0, \phi^2 = 1 \)

Can this non-convex problem be equivalent to a convex model???
YES!!!
Binary formulation of multiphase Chan-Vese model

Wish to obtain global optimization framework for

$$\min_{\phi^1, \phi^2 \in \{0,1\}} \alpha \int_{\Omega} |\nabla \phi^1| dx + \alpha \int_{\Omega} |\nabla \phi^2| dx + E^{data}(\phi^1, \phi^2),$$

with

$$E^{data}(\phi^1, \phi^2) = \int_{\Omega} \left\{ \phi^1 \phi^2 |c_2 - u^0|^\beta + \phi^1 (1 - \phi^2) |c_1 - u^0|^\beta + (1 - \phi^1) \phi^2 |c_4 - u^0|^\beta + (1 - \phi^1)(1 - \phi^2) |c_3 - u^0|^\beta \right\} dx.$$

| Phase 1: $\phi^1 = 1, \phi^2 = 0$ | Phase 2: $\phi^1 = 1, \phi^2 = 1$ |
| Phase 3: $\phi^1 = 0, \phi^2 = 0$ | Phase 4: $\phi^1 = 0, \phi^2 = 1$ |

Can this non-convex problem be equivalent to a convex model???

YES!!!  Why ??
Continuous max-flow formulation

\[
\sup_{p^i_s, p^i_t, p^{12}, q^i; i=1,2} \int_{\Omega} \left( p^1_s(x) + p^2_s(x) \right) \, dx
\]

subject to

\[
\begin{align*}
p^1_s(x) &\leq C(x), \quad p^2_s(x) \leq D(x), \quad p^1_t(x) \leq A(x), \quad p^2_t \leq B(x), \quad |q^i(x)| \leq \alpha \\
-F(x) &\leq p^{12}(x) \leq E(x), \\
\text{div} \, q^1(x) - p^1_s(x) + p^1_t(x) + p^{12}(x) &= 0 \\
\text{div} \, q^2(x) - p^2_s(x) + p^2_t(x) - p^{12}(x) &= 0
\end{align*}
\]
Lagrange multipliers $\lambda_1$ and $\lambda_2$ for flow conservation constraints. Lagrangian functional:

$$\max_{p^i_s, p^i_t, p^{12}, q^i; i=1, 2} \inf_{\lambda_1, \lambda_2} \int_\Omega \left( (1 - \lambda_1(x)) p^1_s(x) + (1 - \lambda_2(x)) p^2_s(x) \right) dx$$

$$+ \int_\Omega \lambda_1(x) p^1_t(x) + \lambda_2(x) p^2_t(x) + (\lambda_1(x) - \lambda_2(x)) p^{12}(x) dx$$

$$+ \int_\Omega \lambda_1(x) \text{div} q^1(x) + \int_\Omega \lambda_2(x) \text{div} q^2(x).$$

subject to

$$p^1_s(x) \leq C(x), \quad p^2_s(x) \leq D(x), \quad p^1_t(x) \leq A(x), \quad p^2_t \leq B(x), \quad |q^i(x)| \leq \alpha$$

$$-F(x) \leq p^{12}(x) \leq E(x),$$
Maximizing Lagrangian for all flows results in

\[
\min_{\lambda_1, \lambda_2} \int_{\Omega} \left( (1-\lambda_1(x))C(x) + (1-\lambda_2(x))D(x) + \lambda_1(x)A(x) + \lambda_2(x)B(x) \right) dx \\
+ \int_{\Omega} \max\{\lambda_1(x)-\lambda_2(x), 0\} E(x) \, dx - \min\{\lambda_1(x)-\lambda_2(x), 0\} F(x) \, dx \\
+ \alpha \int_{\Omega} |\nabla \lambda_1(x)| \, dx + \alpha \int_{\Omega} |\nabla \lambda_2(x)| \, dx.
\]

subject to \( \lambda_1(x), \lambda_2(x) \in [0, 1], \forall x \in \Omega. \)

\[
\begin{align*}
A(x) + B(x) &= |c_2 - u^0(x)|^\beta \\
C(x) + D(x) &= |c_3 - u^0(x)|^\beta \\
A(x) + E(x) + D(x) &= |c_1 - u^0(x)|^\beta \\
B(x) + F(x) + C(x) &= |c_4 - u^0(x)|^\beta
\end{align*}
\]

- Convex, iff \( E(x), F(x) \geq 0 \)
- Theorem: Thresholding optimal \( \lambda_1(x) \) and \( \lambda_2(x) \) will give a binary global solution to multiphase Chan-Vese model
Corollaries

- No approximation: the global minimizer of the max-flow (convex CV) is the global minimizer of the original non-convex CV model.

\[ R(u) = \int_{\Omega} |\nabla u_1| + |\nabla u_2|. \]

- We can also regularize the length of the interface, then Thresholded solution is not guaranteed to be exact.
Corollaries

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Why??

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We can also regularize the length of the interface, then Thresholded solution is not guaranteed to be exact.
A new tight relaxation with product of labels (more than binary) has been given in Goldluecke-Cremers ECCV(2010).
Multiphase problems

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Other multiphase relaxations:
Multiphase problems

Multiphase problem (II)

Layered Graph

\[^1\]Boykov-Kolmogorov (PAMI 2001), Ishikawa (PAMI 2003), Darbon-Segle (JMIV, 2006), Bae-Tai (SSVM 2009)
Multiphase problems

To identify $n$ phases, we need one label function, but $n$ labels.
Figure: Need multi-labels $\phi(x) = i$ in $\Omega_i$, $i = 1, 2, 3, 4$. 

**Multiphase problem**
Increase dimension – only need two phases

\[ |\nabla \phi| = |\nabla u|. \]

Figure: Just need one label: Increase the dimension, we just need \( u(x, \phi) = 0 \) or \( 1 \).
1D signal and multiphase

Figure: Left: Example cut on the graph $G$ corresponding to a 1d image of 6 grid points. Right: Values of $\phi$ corresponding to the cut.
This graph was proposed in Ishikaka (PAMI 2003).
Historical review

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- Using this kind of regularization, segmentation is essentially an generalization of the Quantized ROF model.
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Using this kind of regularization, segmentation is essentially a generalization of the Quantized ROF model.

Lie-Lysaker-T. (2004, 2005) is a formulation of this model with finite number of labels in a continuous domain $x \in \Omega$.

T. Pock and D. Cremers and H. Bischof and A. Chambolle (2010): gives a convex relaxation in case both image domain and the labels are continuous.
Continuous max-flow and cut

Costs: $\rho(u(p), p), C(p, q), i = 1, 2, 3$. 
Discrete min-cut

\[
\min \sum_{v \in P} \rho(u_v, v) + \sum_{(u,v) \in \mathcal{N}} C(u, v)|u_v - u_w|.
\]
Discrete max-flow

\[
\max \sum_{v \in P} p_1(v)
\]

\[
p_i(v) \leq \rho(\ell_i, v), \quad i = 1, 2, \ldots, n,
\]

\[
|q_i(v, w)| \leq C(v, w).
\]
Continuous min-cut:

\[
\min_{u \in U} \int_{\Omega} \rho(u(x), x) \, dx + \int_{\Omega} C(x) |\nabla u| \, dx.
\]

\[
U = \{ u : \Omega \mapsto \{\ell_1, \ell_2, \cdots, \ell_n\}\}.
\]
Continuous min-cut and max-flow

Continuous min-cut:

\[
\min_{u \in U} \int_{\Omega} \rho(u(x), x) \, dx + \int_{\Omega} C(x) |\nabla u| \, dx.
\]

\[U = \{ u : \Omega \mapsto \{ \ell_1, \ell_2, \cdots \ell_n \} \}.\]

Continuous max-flow

\[
\max \int_{\Omega} p_1(x) \, dx
\]
Continuous min-cut and max-flow

Continuous min-cut:

\[
\min_{u \in U} \int_{\Omega} \rho(u(x), x) \, dx + \int_{\Omega} C(x) |\nabla u| \, dx.
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\[U = \{u : \Omega \mapsto \{\ell_1, \ell_2, \cdots \ell_n\}\}.
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Continuous max-flow

\[
\max \int_{\Omega} p_1(x) \, dx
\]

\[p_i(x) \leq \rho(\ell_i, x), \quad i = 1, 2, \cdots n,
\]
Continuous min-cut:

\[
\min_{u \in U} \int_{\Omega} \rho(u(x), x)dx + \int_{\Omega} C(x)|\nabla u|dx.
\]

\[U = \{u : \Omega \mapsto \{\ell_1, \ell_2, \cdots \ell_n\}\}.
\]

Continuous max-flow

\[
\max \int_{\Omega} p_1(x)dx
\]

\[p_i(x) \leq \rho(\ell_i, x), \quad i = 1, 2, \cdots n,
\]

\[|q_i(x)| \leq C(x),
\]
Continuous min-cut and max-flow

Continuous min-cut:

$$\min_{u \in U} \int_{\Omega} \rho(u(x), x) dx + \int_{\Omega} C(x) |\nabla u| dx.$$  

$$U = \{u : \Omega \mapsto \{\ell_1, \ell_2, \cdots \ell_n\}\}.$$  

Continuous max-flow

$$\max \int_{\Omega} p_1(x) dx$$  

$$p_i(x) \leq \rho(\ell_i, x), \quad i = 1, 2, \cdots, n,$$  

$$|q_i(x)| \leq C(x),$$  

$$(\text{div} q_i - p_i + p_{i+1})(x) = 0, \quad q_i \cdot n = 0.$$
**Theorem:** The continuous min-cut and max-flow problems are dual to each other. A "threshold" of any solutions of the "convex" min-cut problem is a global minimizer for the "non-convex" min-cut problem.
Algorithm: Primal-dual algorithm is tested and is fast.  
Primal variables: The flow variables.  
Dual variables: The cut $u$ which turn out to the Lagrangian of the "flow conservation" constraints.
For the number of labels, instead of:

\[ U = \{ u : \Omega \mapsto \{ \ell_1, \ell_2, \ldots, \ell_n \} \} . \]

we use “infinite number of labels”:

\[ U = \{ u : \Omega \mapsto [\ell_{\text{min}}, \ell_{\text{max}}] \} . \]

This is exactly the same problem considered in:
T. Pock and D. Cremers and H. Bischof and A. Chambolle (2010).
Continuous labels

As the number of labels goes to the limit of infinity, the max-flow problem with the flow constraints turns into:

\[
\sup_{p,q} \int_\Omega p(\ell_{\text{min}}, x) \, dx \\
\text{s.t.} \quad p(\ell, x) \leq \rho(\ell, x), \quad |q(\ell, x)| \leq \alpha, \quad \forall x \in \Omega, \ \forall \ell \in [\ell_{\text{min}}, \ell_{\text{max}}] \\
\text{div}_x q(\ell, x) + \partial_\ell p(\ell, x) = 0, \quad \text{a.e.} \ x \in \Omega, \ \ell \in [\ell_{\text{min}}, \ell_{\text{max}}].
\]
Continuous labels

As the number of labels goes to the limit of infinity, the max-flow problem with the flow constraints turns into:

\[
\sup_{p,q} \int_{\Omega} p(\ell_{\text{min}}, x) \, dx
\]

subject to\[
p(\ell, x) \leq \rho(\ell, x), \quad |q(\ell, x)| \leq \alpha, \quad \forall x \in \Omega, \ \forall \ell \in [\ell_{\text{min}}, \ell_{\text{max}}]
\]
\[
div_x q(\ell, x) + \partial_\ell p(\ell, x) = 0, \quad \text{a.e. } x \in \Omega, \ \ell \in [\ell_{\text{min}}, \ell_{\text{max}}].
\]

The convex min-cut problem (the dual problem to the max-flow) is:

\[
\min_{\lambda(\ell,x) \in [0,1]} \int_{\ell_{\text{min}}}^{\ell_{\text{max}}} \int_\Omega \left\{ \alpha |\nabla_x \lambda| - \rho(\ell, x) \partial_\ell \lambda(\ell, x) \right\} \, dx \, d\ell
\]
\[
+ \int_\Omega (1 - \lambda(\ell_{\text{min}}, x)) \rho(\ell_{\text{min}}, x) + \lambda(\ell_{\text{max}}, x) \rho(\ell_{\text{max}}, x) \, dx
\]

subject to\[
\partial_\ell \lambda(\ell, x) \leq 0, \quad \lambda(\ell_{\text{min}}, x) \leq 1, \quad \lambda(\ell_{\text{max}}, x) \geq 0, \quad \forall x \in \Omega, \ \forall \ell \in [\ell_{\text{min}}, \ell_{\text{max}}] \quad (3)
\]
The convex min-cut problem (the dual problem to the max-flow) is:

$$\min_{\lambda(\ell, x) \in [0, 1]} \int_{\ell_{\min}}^{\ell_{\max}} \int_{\Omega} \left\{ \alpha \left| \nabla_x \lambda \right| - \rho(\ell, x) \partial_{\ell} \lambda(\ell, x) \right\} \, dx \, d\ell$$

$$+ \int_{\Omega} (1 - \lambda(\ell_{\min}, x)) \rho(\ell_{\min}, x) + \lambda(\ell_{\max}, x) \rho(\ell_{\max}, x) \, dx$$

subject to

$$\partial_{\ell} \lambda(\ell, x) \leq 0, \quad \lambda(\ell_{\min}, x) \leq 1, \quad \lambda(\ell_{\max}, x) \geq 0,$$
The convex min-cut problem (the dual problem to the max-flow) is:

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+ \int_{\Omega} (1 - \lambda(\ell_{\min}, x)) \rho(\ell_{\min}, x) + \lambda(\ell_{\max}, x) \rho(\ell_{\max}, x) \, dx
\]

subject to

\[
\partial_\ell \lambda(\ell, x) \leq 0, \quad \lambda(\ell_{\min}, x) \leq 1, \quad \lambda(\ell_{\max}, x) \geq 0,
\]

The following is the model from Poct et al (2010): (Note the difference)

\[
\min_{\lambda(\ell, x) \in \{0, 1\}} \int_{\ell_{\min}}^{\ell_{\max}} \int_{\Omega} \{ \alpha |\nabla_x \lambda| + \rho(\ell, x) |\partial_\ell \lambda(\ell, x)| \} \, dx \, d\ell.
\]

subject to

\[
\lambda(\ell_{\min}, x) = 1, \quad \lambda(\ell_{\max}, x) = 0.
\]
Algorithm: Primal-dual algorithm is tested and is fast.  
Primal variables: The flow variables.  
Dual variables: The cut \( u \) which turn out to the Lagrangian of the ”flow conservation” constraints.
Multiphase problem (III)

Graph for characteristic functions$^1$

$^1$Yuan-Bae-T.-Boykov (ECCV’10)
Multi-partitioning problem

Multi-partitioning problem (Pott’s model)

\[
\min_{\{\Omega_i\}} \sum_{i=1}^{n} \int_{\Omega_i} f_i \, dx + \sum_{i=1}^{n} \int_{\partial\Omega_i} g(x) \, ds,
\]

such that \( \bigcup_{i=1}^{n} \Omega_i = \Omega \), \( \bigcap_{i=1}^{n} \Omega_i = \emptyset \)
Multi-partitioning problem

Multi-partitioning problem (Pott’s model)

\[
\min_{\{\Omega_i\}} \sum_{i=1}^{n} \int_{\Omega_i} f_i \, dx + \sum_{i=1}^{n} \int_{\partial \Omega_i} g(x) \, ds,
\]

such that \( \bigcup_{i=1}^{n} \Omega_i = \Omega, \quad \bigcap_{i=1}^{n} \Omega_i = \emptyset \)

Pott’s model in terms of characteristic functions

\[
\min_{u_i(x) \in \{0,1\}} \sum_{i=1}^{n} \int_{\Omega} u_i(x)f_i(x) \, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) \|\nabla u_i\| \, dx, \text{ s.t. } \sum_{i=1}^{n} u_i(x) = 1
\]
Multi-partitioning problem (Pott’s model)

\[
\min_{\{\Omega_i\}} \sum_{i=1}^{n} \int_{\Omega_i} f_i(x) \, dx + \sum_{i=1}^{n} \int_{\partial \Omega_i} g(x) \, ds,
\]

such that \( \bigcup_{i=1}^{n} \Omega_i = \Omega, \quad \bigcap_{i=1}^{n} \Omega_i = \emptyset \)

Pott’s model in terms of characteristic functions

\[
\min_{u_i(x) \in \{0,1\}} \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x) \, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) |\nabla u_i| \, dx, \text{ s.t. } \sum_{i=1}^{n} u_i(x) = 1
\]

\[
u_i(x) = \chi_{\Omega_i}(x) := \begin{cases} 1, & x \in \Omega_i \\ 0, & x \notin \Omega_i \end{cases}, \quad i = 1, \ldots, n
\]
A convex relaxation approach

Relaxed Pott’s model in terms of characteristic functions (primal model)

\[
\min_u E^P(u) = \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x) \, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) |\nabla u_i| \, dx ,
\]
A convex relaxation approach

Relaxed Pott’s model in terms of characteristic functions (primal model)

\[
\min_u E^P(u) = \sum_{i=1}^{n} \int_{\Omega} u_i(x)f_i(x) \, dx + \sum_{i=1}^{n} \int_{\Omega} g(x)|\nabla u_i| \, dx,
\]

s.t. \( u \in \triangle_+ = \{(u_1(x), \ldots, u_n(x)) | \sum_{i=1}^{n} u_i(x) = 1; \ u_i(x) \geq 0 \} \)

- Convex optimization problem
**Dual model:** $C_\lambda := \{ p : \Omega \mapsto \mathbb{R}^2 \mid ||p(x)||_2 \leq g(x), \ p_n|_{\partial \Omega} = 0 \}$,

- Hence the primal-dual model can be optimized pointwise for $u$

$$\min_{u \in \Delta^+} \sum_{i=1}^{n} \int_{\Omega} u_i(x)f_i(x) \, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) |\nabla u_i| \, dx,$$
Dual model: $C_{\lambda} := \{ p : \Omega \mapsto \mathbb{R}^2 \mid |p(x)|_2 \leq g(x), \ p_n|_{\partial \Omega} = 0 \}$,

- Hence the primal-dual model can be optimized pointwise for $u$

$$\min_{u \in \Delta_+} \sum_{i=1}^n \int_{\Omega} u_i(x) f_i(x) \, dx + \sum_{i=1}^n \int_{\Omega} g(x) |\nabla u_i| \, dx,$$

$$\max_{p_i \in C_{\lambda}} \min_{u \in \Delta_+} E(u, p) = \int_{\Omega} \sum_{i=1}^n u_i (f_i + \text{div } p_i) \, dx.$$
Dual model: $C_\lambda := \{ p : \Omega \rightarrow \mathbb{R}^2 \mid \| p(x) \|_2 \leq g(x), \ p_n|_{\partial \Omega} = 0 \}$,

Hence the primal-dual model can be optimized pointwise for $u$

$$\min_{u \in \Delta_+} \sum_{i=1}^{n} \int_{\Omega} u_i(x)f_i(x) \, dx + \sum_{i=1}^{n} \int_{\Omega} g(x)|\nabla u_i| \, dx,$$

$$\max_{p_i \in C_\lambda} \min_{u \in \Delta_+} E(u, p) = \int_{\Omega} \sum_{i=1}^{n} u_i(f_i + \text{div } p_i) \, dx$$

$$= \max_{p_i \in C_\lambda} \int_{\Omega} \min_{u(x) \in \Delta_+} \sum_{i=1}^{n} u_i(x)(f_i(x) + \text{div } p_i(x)) \, dx$$
Dual model: \( C_\lambda := \{ p : \Omega \mapsto \mathbb{R}^2 \mid |p(x)|_2 \leq g(x), \ p_n|_{\partial \Omega} = 0 \} \),

- Hence the primal-dual model can be optimized pointwise for \( u \)

\[
\min_{u \in \Delta_+} \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x) \, dx \, \bigg( \sum_{i=1}^{n} \int_{\Omega} g(x) |\nabla u_i| \, dx \bigg),
\]

\[
\max_{p_i \in C_\lambda} \min_{u \in \Delta_+} E(u, p) = \int_{\Omega} \sum_{i=1}^{n} u_i(f_i + \text{div} \, p_i) \, dx
\]

\[
= \max_{p_i \in C_\lambda} \int_{\Omega} \min_{u(x) \in \Delta_+} \sum_{i=1}^{n} u_i(x)(f_i(x) + \text{div} \, p_i(x)) \, dx
\]

\[
= \max_{p_i \in C_\lambda} \int_{\Omega} \{ \min(f_1 + \text{div} \, p_1, \ldots, f_n + \text{div} \, p_n) \} \, dx
\]
Dual model: \( C_\lambda := \{ p : \Omega \mapsto \mathbb{R}^2 | \| p(x) \|_2 \leq g(x), \ p_n|_{\partial \Omega} = 0 \} \),

Hence the primal-dual model can be optimized pointwise for \( u \)

\[
\min_{u \in \Delta_+} \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x) \, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) \| \nabla u_i \| \, dx,
\]

\[
\max_{p_i \in C_\lambda} \min_{u \in \Delta_+} E(u, p) = \int_{\Omega} \sum_{i=1}^{n} u_i(f_i + \text{div} \, p_i) \, dx
\]

\[
= \max_{p_i \in C_\lambda} \int_{\Omega} \min_{u(x) \in \Delta_+} \sum_{i=1}^{n} u_i(x)(f_i(x) + \text{div} \, p_i(x)) \, dx
\]

\[
= \max_{p_i \in C_\lambda} \int_{\Omega} \{ \min(f_1 + \text{div} \, p_1, \ldots, f_n + \text{div} \, p_n) \} \, dx
\]

\[
= \max_{p_i \in C_\lambda} E^D(p)
\]
1. $n$ copies $\Omega_i$, $i = 1, \ldots, n$, of $\Omega$;
2. For $\forall x \in \Omega$, the same source flow $p_s(x)$ from the source $s$ to $x$ at $\Omega_i$, $i = 1, \ldots, n$, simultaneously;
3. For $\forall x \in \Omega$, the sink flow $p_i(x)$ from $x$ at $\Omega_i$, $i = 1, \ldots, n$, of $\Omega$ to the sink $t$. $p_i(x)$, $i = 1, \ldots, n$, may be different one by one;
4. The spatial flow $q_i(x)$, $i = 1, \ldots, n$ defined within each $\Omega_i$. 

Continuous Max-Flow Model (CMF-PM)
Max-flow on this graph

Max-Flow:

\[
\max_{p_s, p, q} \{ P(p_s, p, q) = \int_{\Omega} p_s \, dx \}
\]

\[
|q_i(x)| \leq g(x), \quad p_i(x) \leq f_i(x),
\]

\[
(\text{div} q_i - p_s + p_i)(x) = 0, \quad i = 1, 2, \ldots, n.
\]

Note that

\[
p_s(x) = \text{div} q_i(x) + p_i(x), \quad i = 1, 2 \ldots, n.
\]

Thus

\[
p_s(x) = \min(f_1 + \text{div} p_1, \ldots, f_n + \text{div} p_n).
\]

Therefore, the maximum of \( \int_{\Omega} p_s(x) \) is:

\[
\max \int_{\Omega} \min(f_1 + \text{div} p_1, \ldots, f_n + \text{div} p_n) \, dx.
\]
(Convex) min-cut on this graph

\[
\max_{p_s, p, q} \min_u \{ E(p_s, p, q, u) = \int_\Omega p_s dx + \sum_{i=1}^m u_i (\text{div} q_i - p_s + p_i) dx \} \\
\text{s.t. } p_i(x) \leq f_i(x), \quad |q_i(x)| \leq g(x).
\]

Rearranging the energy functional \( E(\cdot) \), we that

\[
E(p_s, p, q, u) = \int_\Omega (1 - \sum_{i=1}^m u_i) p_s + \sum_{i=1}^m u_i p_i + \sum_{i=1}^m u_i \text{div} q_i . dx.
\]

The following constraint are automatically satisfied from the optimization:

\[
u_i(x) \leq 0, \sum_{i=1}^m u_i = 1.
\]
We show a number of non-convex problems can be solved exactly through convex relaxation. They can be interpreted as continuous max-flow and min-cut problems. It is interesting to observe that the Lagrangian multiplier for the flow conservation is the ”cut”.

A number of the models has “binary” global minimizer. However, some of them have duality gap between the max-flow and (non-convex) min-cut.

If we replace the isotropic TV by antisotropic TV, then all the models we have investigated has a discrete “graph”. However, some recent (ongoing) work show that some continuous max-flow models do not have a discrete graph.

The CV (Chan-Vese) model has special properties in term of global minimization through max-flow and min-cut approach. Two-phase and four-phase problems have global binary minimizers, but not \(2^n\)-phases (\(n \geq 3\)).
Summary

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