Minors & Algorithms

Erik Demaine
M.I.T.
Minors

- $H$ is a **minor** of $G$ if $G$ can reach $H$ via
  - edge deletions
  - edge contractions

$K_{3,3}$ is a minor of
Graph Minors. I. Excluding a Forest

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The path-width of a graph is the minimum value of \(\lambda\) obtained from a sequence of graphs \(G_1, \ldots, G_s\) each containing \(n\) vertices, by identifying some vertices of \(G_i\) pairwise with \(G_{i+1}\). For every forest \(H\) it is proved that there is a number \(k\) such that no minor isomorphic to \(H\) has path-width \(\leq k\). This, together with the papers, yields a “good” algorithm to test for the presence of every minor, and implies that if \(P\) is any property of graphs that does not have property \(P\), then the set of minor-minimal graphs not having \(P\) is finite.

Graph Minors. XX. Wagner’s conjecture

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Abstract

We prove Wagner’s conjecture, that for every infinite set of finite graphs, one of its members is isomorphic to a minor of another.

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Keywords: Graph, Minor, Surface embedding, Well-quasi-ordering
Wagner’s Conjecture
[Robertson & Seymour 2004]

- Minor relation is a **well-quasi-ordering**: Every *infinite* graph sequence $G_1, G_2, G_3, \ldots$ has some $G_i$ a minor of $G_j$ (with $i < j$)

...or equivalently...

- Every **minor-closed** graph family is characterized by a **finite** set of excluded minors
  - **Proof by contradiction**: Infinite excluded minors would have a minor relation among them
  - In some sense necessarily **nonconstructive** [Friedman, Robertson, Seymour 1987]
Wagner’s Conjecture
[Robertson & Seymour 2004]

- Every **minor-closed** graph family is characterized by a **finite** set of excluded minors

- **Examples:**
  - Forests: $K_3$
  - Outerplanar: $K_4 \& K_{2,3}$
  - Series-parallel: $K_4$
  - Planar: $K_5 \& K_{3,3}$
  - Linklessly embeddable: Petersen family [Robertson & Seymour 1995]
Algorithms via Minors

• Theorem: Polynomial test for a fixed minor \( H \)
  - \( f(H) n^3 \) [GM XII, 1995]
  - \( f(H) n^2 \) [Kawarabayashi, Kobayashi, Reed 2012]

• Corollary: Any minor-closed graph property has a polynomial-time decision algorithm
  - But algorithm is existential: in general, don’t know which excluded minors to test for
Warning:
Graph Minor Constants

• **Example**: Planar minor testing [GM5 1986], analyzed by David Johnson [1987]
  - Running time $= O(n^3)$
  - Lead constant $= 2^{2^{2^x}}$, $x = 2^{2^{2^{\cdots}}} 2^{2^{\cdots}} \frac{1}{2} |V(H)|$

“1. The Case of the Hidden Constants

… Unfortunately, for any instance $G = (V,E)$ that one could fit into the known universe, one would easily prefer $|V|^{70}$ to even constant time, if that constant had to be one of Robertson and Seymour’s.”

The NP-Completeness Column: An Ongoing Guide

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Fixed-Parameter Algorithms via Minors

• Graph parameter is **minor-closed** if it only decreases when deleting or contracting
  ▪ **Examples**: vertex cover, feedback vertex set

• **Corollary**: Any minor-closed graph parameter has an \( f(k) n^2 \) algorithm for deciding \( \leq k \)
  ▪ “Fixed-parameter tractable”
  ▪ But a **different**, unknown algorithm for each \( k \)
  ▪ Not really a (uniform) fixed-parameter algorithm

[Fellows & Langston 1988]
Fixed-Parameter Algorithms via Minors

• **Theorem:** Any minor-closed graph parameter
  - nonzero on some planar graph,
  - at least the sum over connected components in a disconnected graph, and
  - fixed-parameter tractable with respect to treewidth

  can be solved in $f(k) n$ by a known algorithm

[Demaine & Hajiaghayi 2007]
What Are $H$-minor-free Graphs Like?

- Bounded average degree (& degeneracy): $O \left( |V(H)| \sqrt{\log |V(H)|} \right)$
  [Kostochka 1982; Thomason 1984]

- $O(\sqrt{n})$ treewidth [Grohe 2003]

- $\Rightarrow O(\sqrt{n})$ balanced separators
  (like Lipton-Tarjan planar separator theorem)
  [Alon, Seymour, Thomas 1990]

- Kinda like planar graphs?
**Structure Theorem:**

- $H$-minor-free graph looks like a tree of "almost embeddable" graphs:
  - Base graph drawn on surface of genus $f(H)$
  - $f(H)$ vortex faces filled with graphs of pathwidth $f(H)$
  - $f(H)$ apex vertices connected to anything
H-Minor-Free Graphs
[Robertson & Seymour 2003]

Structure Theorem:
• $H$-minor-free graph looks like a tree of “almost embeddable” graphs

Structure Algorithms:
• $n^f(H)$ [Demaine, Hajiaghayi, Kawarabayashi — FOCS 2005]
• $O(n^3)$ [Kawarabayashi & Wollan — STOC 2011]
• $O(n^2)$ [Grohe, Kawarabayashi, Reed — SODA 2013]
Grid Minors vs. Treewidth

- Every $H$-minor-free graph of treewidth $\geq f(H) \cdot r$ has an $r \times r$ grid minor [Demaine & Hajiaghayi 2005]
  - Previous bounds exponential in $r$ (and $H$) [GM5, ...]
  - $f(H) = |V(H)|^{O(|E(H)|)}$ [Kawarabayashi & Kobayashi 2012]

- Every graph of treewidth $\geq c_1 \cdot r^{c_2}$ has an $r \times r$ grid minor [Chekuri & Chuzhoy 2013]
  - $c_2 > 2$ necessary [Robertson, Seymour, Thomas 1994]
  - Previous bounds exponential in $r$ [RST94]
Almost-embeddable graphs:
- Remove apices by deletion
- Contract each vortex to a vertex
- Pull apart each handle
  [Demaine, Fomin, Hajiaghayi, Thilikos 2004]
- Planar graph has large grid minor  [RST94]

Tree takes max of treewidths, so one almost-embeddable graph has full treewidth
- But may not be $H$-minor-free from tree joins
- “Approximate” graph to make minor of original graph $\Rightarrow H$-minor-free  [Demaine & Hajiaghayi 2005]
Graph Minor Hierarchy

- planar
- bounded genus
  - apex-minor-free
  - $H$-minor-free
Apex-Minor-Free Graphs

- Bounded local treewidth = radius-$r$ neighborhood around every vertex $v$ has treewidth $\leq f(r)$
  - **Examples:** planar and bounded-genus graphs
- Minor-closed graph family has bounded local treewidth $\iff$ excludes an apex graph = minor of planar graph + one vertex
  - [Eppstein 2000]
  - Call such graphs apex-minor-free
Apex-Minor-Free Graphs

- **Structure Theorem:** Apices attach only to vortices, allowing bounded-treewidth vortices
  - Additive +2 approximation for chromatic number (extending from bounded genus [Thomassen 1997])

- **Bounded local treewidth**
  - \( f(r) = 2^{2^O(r)} \) [Eppstein 2000]
  - \( f(r) = 2^O(r) \) [Demaine & Hajiaghayi 2004a]
  - \( f(r) = O(r) \) [Demaine & Hajiaghayi 2004b]
  - [EPTASs: \( 2^{2^{O(1/\varepsilon)}} n^{O(1)} \rightarrow 2^{O(1/\varepsilon)} n^{O(1)} \) ]
• Equivalently, $H$ is a **minor** of $G$ if
  
  - Every vertex $v$ in $H$ has a corres. tree $T_v$ in $G$
  - Any edge $(v, w)$ in $H$ has a corresponding edge connecting $T_v$ to $T_w$ in $G$

$K_{3,3}$ is a minor of

$\text{delete}$ $\text{contract}$

“model”
Odd Minors

• \( H \) is an **odd minor** of \( G \) if we can also **2-color** the vertices in \( G \) so that
  - Tree edges are **bichromatic**
  - \( T_v \)-to-\( T_w \) edges are **monochromatic**

\[ K_{3,3} \] is an odd minor of
Odd Minors

- $H$ is an **odd minor** of $G$ if we can also **2-color** the vertices in $G$ so that
  - Tree edges are **bichromatic**
  - $T_v$-to-$T_w$ edges are **monochromatic**

is an odd minor of $K_{3,3}$
Odd Minors

- $H$ is an **odd minor** of $G$ if we can also **2-color** the vertices in $G$ so that
  - Tree edges are **bichromatic**
  - $T_v$-to-$T_w$ edges are **monochromatic**

$K_{3,3}$ is an odd minor of

is an odd minor of

**delete**

**contract**
Odd-Minor-Free Graphs
[Demaine, Hajiaghayi, Kawarabayashi 2010]

- **Structure Theorem:** Every odd-$H$-minor-free graph can be written as a tree of graphs joined along $f(H)$-size cliques, where each term is
  - Bounded-genus graph
    + $f(H)$ vortices
    + $f(H)$ apices
    (joined at only 3 surface vertices)
  - Bipartite graph
    + $f(H)$ apices
    (joined at only 1 bipartite vertex)
- $n^{f(H)}$ algorithm
  - $f(H) n^{O(1)}$ simpler decomp. [Tazari 2012]
any problem satisfying:
- closed under contraction [& deletion]
- \( \text{OPT} \) costly on \( r \times r \) grid

Bidimensionality Overview

- subexponential fixed-parameter algorithm: \( 2^{O(\sqrt{\text{OPT}})} n^{O(1)} \)
- efficient PTAS: \( f(\varepsilon) n^{O(1)} \) time
- linear kernelization: \( n \to O(\text{OPT}) \)

... on \( H \)-minor-free graphs
History of Subexponential FPT

- In the early days, most fixed-parameter algorithms ran in $2^{O(k^c)} \cdot n^{O(1)}$ time, for $c \geq 1$

- **Natural question**: Is $2^{o(k)} \cdot n^{O(1)}$ possible?
History of Subexponential FPT

- **Vertex Cover**: choose smallest set of vertices to cover every edge (on either endpoint)

- Vertex Cover can be kernelized to \( \leq 2k \) vertices

- Lipton & Tarjan’s separator approach [1980] solves Planar Independent Set in \( 2^{O(\sqrt{n})} \) time
  - Min. Vertex Cover + Max. Independent Set = \( |V| \)
  - \( \Rightarrow n^{O(1)} + 2^{O(\sqrt{k})} \) time
History of Subexponential FPT

- What about problems like these?
  - **Dominating Set**: Cover all vertices with $k$ vertex neighborhoods
  - **Feedback Vertex Set**: Cover cycles with $k$ vertices
  - **Long Path**: Is there a simple path of length $\geq k$?

- No linear (or even polynomial) kernel was known for these problems then

- We now know that some, e.g. Long Path, have no polynomial kernel (unless \( \text{NP} \subseteq \text{coNP/poly} \)), even for planar graphs
  
  [Bodlaender, Downey, Fellows, Hermelin 2009]
History of Subexponential FPT

- $2^{O(\sqrt{k})}n^{O(1)}$ for planar Dominating Set
  [Alber, Bodlaender, Fernau, Kloks, Niedermeier 2000]

- Same approach extended to Dominating Set variations, Feedback Vertex Set, etc.
Bidimensionality Overview

- Any problem satisfying:
  - Closed under contraction [\& deletion]
  - OPT costly on $r \times r$ grid
- Minor $H$
- Parameter-treewidth bound
- Subexponential fixed-parameter algorithm
- Efficient PTAS
- Linear kernel
- ... on $H$-minor-free graphs
Bidimensionality (version 1)

[Demaine, Fomin, Hajiaghayi, Thilikos 2004]

- Parameter $k = k(G)$ is **bidimensional** if
  - **Closed under minors:**
    $k$ only decreases when deleting or contracting edges
  - **Large on grids:**
    For the $r \times r$ grid, $k = \Omega(r^2)$
**Example 1: Vertex Cover**

- $k =$ minimum number of vertices required to cover every edge (on either endpoint)

- Closed under minors:

  ⇒ still a cover (only fewer edges)

  ⇒ still a cover, possibly 1 smaller
Example 1: Vertex Cover

- $k =$ minimum number of vertices required to cover every edge (on either endpoint)

- Large on grids:
  - Matching of size $\Omega(r^2)$
  - Every edge must be covered by a different vertex
Example 2: Feedback Vertex Set

- $k =$ minimum number of vertices required to cover every cycle (on some vertex)

- Closed under minors:
  - $\Rightarrow$ still a cover (only break cycles)
  - $\Rightarrow$ still a cover, possibly 1 smaller
**Example 2: Feedback Vertex Set**

- $k =$ minimum number of vertices required to cover every cycle (on some vertex)

- **Large on grids:**
  - $\Omega(r^2)$ vertex-disjoint cycles
  - Every cycle must be covered by a different vertex
**Bidimensional ⇒ Relate Parameter & Treewidth**

- **Theorem 1**: If a parameter $k$ is bidimensional, then it satisfies parameter-treewidth bound
  \[ \text{treewidth} = O(\sqrt{k}) \]
  in $H$-minor-free graphs
  [Demaine, Fomin, Hajiaghayi, Thilikos 2004; Demaine & Hajiaghayi 2005]

- **Proof sketch**:
  Treewidth $w \Rightarrow \Omega(w) \times \Omega(w)$ grid minor
  \[ \Rightarrow k = \Omega(w^2) \]  
  [bidimensional]
Theorem 2: If a parameter $k$ is
- bidimensional, and
- fixed-parameter tractable on graphs of bounded treewidth: $h(\text{treewidth}) n^{O(1)}$ time
then it has a subexponential fixed-parameter algorithm, running in $h(\sqrt{k}) n^{O(1)}$ time, in $H$-minor-free graphs

Proof sketch:
- If (approx.) treewidth $= \omega(\sqrt{k})$, answer NO
- Else run bounded-treewidth algorithm
any problem satisfying:
- closed under contraction [\& deletion]
- \( \text{OPT costly on } r \times r \text{ grid} \)

minor \( H \)

\begin{itemize}
\item parameter-treewidth bound
\item subexponential fixed-parameter algorithm
\item efficient PTAS
\item linear kernel
\end{itemize}

\( \ldots \text{ on } H\text{-minor-free graphs} \)
any problem satisfying:

closed under contraction [& deletion]

OPT costly on $r \times r$ grid

minor $H$

parameter-treewidth bound

subexponential fixed-parameter algorithm

efficient PTAS

linear kernel

... on $H$-minor-free graphs
Theorem 4: If a parameter is
- bidimensional,
- satisfies the “separation property”, and
- has “finite integer index”,
then it has a linear kernel in $H$-minor-free graphs

[Fomin, Lokshtanov, Saurabh, Thilikos 2009]
Bidimensional \Rightarrow Linear Kernel

- **Protrusion:**
  - Idea: Kernelize protrusions

- **Idea:** Kernelize protrusions
Bidimensionality (version 2)
[Demaine, Fomin, Hajiaghayi, Thilikos 2004]

- Parameter $k$ is contraction-bidimensional if
  - Closed under contractions: $k$ only decreases when contracting edges
  - Large on gridoids:
    - For $r \times r$ “grid-like graphs”, $k = \Omega(r^2)$
      - Triangulated + few extra edges
Bidimensionality (version 3)
[Fomin, Golovach, Thilikos 2009]

- Parameter $k$ is contraction-bidimensional if
  - Closed under contractions: $k$ only decreases when contracting edges
  - Large on $\Gamma$ graphs:
    For naturally triangulated
    $r \times r$ grid graphs, $k = \Omega(r^2)$
Contraction-Bidimensional Problems

- Minimum maximal matching
- Face cover (planar graphs)
- Dominating set
- Edge dominating set
- $r$-dominating set
- Connected ... dominating set
- Unweighted TSP tour
- Chordal completion (fill-in)
Theorem 6: As before, obtain
- parameter-treewidth bound
- subexponential FPT
- efficient PTASs
- linear kernel

for contraction-bidimensional parameters in any graph family excluding an apex minor

Separators via Bidimensionality

- **Number of vertices** is minor-bidimensional
- Parameter-treewidth bound
  \[ \Rightarrow \text{treewidth} = O(\sqrt{k}) = O(\sqrt{n}) \]
- **Corollary:** For any fixed graph \( H \), every \( H \)-minor-free graph has treewidth \( O(\sqrt{n}) \) and hence separator of size \( O(\sqrt{n}) \)

[Alon, Seymour, Thomas 1990; Grohe 2003]
Local Treewidth via Bidimensionality

- **Diameter** is a contraction-bidimensional parameter except $\Omega(r)$, not $\Omega(r^2)$, in $r \times r$ grid
- $\Rightarrow$ Treewidth $= O(\text{diameter})$ in apex-minor-free graphs
- $\Rightarrow$ Treewidth of radius-$r$ neighborhood $= O(r)$

[Demaine & Hajiaghayi 2004]
General Graphs via Bidimensionality

- **Theorem**: Any minor-closed graph parameter
  - nonzero on some planar graph $X$,
  - at least the sum over connected components in a disconnected graph, and
  - fixed-parameter tractable with respect to treewidth can be solved in $f(k)n$ by a known algorithm [Demaine & Hajiaghayi 2007]

- **Proof sketch**:
  - Treewidth $> c_1(xk)^{c_2} \Rightarrow (xk) \times (xk)$ grid minor
    $\Rightarrow k^2$ of $x \times x$ grid minors, with $X$ minor
    $\Rightarrow \text{OPT} \geq k^2 \Rightarrow \text{answer NO}$
• **Conjecture:** Algorithms for contraction-bidimensional problems generalize to $H$-minor-free graphs, even for a non-apex graph $H$
  - True for e.g. dominating set subexponential FPT (but parameter-treewidth bound *false*) [Demaine, Fomin, Hajiaghayi, Thilikos 2004]
  - Stronger form of contraction-bidimensionality works [Fomin, Golovach, Thilikos 2009]
Beyond $H$-Minor-Free Graphs

- **Conjecture**: Most of bidimensionality theory generalizes to fixed powers of $H$-minor-free graphs, e.g., map graphs
  - Treewidth $\Omega(r^7)$
    $\Rightarrow r \times r$ grid minor
    [Demaine, Hajiaghayi, Kawarabayashi 2009]
  - Subexponential FPT for dominating set in map graphs
    [Demaine, Fomin, Hajiaghayi, Thilikos 2003]
  - Subexponential FPT & EPTASs for map graphs by “cleaning” cliques [Fomin, Lokshtanov, 2012]
Beyond *H*-Minor-Free Graphs

- **Excluded topological minors**
  - Linear kernel for (connected) dominating set
    [Fomin, Lokshtanov, Saurabh, Thilikos 2013]
  - General framework for linear kernels
    [Langer, Reidl, Rossmanith, Sikdar 2012]

- **Directed graphs**
  - Subexponential fixed-parameter algorithms for e.g. Directed Hamiltonian Path via bidimensionality
    [Dorn, Fomin, Lokshtanov, Raman, Saurabh 2010]
  - PTASs?
Beyond: Subset Problems

- What if only some of the nodes are “critical”? 
- Steiner tree, subset TSP, etc. have PTASs up to bounded-genus graphs 
  [Borradaile, Mathieu, Klein 2007; Borradaile, Demaine, Tazari 2009]
- Steiner forest has PTAS in planar graphs 
  [Bateni, Hajiaghayi, Marx 2010]
- **Wanted:** General framework
Contractions not Well-Quasi-Ordered
[Demaine, Hajiaghayi, Kawarabayashi 2009]

• No $K_{2,k}$ can be contracted into $K_{2,j}$

• Well-quasi-ordered for
  - triangulated planar graphs
  - 2-connected outerplanar graphs
  - trees

• Minor-closed $\Rightarrow$ finite excluded contractions
Simplifying Graph Decomposition
[Demaine, Hajiaghayi, Kawarabayashi 2005; DeVos et al. 2004]

- $H$-minor-free graphs can have vertices or edges partitioned into $k$ pieces such that deleting any one piece results in bounded treewidth
  - For PTAS, set $k \approx 1/\varepsilon$
  - $n^{f(H)}$ algorithm
  - $f(H)n^{O(1)}$ algorithm [Tazari 2012]

- Previously known for
  - planar graphs [Baker 1994]
  - apex-minor-free [Eppstein 2000]
Odd-$H$-minor-free graphs can have their vertices or edges partitioned into 2 pieces such that deleting any one piece results in bounded treewidth

- $n^{f(H)}$ algorithm
- $f(H) n^{O(1)}$ algorithm

[Tazari 2012]
Contraction Decomposition
[Demaine, Hajiaghayi, Kawarabayashi 2011]

- **Theorem**: $H$-minor-free graphs can have their edges partitioned into $k$ pieces such that contracting any one piece results in bounded treewidth
  - Polynomial-time algorithm
  - Previously known for
    - planar [Klein 2005, 2006]
    - bounded-genus [Demaine, Hajiaghayi, Mohar 2007]
    - apex-minor-free [Demaine, Hajiaghayi, Kawarabayashi 2009]
Applications of Contraction Decomposition

- PTAS for Traveling Salesman Problem in weighted $H$-minor-free graphs
  - Decade-old problem by [Grohe]
- PTAS for minimum-size $c$-edge-connected submultigraph
- Fixed-parameter algorithm for $k$-cut & bisection

[Demaine, Hajiaghayi, Kawarabayashi 2011]
Fixed-Parameter Algorithms via Contraction Decomposition

- **Bisection**: Cut graph into equal halves with $\leq k$ edges between them
- FPT in $H$-minor-free graphs:
  - Closed under contraction (but not minors)
  - Contraction decomposition with $k + 1$ layers avoids OPT in some contraction
  - Vertices become weighted
  - Can still solve weighted bounded-treewidth
  - $\Rightarrow$ Solve in $2^{\tilde{O}(k)} n + n^{O(1)}$ time
Fixed-Parameter Algorithms via Contraction Decomposition

- **$k$-cut**: Remove fewest edges to make at least $k$ connected components
- FPT in $H$-minor-free graphs:
  - Average degree $c_H = O(H \sqrt{\lg H})$
  - $\Rightarrow$ OPT $\leq c_H \cdot k$
  - $\Rightarrow$ Contraction decomposition with $c_H \cdot k + 1$ layers avoids OPT in some contraction
  - $\Rightarrow$ Solve in $2^{\tilde{O}(k)} \cdot n + n^{O(1)}$ time
- Generalization to arbitrary graphs
  [Kawarabayashi & Thorup 2011]
Start with **tight** precolored edges:

- Treewidth of radius-\( r \) neighborhood is \( f(r) \)
- \( c = O(1) \) induced connected components

Color edges at radial distance \( r \) with color \( r \mod k \)
Radial Coloring for Bounded Genus

[Demaine, Hajiaghayi, Kawarabayashi 2011]

• Contract color \( i \)
  ▪ Each connected component → articulation point
  ▪ Split these apart → **blobs**, connected in DAG

• Outdegree > 1 \( \Rightarrow \text{split} \)

• Indegree > 1 \( \Rightarrow \text{rejoin} \)
  ▪ \( \Rightarrow \) Indegree \( \leq c + g \)
    (initial component or handle)
Radial Coloring for Bounded Genus

[Demaine, Hajiaghayi, Kawarabayashi 2011]

- Split nonroot blob into \( \leq c + g \) chunks by closest BFS seed
- Each chunk has radial diameter \( O(k) \)
- Nonroot blob has radial diameter, so treewidth \( O(k (c + g)) \)  
  [Eppstein 2000]
- Blob DAG = tree + \( \leq g \) extra edges
  - Tree of 1-sums doesn’t affect treewidth
  - Add extra edges to all bags \( \Rightarrow +O(g) \)
Contraction Decomp. for $H$-Minor-Free

- $H$-minor-free graph = “tree” of “almost-embeddable graphs” [GM16]

- Each almost-embeddable graph has contraction decomposition:
  - Bounded genus done
  - Apices easy: increase treewidth of anything by $O(1)$
  - Vortices similar

[Demaine, Hajiaghayi, Mohar 2007]
Tree “joins” between two graphs are clique sums:

- Attach at clique of same size
- Delete any desired edges

\[ G = G_1 \oplus G_2 \]
Clique sums either:

- Vortices $\leftrightarrow$ apices
  $\Rightarrow$ no contraction necessary
- $K_3$ in bounded-genus part
  (hard part)

$G = G_1 \oplus G_2$
Solution Idea
[Demaine, Hajiaghayi, Kawarabayashi 2011]

- Find a path in child $G_2$ to simulate the effect of each deleted edge in $K_3$
  - Can make pieces $G_i$ 3-edge-connected
- Need these paths to all contract together, by putting them all in piece 1
Solution Idea

[Demaine, Hajiaghayi, Kawarabayashi 2011]

• **Claim:** Precoloring $O(1)$ **shortest paths** doesn’t hurt contraction decomposition

• Shortcut paths to shortest paths (in radial graph $\approx$ primal + dual)
  - Two paths $a \rightarrow b$; one path $c \rightarrow a-b$ path
**Theorem:** Treewidth of radius-\(r\) neighborhood of \(O(1)\) shortest paths is \(f(r)\)

- If not, large grid within the neighborhood
- **Intuition:** Then there should be a shortcut: \(2r \ll f(r)\)
Theorem: Treewidth of radius-$r$ neighborhood of $O(1)$ shortest paths is $f(r)$

- Path is $r$-dive into $\Theta(r) \times \Theta(r)$ wall
- $\Rightarrow r$-rainbow
- By Menger’s Theorem, get $\Theta(r)$ “cross paths”
- Build $\Omega(r) \times \Omega(r)$ wall $\geq 2r$ away from shortest path
- Repeat for each path
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