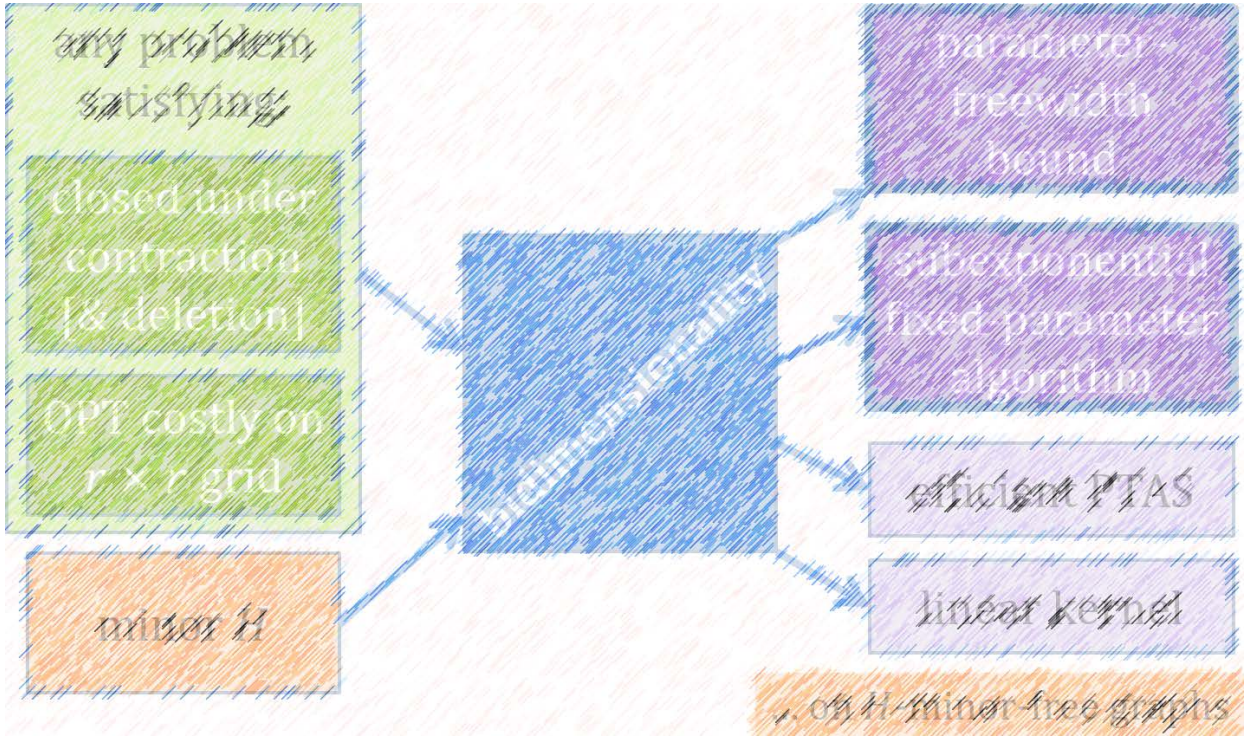
Minors & Algorithms

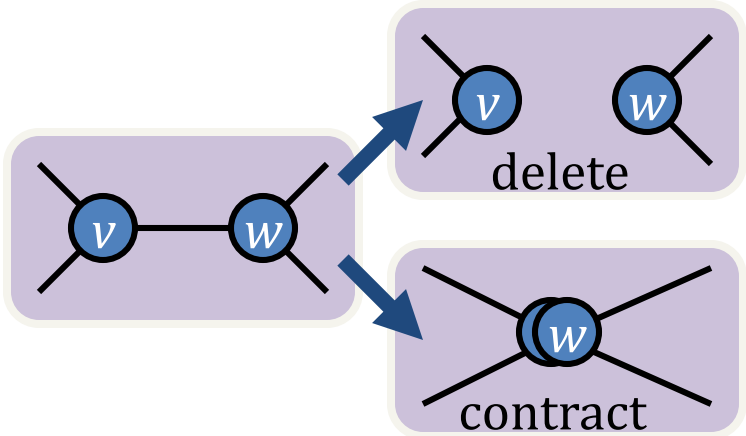


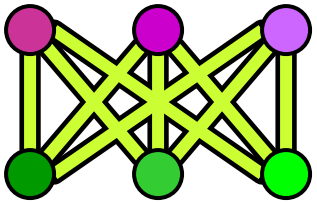
Erik Demaine

M.I.T.

Minors

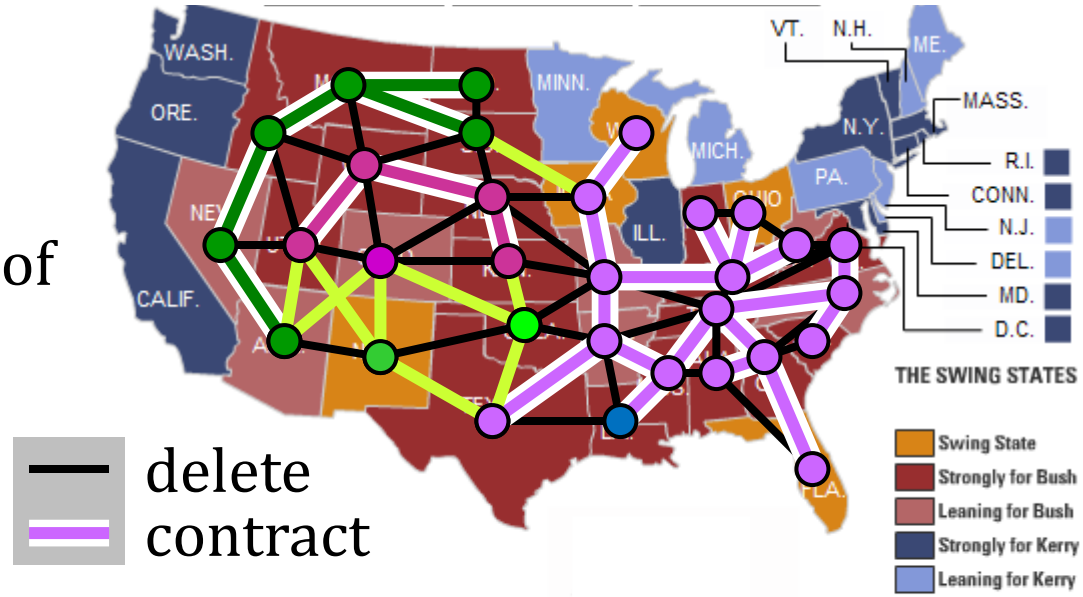
- H is a **minor** of G if G can reach H via
  - edge deletions
  - edge contractions





$K_{3,3}$

is a minor of



Graph Minors [Robertson & Seymour 1983–]

JOURNAL OF COMBINATORIAL THEORY, Series B 35, 39–61 (1983)

Graph Minors. I. Excluding a Forest

NEIL ROBERTSON AND P. D. SEYMOUR*

Department of Mathematics, Ohio Stat
231 W. 18th Ave, Columbus, Ohio

Communicated by the Editor

Received May 3, 1982

The path-width of a graph is the minimum value of
obtained from a sequence of graphs $G_1, \ldots, G_r$ each
vertices, by identifying some vertices of G_i pairwise wi
For every forest H it is proved that there is a number
no minor isomorphic to H has path-width $\leqslant k$. This, t
papers, yields a "good" algorithm to test for the prese
minor, and implies that if P is any property of graphs
not have property P, then the set of minor-minimal g
finite.

ELSEVIER

Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Combinatorial Theory, Series B 92 (2004) 325–357

Journal of Combinatorial Theory Series B

www.elsevier.com/locate/jctb

Graph Minors. XX. Wagner's conjecture

Neil Robertson[a,1], P.D. Seymour[b,2]

[a] *Department of Mathematics, Ohio State University, 231 West 18th Ave., Columbus, OH 43210, USA*
[b] *Telcordia Technologies, 445 South St., Morristown, NJ 07960, USA*

Received 13 February 2001

Abstract

We prove Wagner's conjecture, that for every infinite set of finite graphs, one of its members is isomorphic to a minor of another.

Keywords: Graph; Minor; Surface embedding; Well-quasi-ordering

Wagner's Conjecture
[Robertson & Seymour 2004]

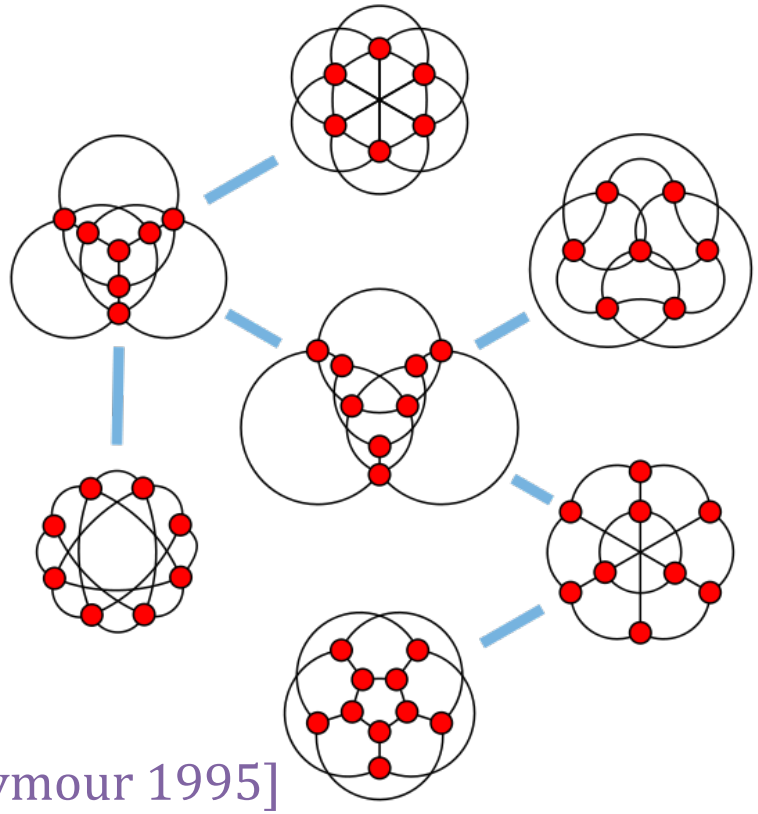
- Minor relation is a **well-quasi-ordering**: Every **infinite** graph sequence $G_1, G_2, G_3, \ldots$ has some G_i a minor of G_j (with $i < j$)

...or equivalently...

- Every **minor-closed** graph family is characterized by a **finite** set of excluded minors
  - <u>Proof by contradiction:</u> Infinite excluded minors would have a minor relation among them
  - In some sense necessarily **nonconstructive** [Friedman, Robertson, Seymour 1987]

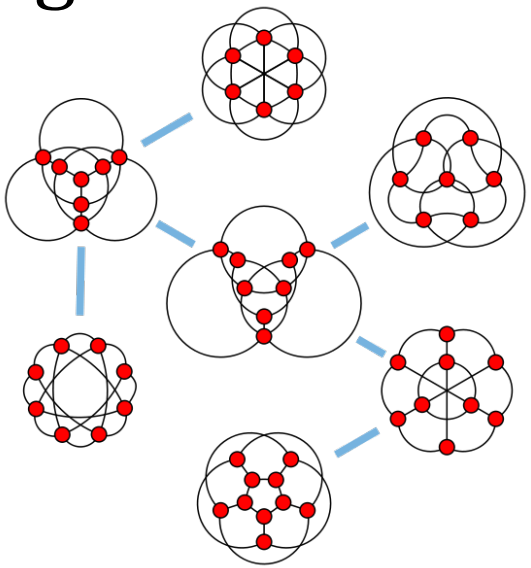
Wagner's Conjecture
[Robertson & Seymour 2004]

- Every **minor-closed** graph family is characterized by a **finite** set of excluded minors
- <u>Examples:</u>
  - **Forests:** K_3
  - **Outerplanar:** K_4 & $K_{2,3}$
  - **Series-parallel:** K_4
  - **Planar:** K_5 & $K_{3,3}$ [Wagner's 1937]
  - **Linklessly embeddable:** Petersen family [Robertson & Seymour 1995]



Algorithms via Minors

- <u>Theorem:</u> Polynomial test for a fixed minor H
  - $f(H)\, n^3$ [GM XII, 1995]
  - $f(H)\, n^2$ [Kawarabayashi, Kobayashi, Reed 2012]
- <u>Corollary:</u> Any minor-closed graph property has a polynomial-time decision algorithm
  - But algorithm is **existential:** in general, don't know *which* excluded minors to test for



Warning:
Graph Minor Constants

JOURNAL OF COMBINATORIAL THEORY, Series B 41, 92–114 (1986)

Graph Minors. V. Excluding a Planar Graph

NEIL ROBERTSON

Department of Mathematics, The Ohio State University, Columbus, Ohio 43210

AND

P. D. SEYMOUR*

Bell Communications Research, Morristown, New Jersey 07960

Communicated by the Managing Editors

Received February 23, 1984

- <u>Example:</u> Planar minor testing [GM5 1986], analyzed by David Johnson [1987]
  - Running time $= O(n^3)$
  - Lead constant $\approx 2^{2^{2^{x}}}, x = 2^{2^{2^{\cdot^{\cdot^{\cdot}}}}} \Big\} 2^{2^{2^{\cdot^{\cdot^{\cdot}}}}} \Big\} \frac{1}{2}|V(H)|$

"1. THE CASE OF THE HIDDEN CONSTANTS

···Unfortunately, for any instance $G = (V,E)$ that one could fit into the known universe, one would easily prefer $|V|^{70}$ to even *constant* time, if that constant had to be one of Robertson and Seymour's."

The NP-Completeness Column: An Ongoing Guide

DAVID S. JOHNSON

AT&T Bell Laboratories, Murray Hill, New Jersey 07974

Fixed-Parameter Algorithms via Minors

- Graph parameter is **minor-closed** if it only decreases when deleting or contracting
  - <u>Examples:</u> vertex cover, feedback vertex set
- <u>Corollary:</u> Any minor-closed graph parameter has an $f(k)\, n^2$ algorithm for deciding $\leq k$
  - "Fixed-parameter tractable"
  - But a **different**, unknown algorithm for each k
  - Not really a (uniform) fixed-parameter algorithm

[Fellows & Langston 1988]

Fixed-Parameter Algorithms via Minors

- <u>Theorem:</u> Any minor-closed graph parameter
  - nonzero on some planar graph,
  - at least the sum over connected components in a disconnected graph, and
  - fixed-parameter tractable with respect to treewidth

can be solved in $f(k)\,n$ by a known algorithm [Demaine & Hajiaghayi 2007]

What Are H-minor-free Graphs Like?

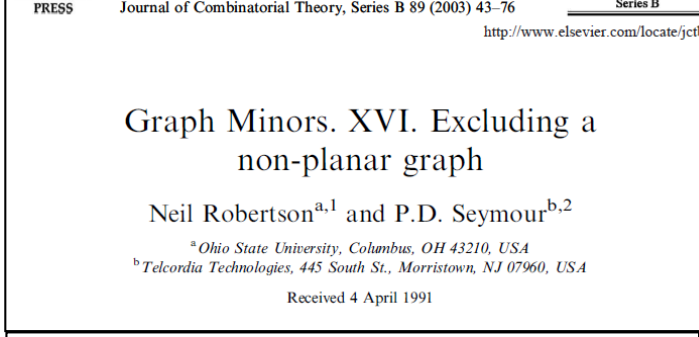
- Bounded average **degree** (& degeneracy):
$$O\left(|V(H)|\sqrt{\log|V(H)|}\right)$$
[Kostochka 1982; Thomason 1984]

- $O(\sqrt{n})$ **treewidth** [Grohe 2003]
- $\Rightarrow O(\sqrt{n})$ **balanced separators**
(like Lipton-Tarjan planar separator theorem)
[Alon, Seymour, Thomas 1990]

- Kinda like planar graphs?

H-Minor-Free Graphs

[Robertson & Seymour 2003]


PRESS Journal of Combinatorial Theory, Series B 89 (2003) 43–76 Series B

http://www.elsevier.com/locate/jctb

Graph Minors. XVI. Excluding a non-planar graph

Neil Robertson[a,1] and P.D. Seymour[b,2]

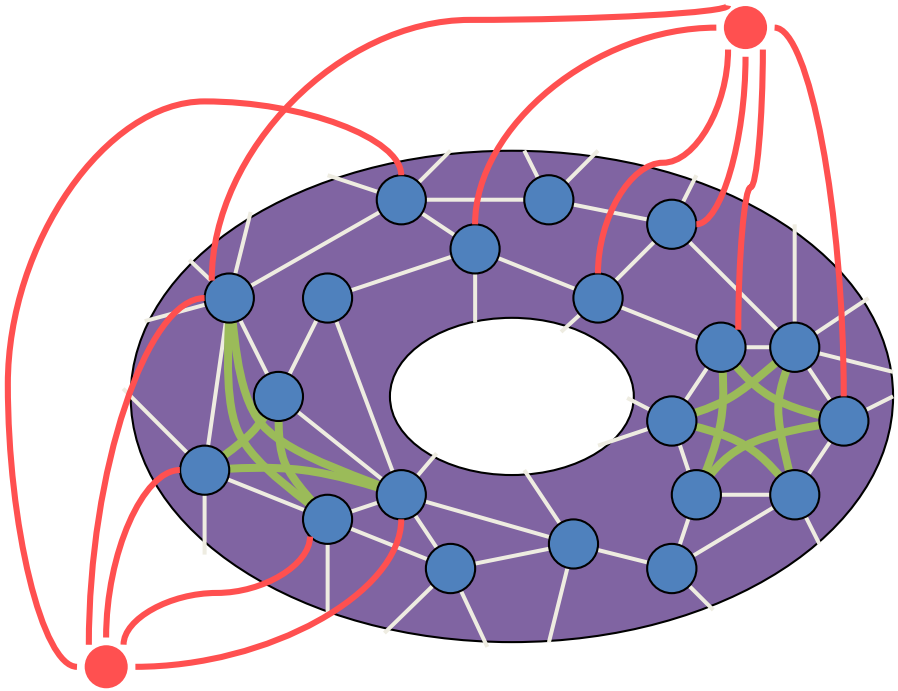
[a] Ohio State University, Columbus, OH 43210, USA
[b] Telcordia Technologies, 445 South St., Morristown, NJ 07960, USA

Received 4 April 1991

This paper contains the corner stone theorem of the series.

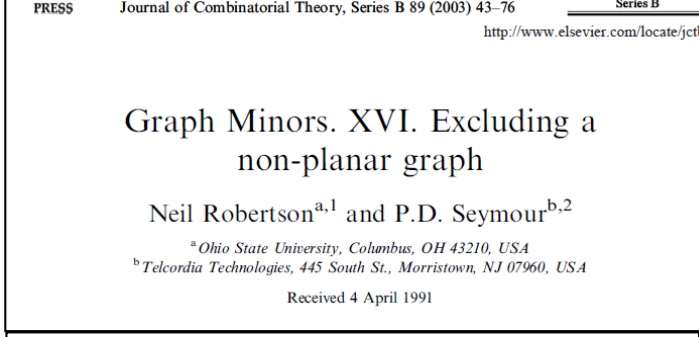
Structure Theorem:

- H-minor-free graph looks like a tree of "**almost embeddable**" graphs:
  - Base graph drawn on surface of genus $f(H)$
  - $f(H)$ **vortex** faces filled with graphs of pathwidth $f(H)$
  - $f(H)$ **apex** vertices connected to anything



H-Minor-Free Graphs

[Robertson & Seymour 2003]


PRESS Journal of Combinatorial Theory, Series B 89 (2003) 43–76 Series B

http://www.elsevier.com/locate/jctb

Graph Minors. XVI. Excluding a non-planar graph

Neil Robertson[a,1] and P.D. Seymour[b,2]

[a] Ohio State University, Columbus, OH 43210, USA
[b] Telcordia Technologies, 445 South St., Morristown, NJ 07960, USA

Received 4 April 1991

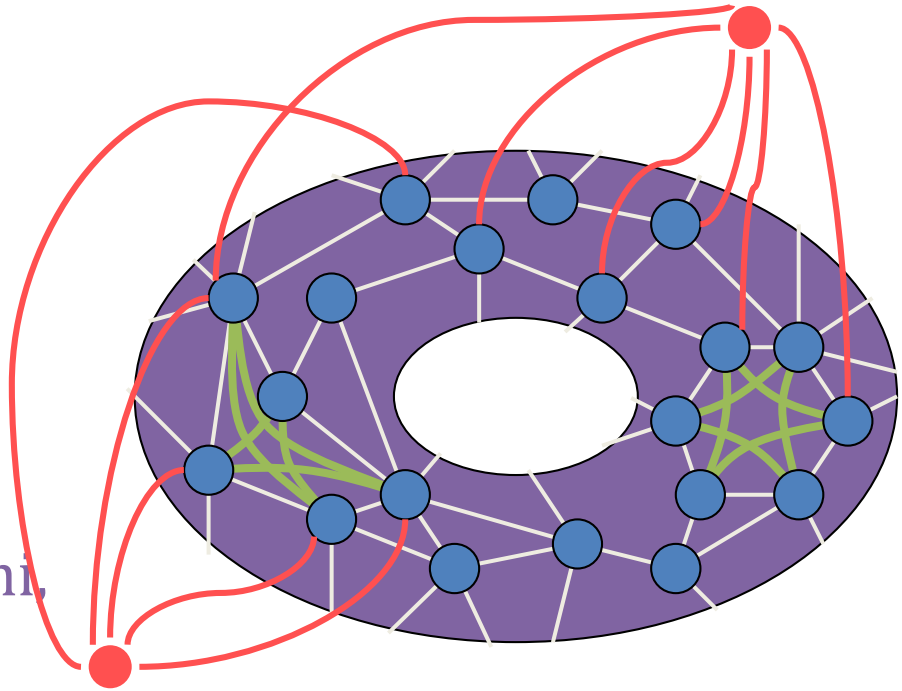
This paper contains the corner stone theorem of the series.

Structure Theorem:

- H-minor-free graph looks like a tree of "**almost embeddable**" graphs

Structure Algorithms:

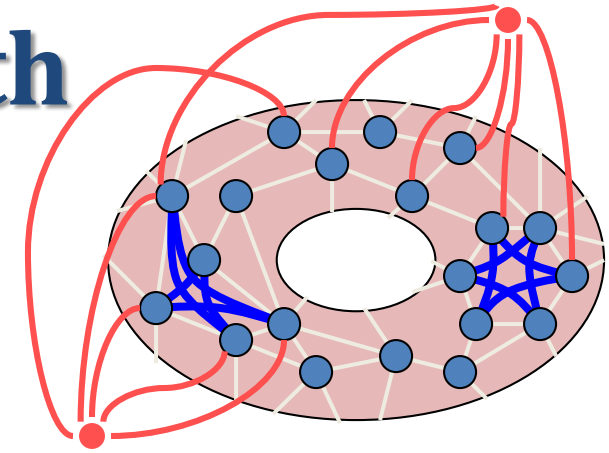
- $n^{f(H)}$ [Demaine, Hajiaghayi, Kawarabayashi — FOCS 2005]
- $O(n^3)$ [Kawarabayashi & Wollan — STOC 2011]
- $O(n^2)$ [Grohe, Kawarabayashi, Reed — SODA 2013]



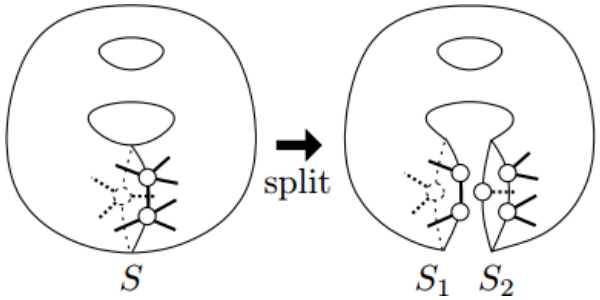
Grid Minors vs. Treewidth

- Every H-minor-free graph of treewidth $\geq f(H)\, r$ has an $r \times r$ grid minor [Demaine & Hajiaghayi 2005]
  - Previous bounds exponential in r (and H) [GM5, ...]
  - $f(H) = |V(H)|^{O(|E(H)|)}$ [Kawarabayashi & Kobayashi 2012]
- Every graph of treewidth $\geq c_1\, r^{c_2}$ has an $r \times r$ grid minor [Chekuri & Chuzhoy 2013]
  - $c_2 > 2$ necessary [Robertson, Seymour, Thomas 1994]
  - Previous bounds exponential in r [RST94]

Grid Minors vs. Treewidth in *H*-Minor-Free Graphs

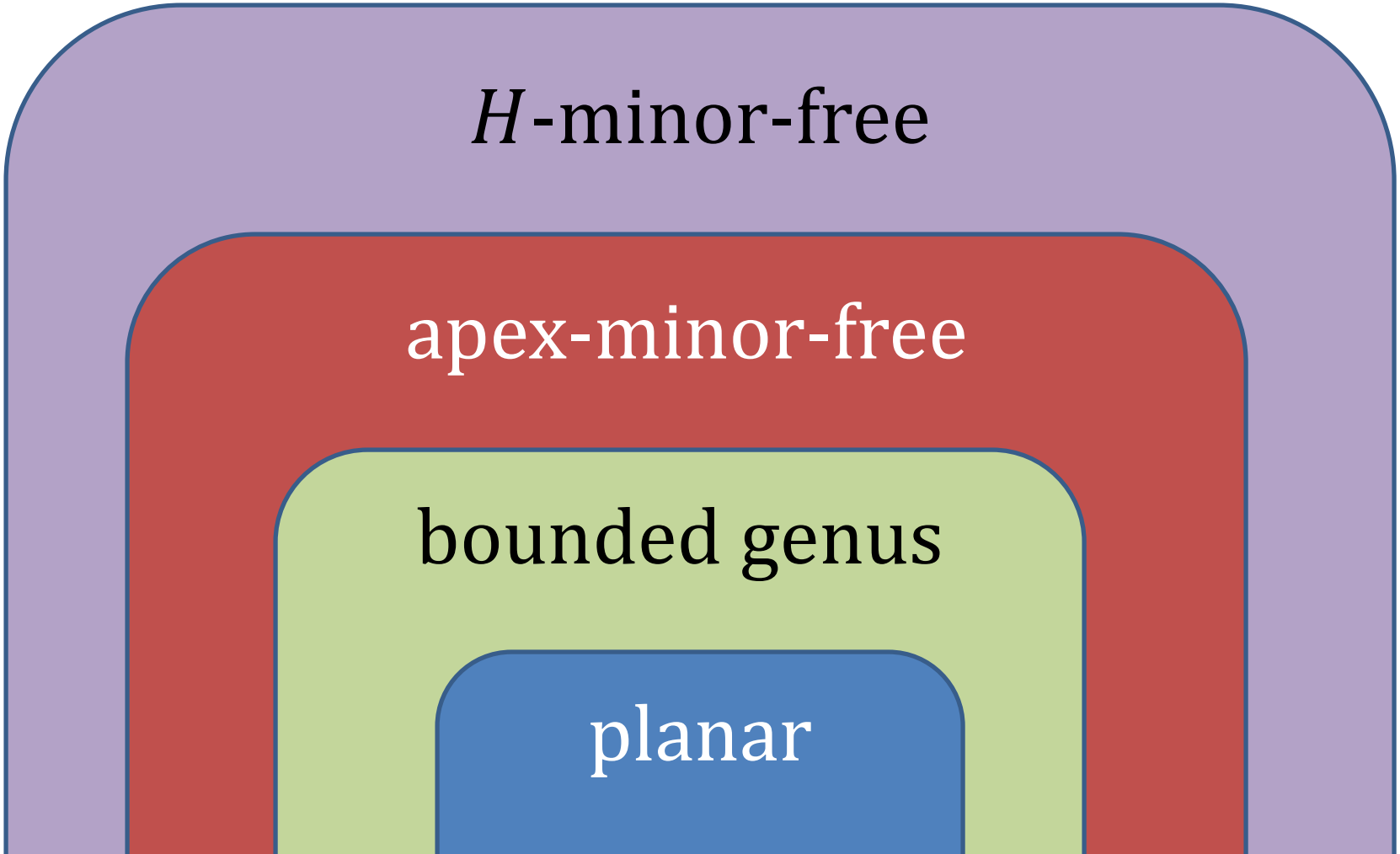


- Almost-embeddable graphs:
  - Remove apices by deletion
  - Contract each vortex to a vertex
  - Pull apart each handle
    [Demaine, Fomin, Hajiaghayi, Thilikos 2004]
  - Planar graph has large grid minor [RST94]

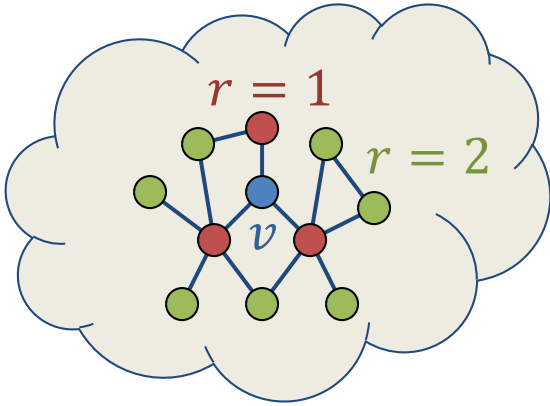


- Tree takes max of treewidths, so one almost-embeddable graph has full treewidth
  - But may not be H-minor-free from tree joins
  - "Approximate" graph to make minor of original graph ⇒ H-minor-free [Demaine & Hajiaghayi 2005]

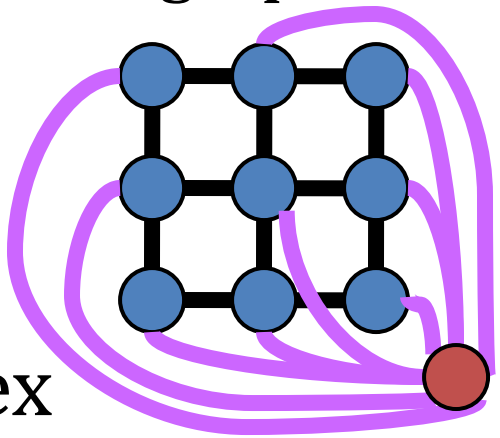
Graph Minor Hierarchy



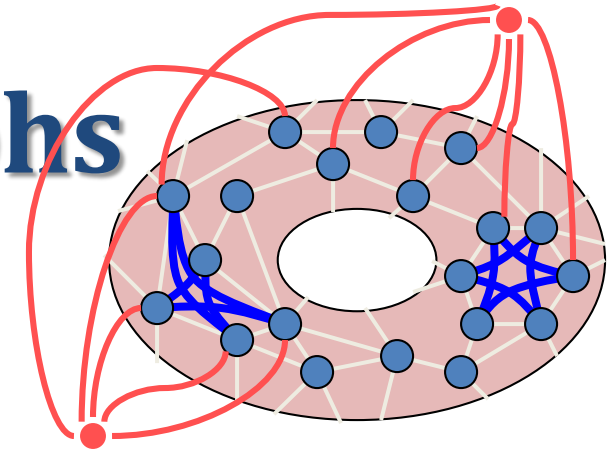
Apex-Minor-Free Graphs



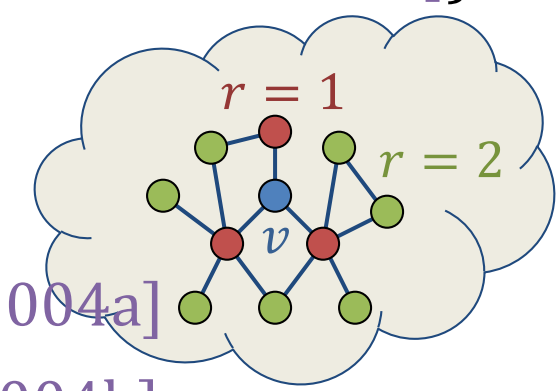
- **Bounded local treewidth =** radius-r neighborhood around every vertex v has treewidth $\leq f(r)$
  - <u>Examples:</u> **planar** and **bounded-genus** graphs
- Minor-closed graph family has bounded local treewidth $\Leftrightarrow$ excludes an **apex graph** = minor of planar graph + one vertex [Eppstein 2000]
  - Call such graphs **apex-minor-free**



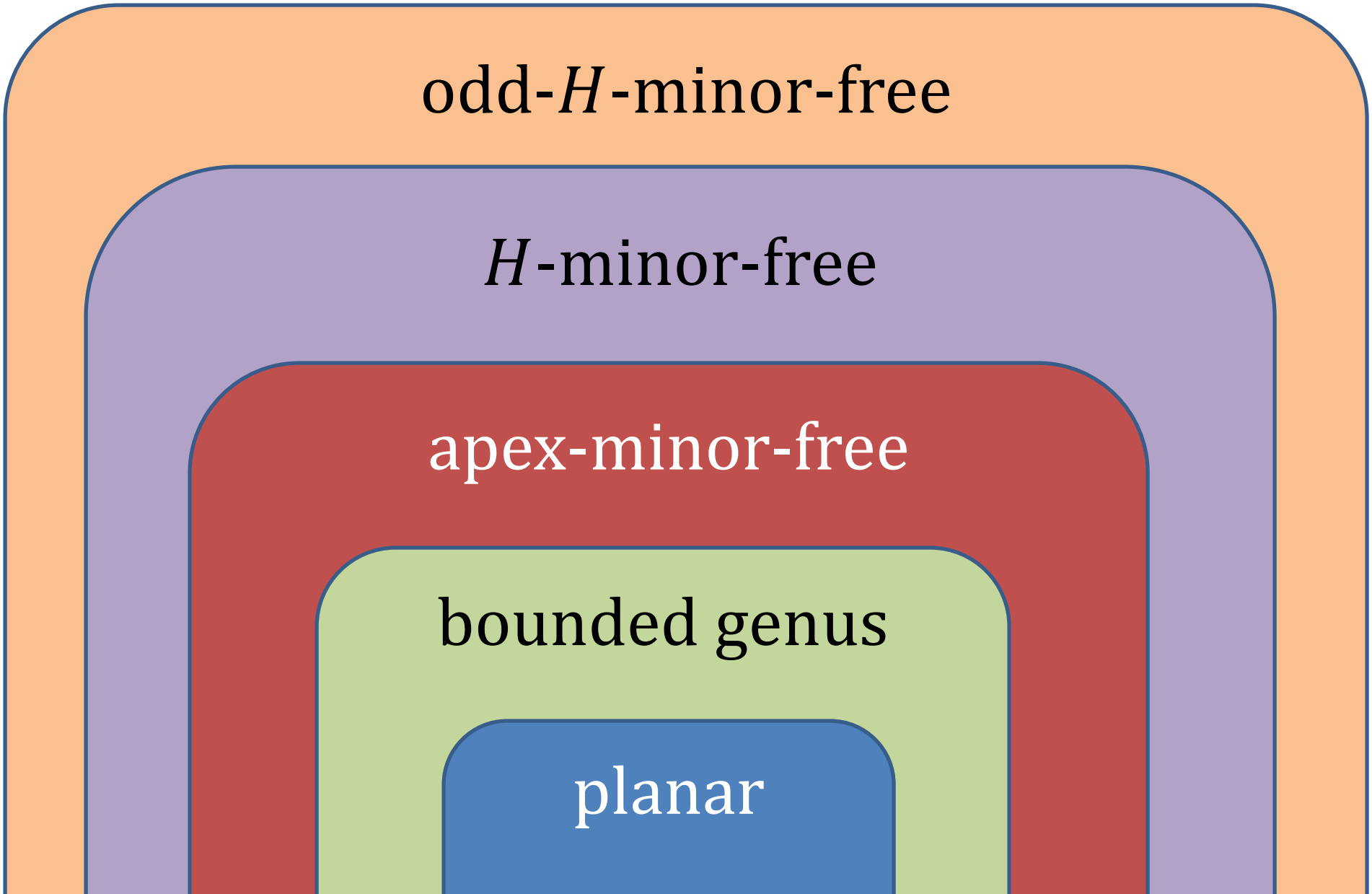
Apex-Minor-Free Graphs



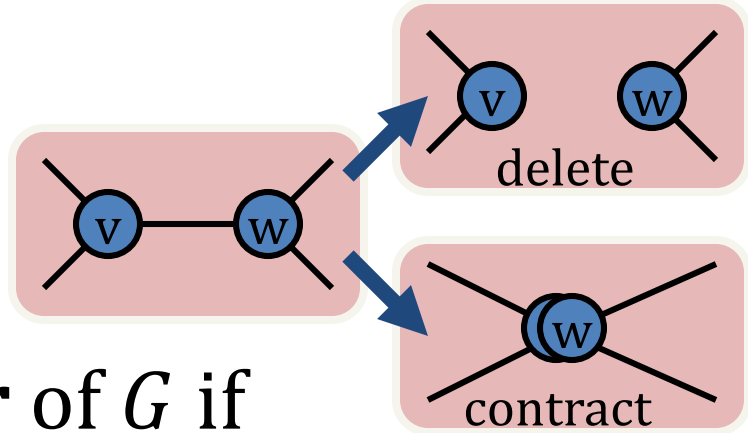
- <u>Structure Theorem:</u>
  Apices attach only to vortices,
  allowing bounded-treewidth vortices
  [Demaine, Hajiaghayi, Kawarabayashi 2009]
  - Additive +2 approximation for chromatic number (extending from bounded genus [Thomassen 1997])
- Bounded local treewidth
  - $f(r) = 2^{2^{O(r)}}$ [Eppstein 2000]
  - $f(r) = 2^{O(r)}$ [Demaine & Hajiaghayi 2004a]
  - $f(r) = O(r)$ [Demaine & Hajiaghayi 2004b]
  - [EPTASs: $2^{2^{2^{O(1/\varepsilon)}}} n^{O(1)} \to 2^{O(1/\varepsilon)} n^{O(1)}$]



Beyond Graph Minors

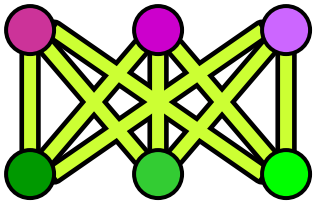


Minors



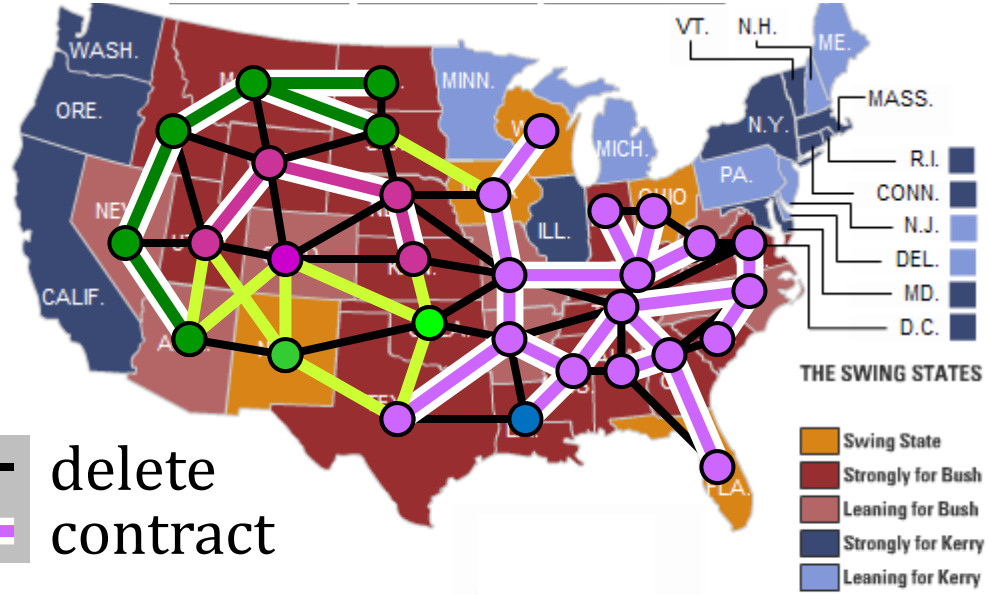
- Equivalently, H is a **minor** of G if
  - Every vertex v in H has a corresp. tree T_v in G
  - Any edge (v, w) in H has a corresponding edge connecting T_v to T_w in G

"model"



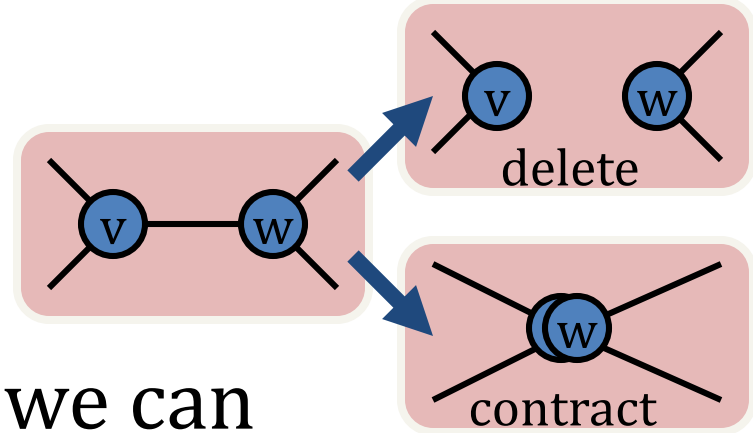
$K_{3,3}$

is a minor of

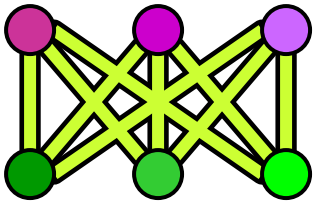


delete
contract

Odd Minors

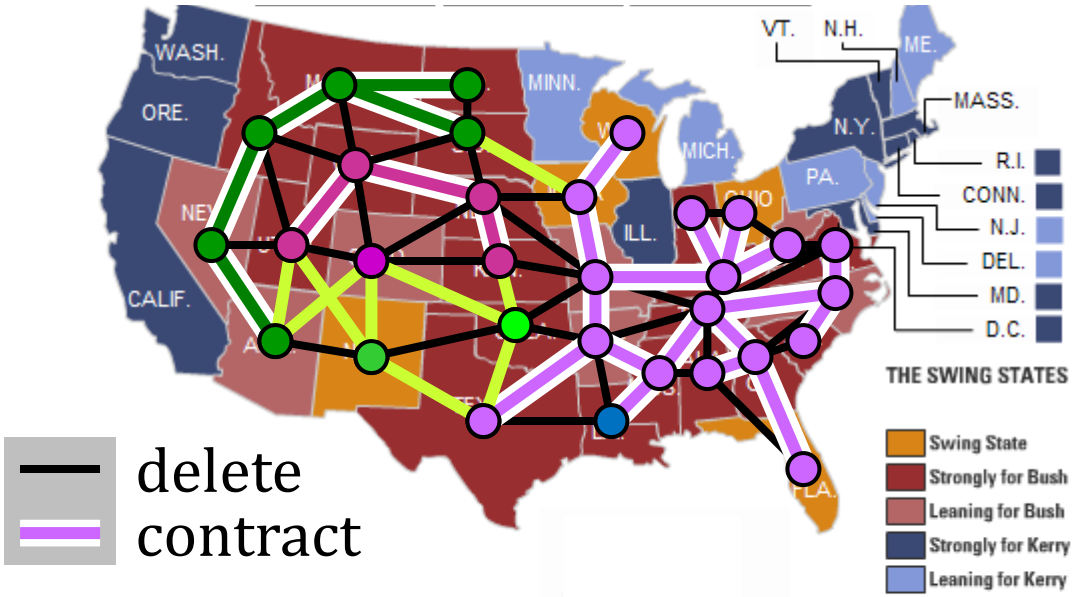


- H is an **odd minor** of G if we can also **2-color** the vertices in G so that
  - Tree edges are **bichromatic**
  - T_v-to-T_w edges are **monochromatic**

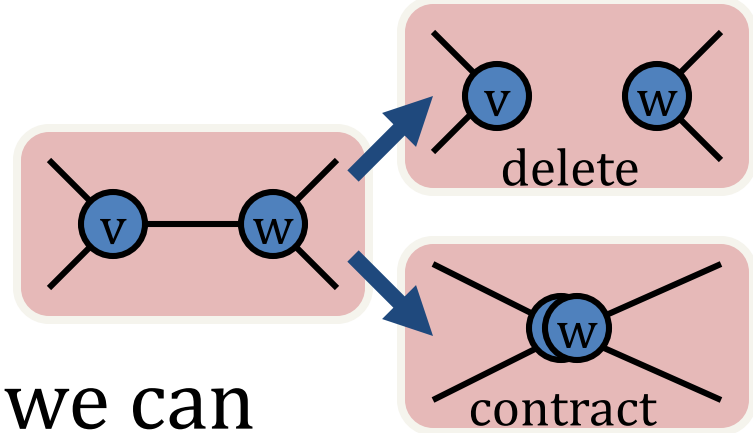


$K_{3,3}$

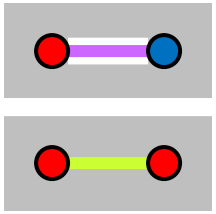
is an odd minor of

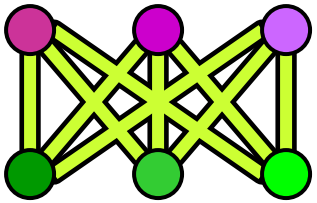


Odd Minors



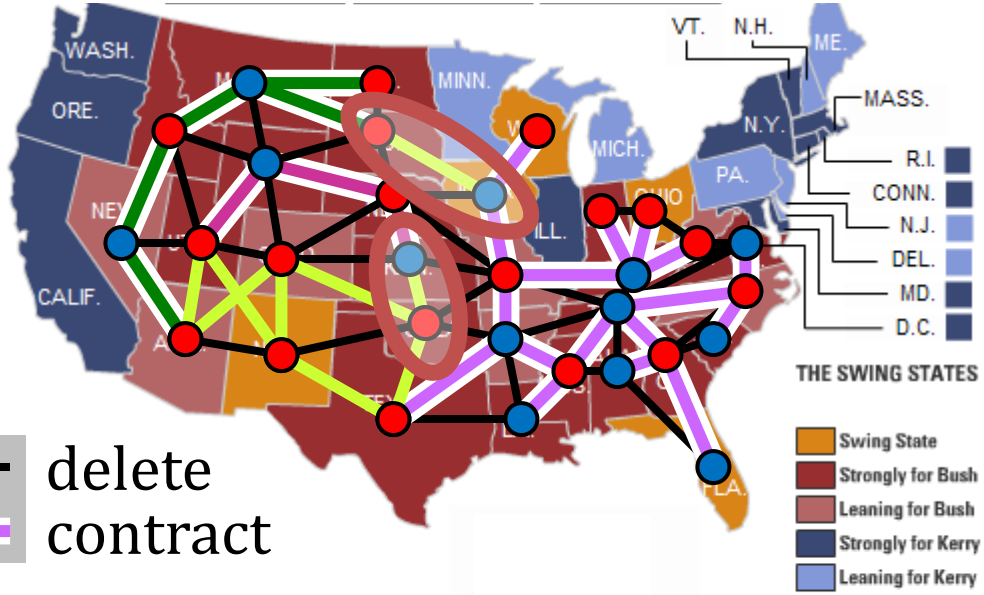
- H is an **odd minor** of G if we can also **2-color** the vertices in G so that
  - Tree edges are **bichromatic**
  - T_v-to-T_w edges are **monochromatic**





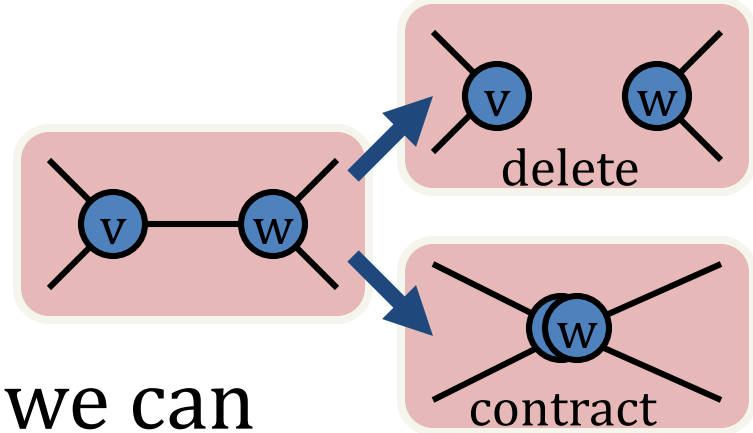
$K_{3,3}$

is an odd minor of

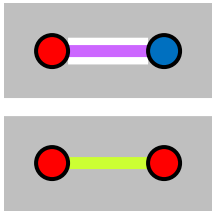


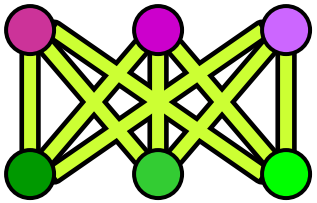
delete
contract

Odd Minors



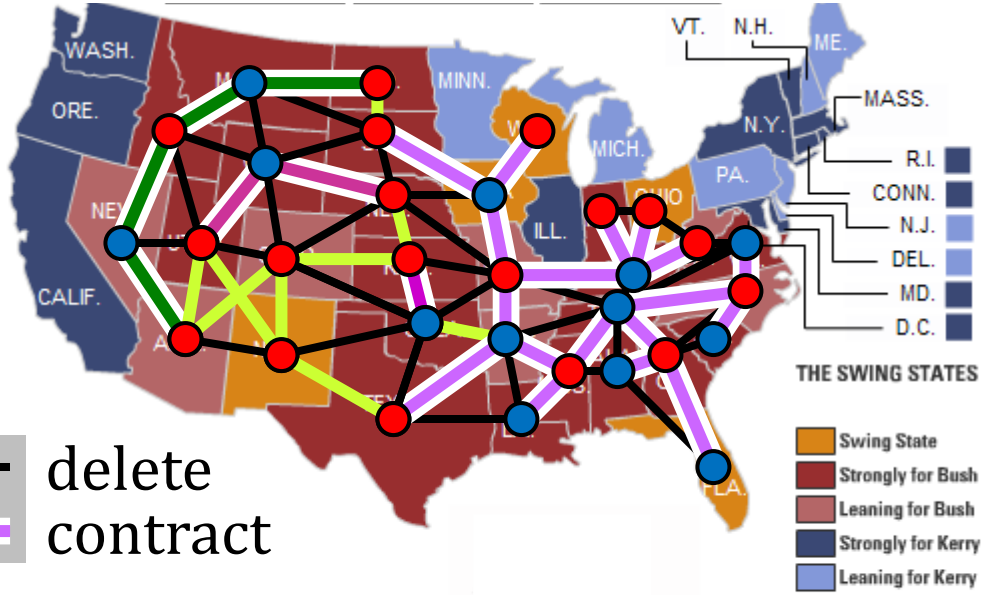
- H is an **odd minor** of G if we can also **2-color** the vertices in G so that
  - Tree edges are **bichromatic**
  - T_v-to-T_w edges are **monochromatic**

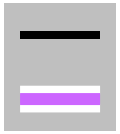




$K_{3,3}$

is an odd minor of



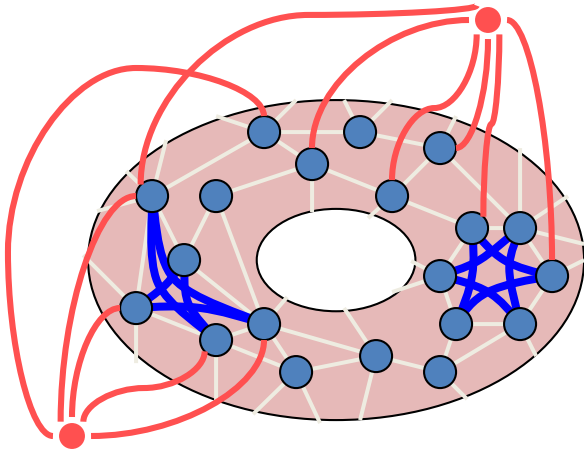


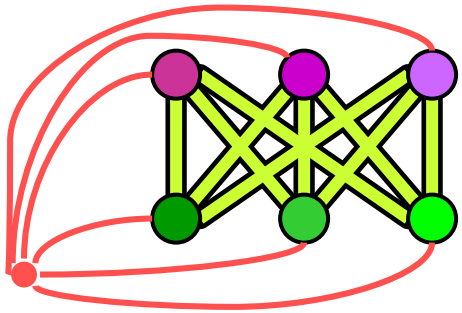
delete
contract

Odd-Minor-Free Graphs

[Demaine, Hajiaghayi, Kawarabayashi 2010]

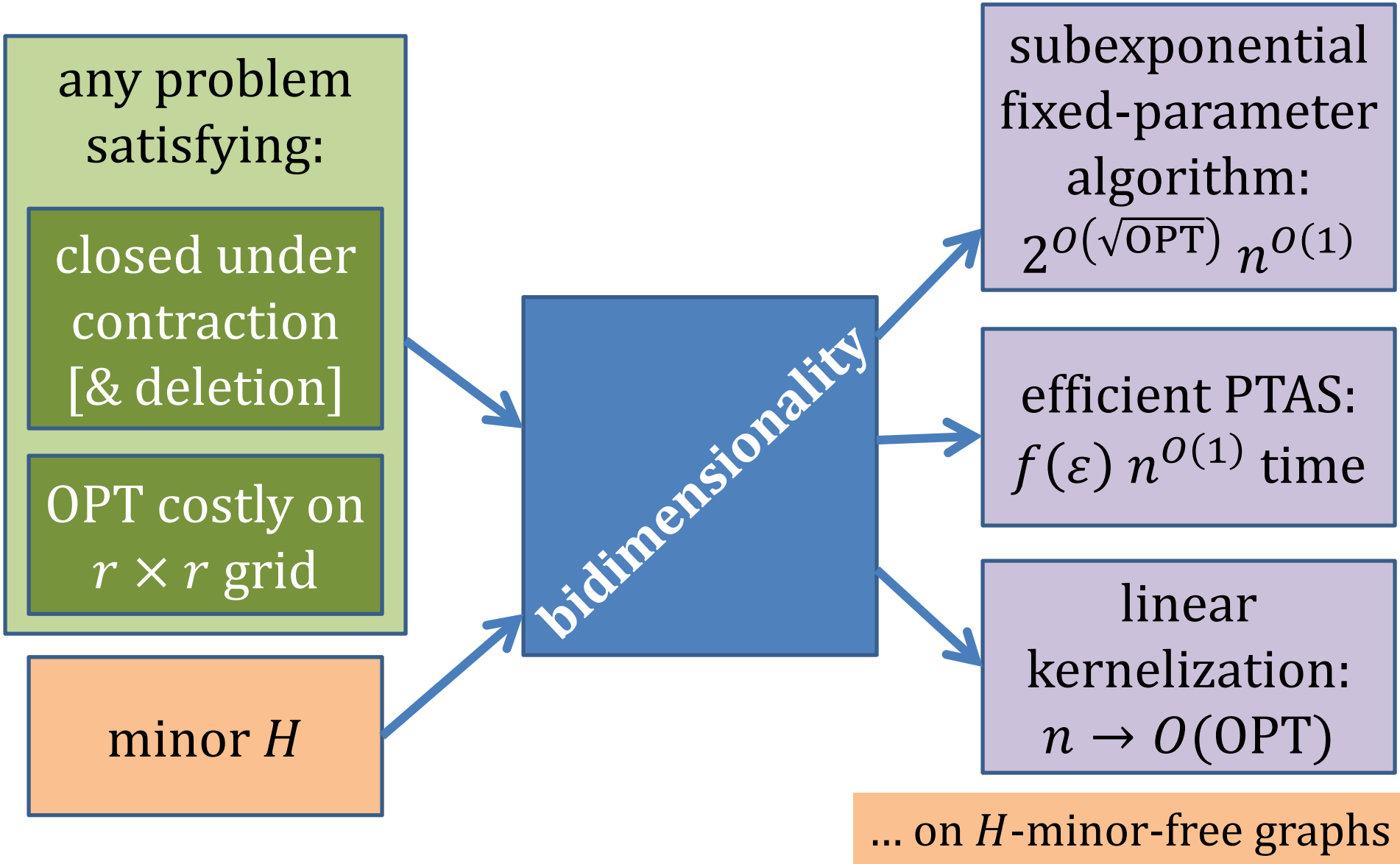
- <u>Structure Theorem:</u> Every **odd**-H-minor-free graph can be written as a tree of graphs joined along $f(H)$-size cliques, where each term is
  - Bounded-genus graph
    + $f(H)$ vortices
    + $f(H)$ apices
    (joined at only 3 surface vertices)
  - **Bipartite graph**
    + $f(H)$ apices
    (joined at only 1 bipartite vertex)





- $n^{f(H)}$ algorithm
  - $f(H)\, n^{O(1)}$ simpler decomp. [Tazari 2012]

Bidimensionality Overview



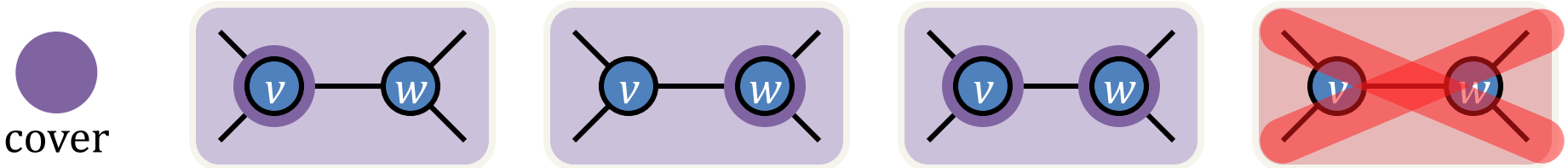
History of Subexponential FPT

- In the early days, most fixed-parameter algorithms ran in $2^{O(k^c)}\, n^{O(1)}$ time, for $c \geq 1$

- <u>Natural question:</u> Is $2^{o(k)}\, n^{O(1)}$ possible?

History of Subexponential FPT

- **Vertex Cover:** choose smallest set of vertices to cover every edge (on either endpoint)



- Vertex Cover can be kernelized to $\leq 2k$ vertices
- Lipton & Tarjan's separator approach [1980] solves Planar Independent Set in $2^{O(\sqrt{n})}$ time
  - Min. Vertex Cover + Max. Independent Set $= |V|$
- $\Rightarrow n^{O(1)} + 2^{O(\sqrt{k})}$ time

History of Subexponential FPT

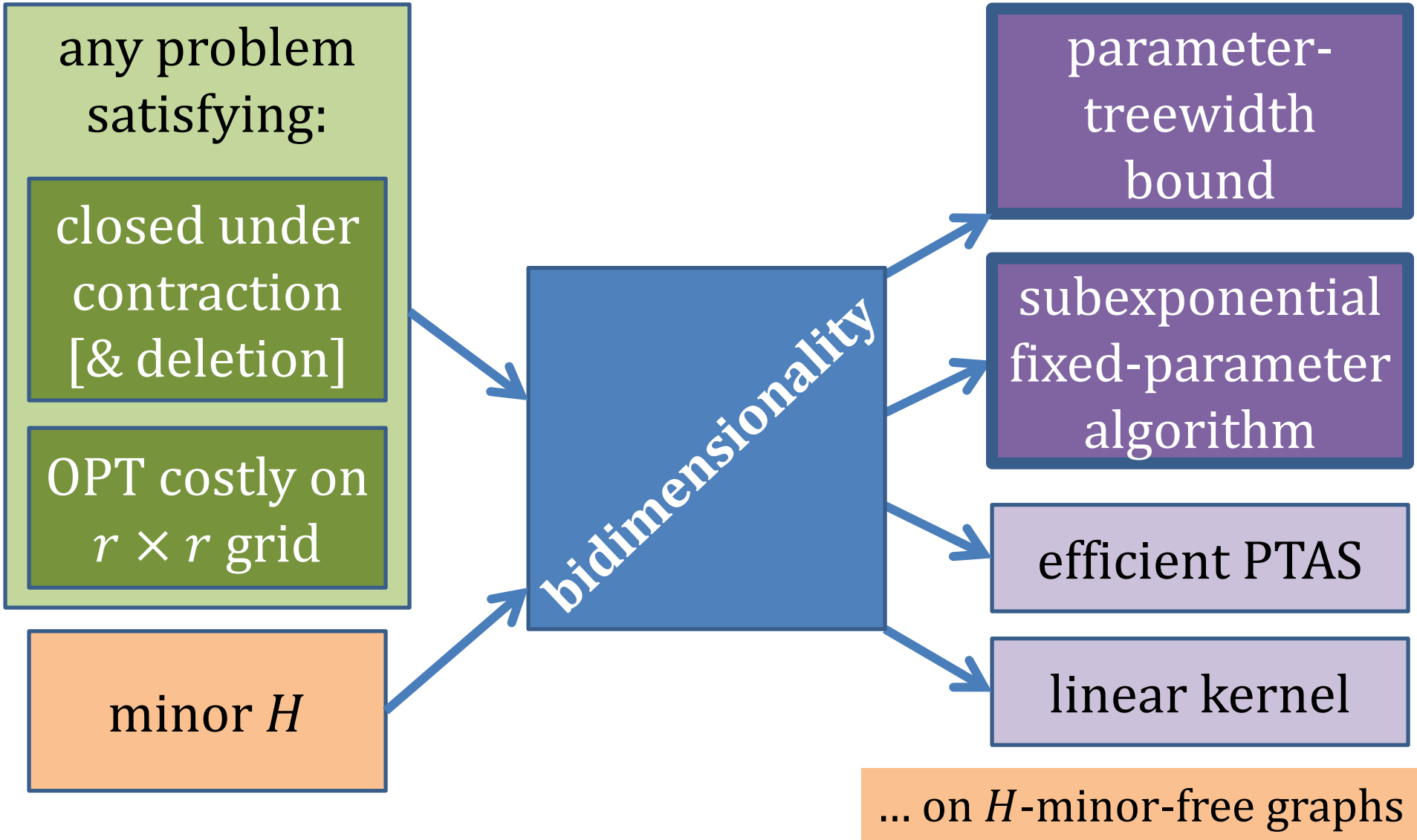
- What about problems like these?
  - **Dominating Set**: Cover all vertices with k vertex neighborhoods
  - **Feedback Vertex Set**: Cover cycles with k vertices
  - **Long Path**: Is there a simple path of length $\geq k$?
- No linear (or even polynomial) kernel was known for these problems then
- We now know that some, e.g. Long Path, have no polynomial kernel (unless NP $\subseteq$ coNP/poly), even for planar graphs [Bodlaender, Downey, Fellows, Hermelin 2009]

History of Subexponential FPT

- $2^{O(\sqrt{k})}\, n^{O(1)}$ for planar Dominating Set
  [Alber, Bodlaender, Fernau, Kloks, Niedermeier 2000]

- Same approach extended to Dominating Set variations, Feedback Vertex Set, etc.
  [Alber et al. 2002; Kanj & Perković 2002; Fomin & Thilikos 2003; Alber, Fernau, Niedermeier 2004; Chang, Kloks, Lee 2001; Kloks, Lee, Liu 2002; Gutin, Kloks, Lee 2001]

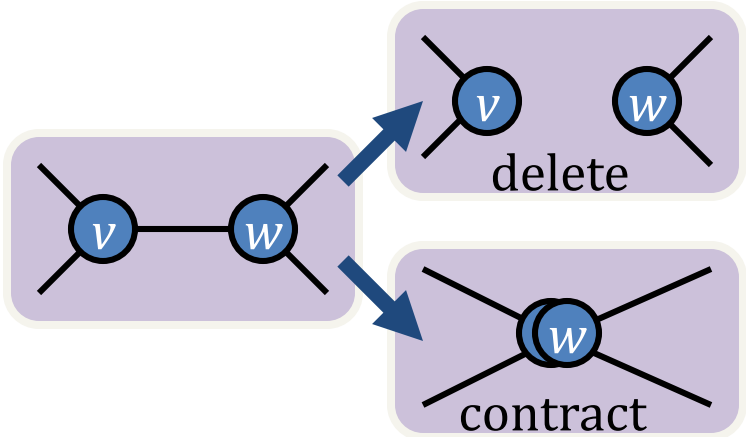
Bidimensionality Overview



Bidimensionality (version 1)

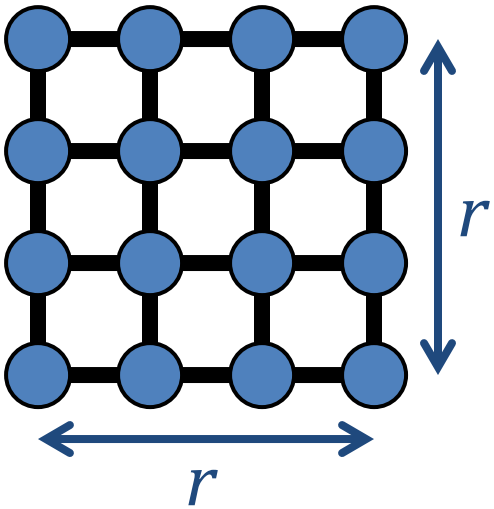
[Demaine, Fomin, Hajiaghayi, Thilikos 2004]

- Parameter $k = k(G)$ is **bidimensional** if
  - **Closed under minors:** k only decreases when deleting or contracting edges



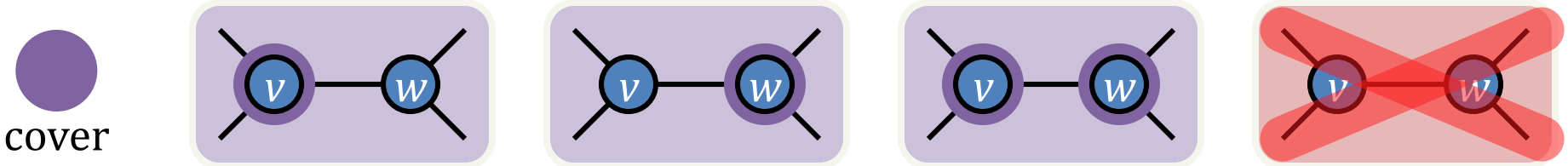
and

  - **Large on grids:** For the $r \times r$ grid, $k = \Omega(r^2)$

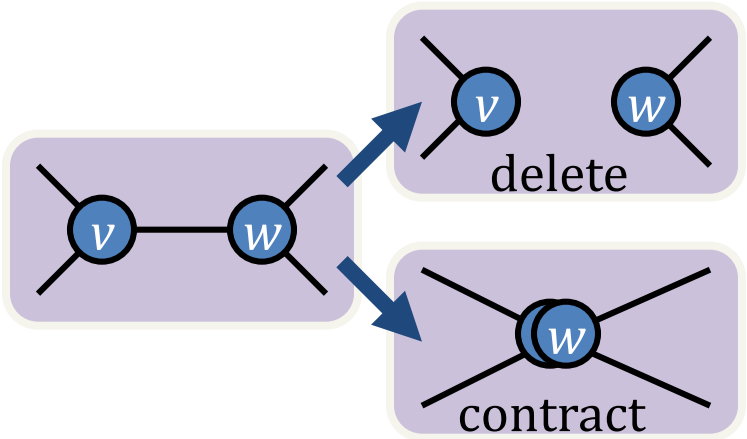


Example 1: Vertex Cover

- k = minimum number of vertices required to cover every edge (on either endpoint)



- **Closed under minors:**

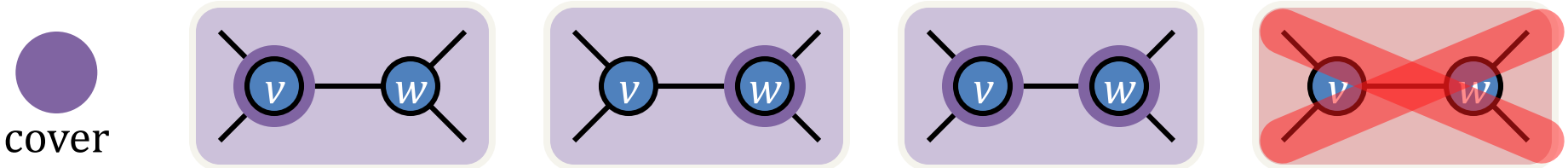


⇒ still a cover (only fewer edges)

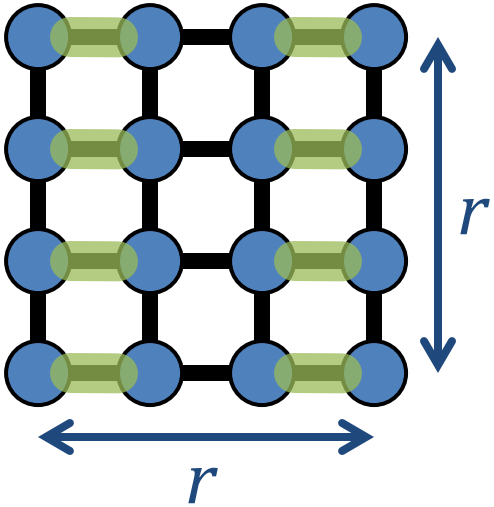
⇒ still a cover, possibly 1 smaller

Example 1: Vertex Cover

- k = minimum number of vertices required to cover every edge (on either endpoint)

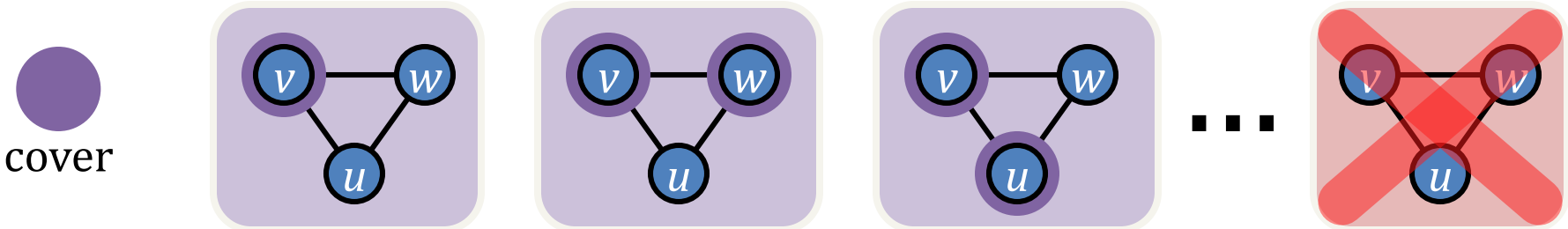


- **Large on grids:**
  - Matching of size $\Omega(r^2)$
  - Every edge must be covered by a different vertex

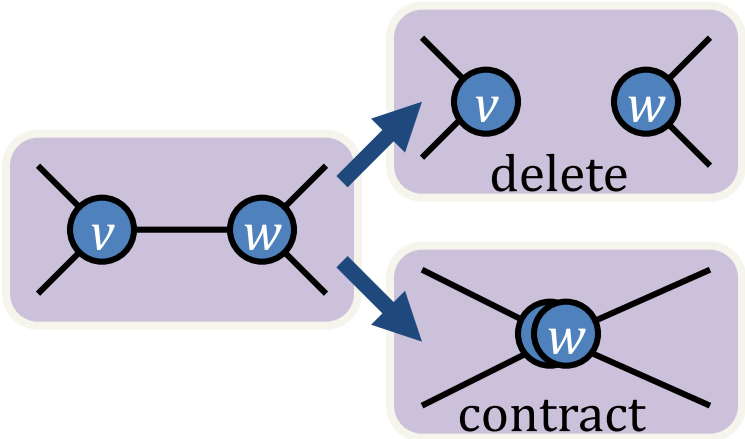


Example 2: Feedback Vertex Set

- k = minimum number of vertices required to cover every cycle (on some vertex)



- **Closed under minors:**

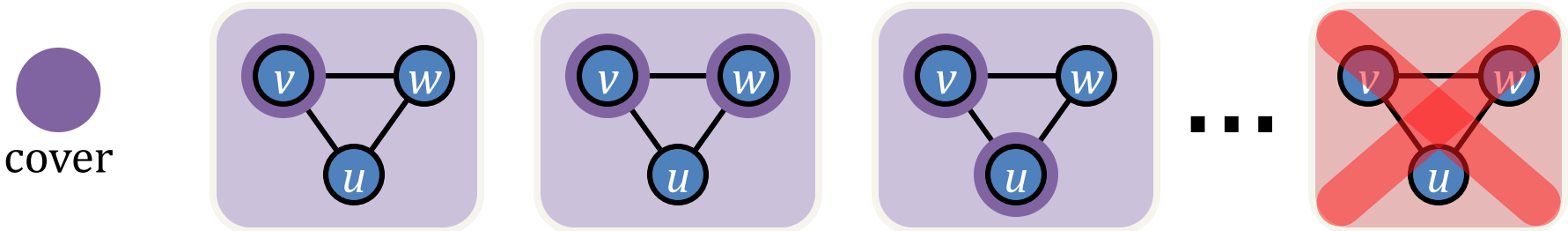


⇒ still a cover (only break cycles)

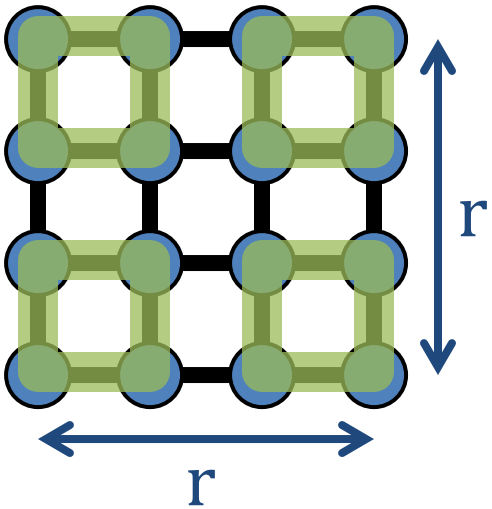
⇒ still a cover, possibly 1 smaller

Example 2: Feedback Vertex Set

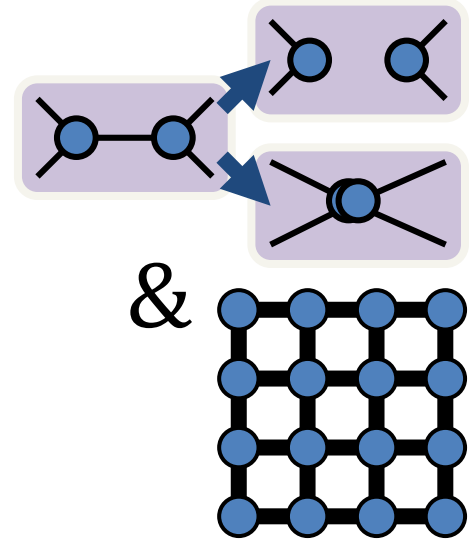
- k = minimum number of vertices required to cover every cycle (on some vertex)



- **Large on grids:**
  - $\Omega(r^2)$ vertex-disjoint cycles
  - Every cycle must be covered by a different vertex



Bidimensional ⇒ Relate Parameter & Treewidth



- <u>Theorem 1:</u> If a parameter k is bidimensional, then it satisfies **parameter-treewidth bound**

$$\text{treewidth} = O\left(\sqrt{k}\right)$$

in H-minor-free graphs

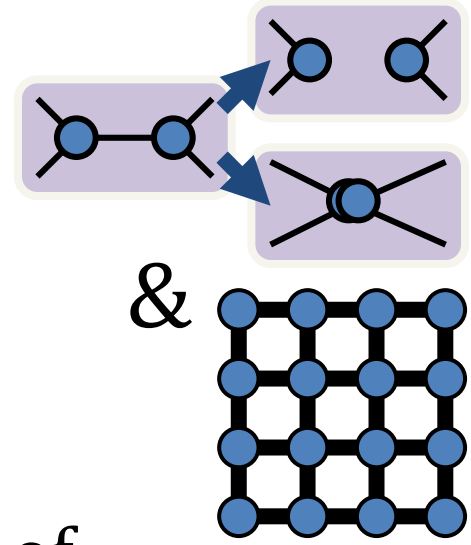
[Demaine, Fomin, Hajiaghayi, Thilikos 2004; Demaine & Hajiaghayi 2005]

- <u>Proof sketch:</u>

Treewidth $w \Rightarrow \Omega(w) \times \Omega(w)$ grid minor

$\Rightarrow k = \Omega(w^2)$ *[bidimensional]*

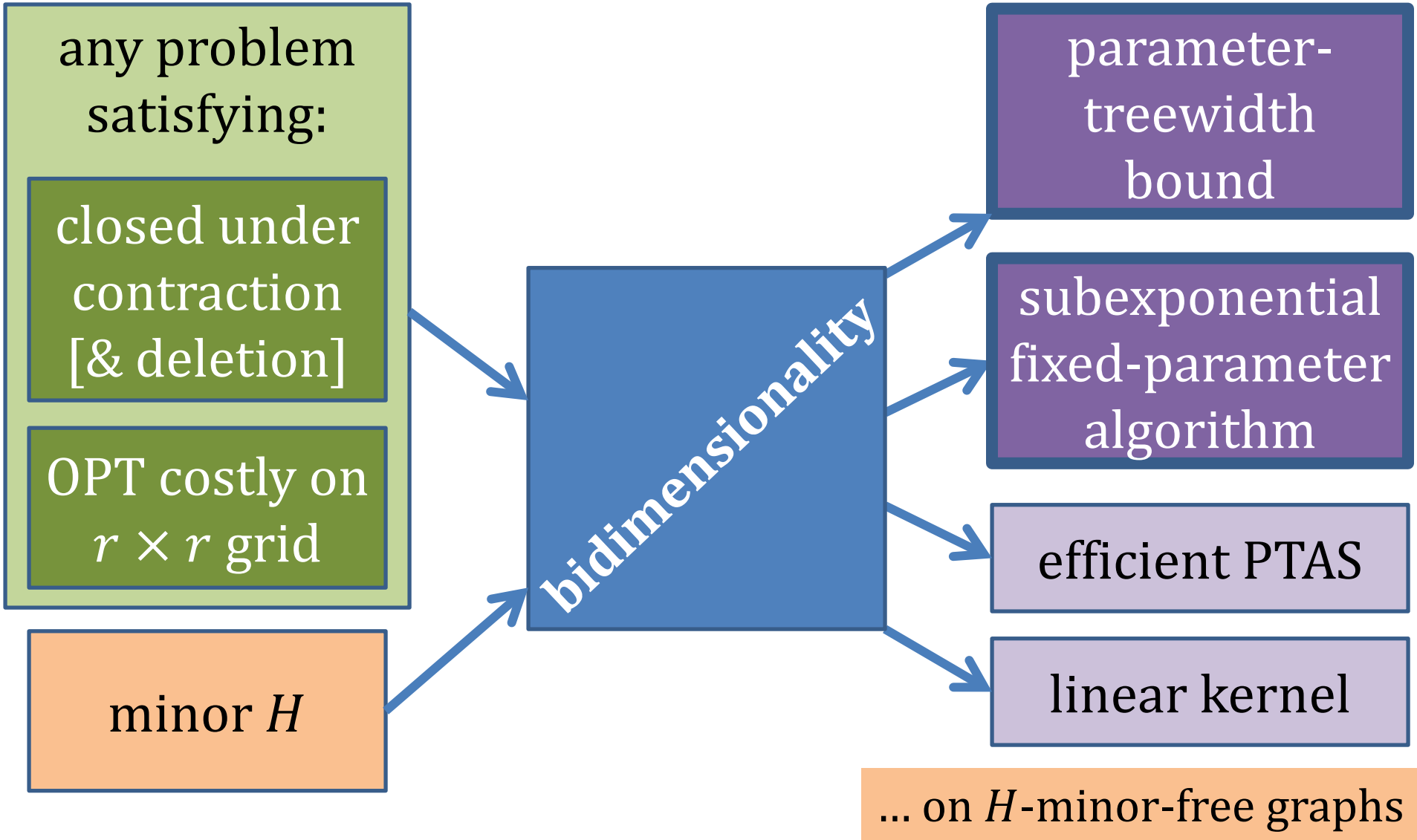
Bidimensional ⇒ Subexponential FPT



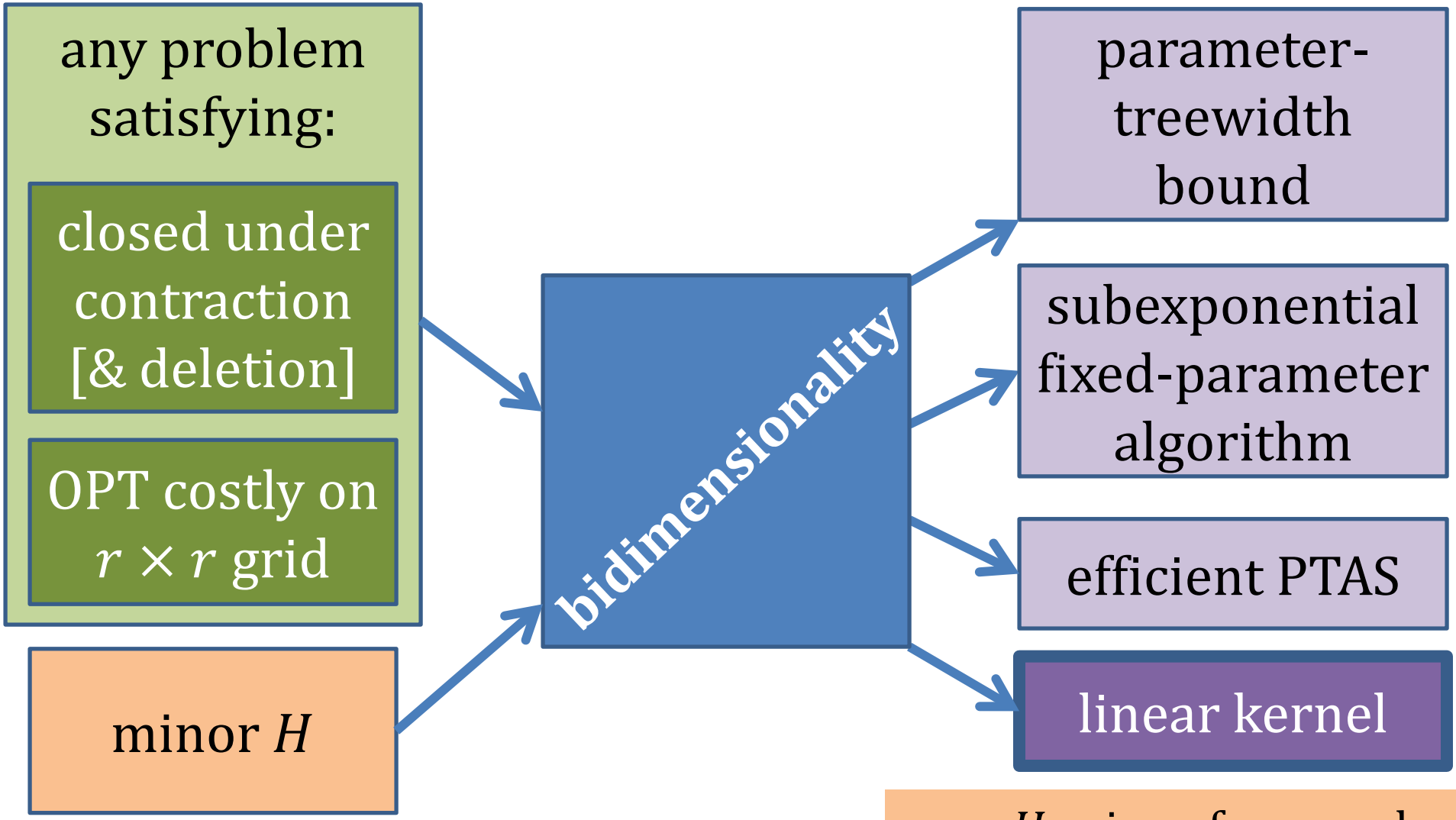
- <u>Theorem 2:</u> If a parameter k is
  - bidimensional, and
  - fixed-parameter tractable on graphs of bounded treewidth: $h(\text{treewidth})\, n^{O(1)}$ time

  then it has a **subexponential** fixed-parameter algorithm, running in $h\left(\sqrt{k}\right) n^{O(1)}$ time, in H-minor-free graphs
- <u>Proof sketch:</u>
  - If (approx.) treewidth $= \omega\left(\sqrt{k}\right)$, answer NO
  - Else run bounded-treewidth algorithm

Bidimensionality Overview

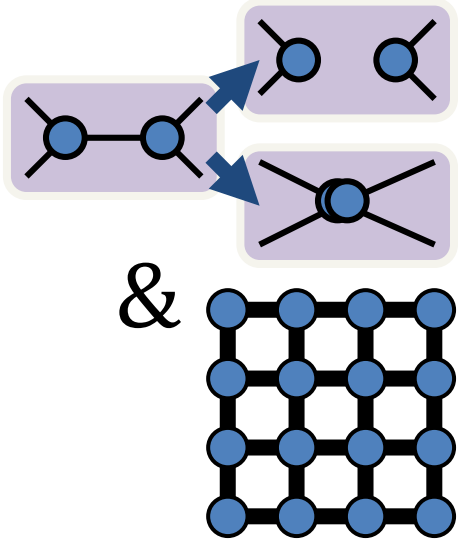


Bidimensionality Overview



... on H-minor-free graphs

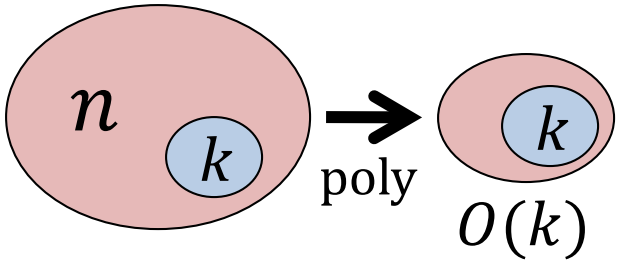
Bidimensional ⇒ Linear Kernel



- <u>Theorem 4:</u> If a parameter is
  - bidimensional,
  - satisfies the "separation property", and
  - has "finite integer index",

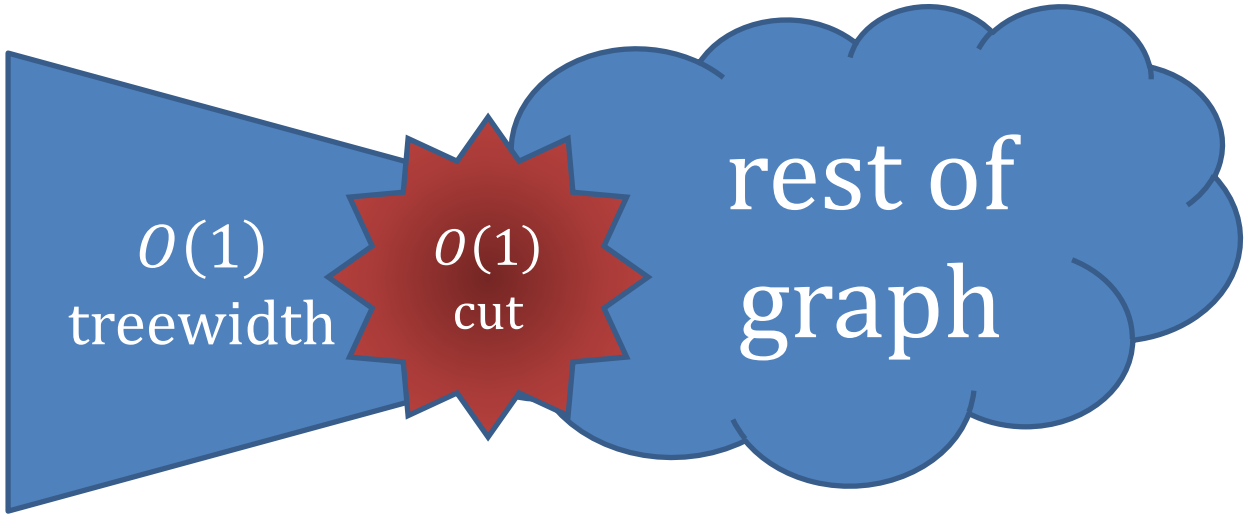
  then it has a **linear kernel** in H-minor-free graphs

  [Fomin, Lokshtanov, Saurabh, Thilikos 2009]

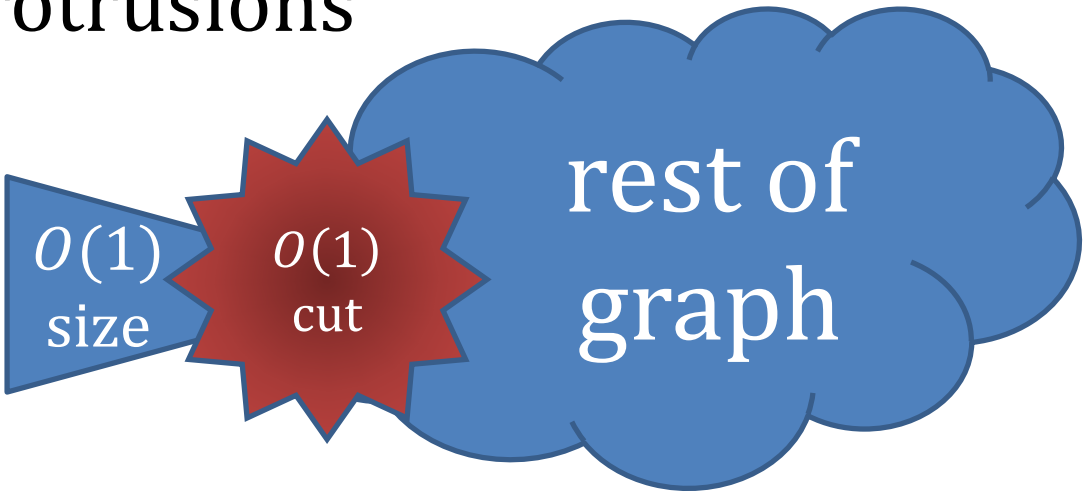


Bidimensional ⇒ Linear Kernel

- **Protrusion:**



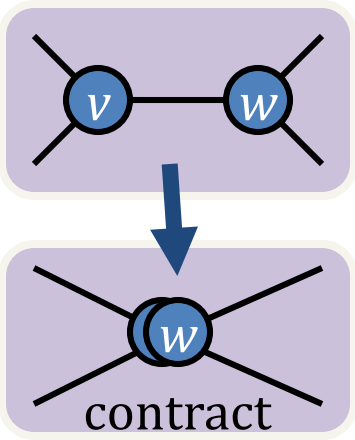
- <u>Idea:</u> Kernelize protrusions



Bidimensionality (version 2)

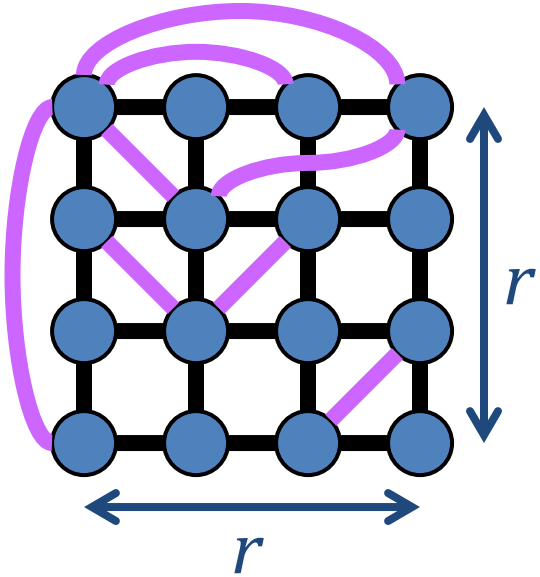
[Demaine, Fomin, Hajiaghayi, Thilikos 2004]

- Parameter k is **contraction-bidimensional** if
  - **Closed under contractions:** k only decreases when contracting edges



and

  - **Large on gridoids:** For $r \times r$ "grid-like graphs", $k = \Omega(r^2)$
    - Triangulated + few extra edges



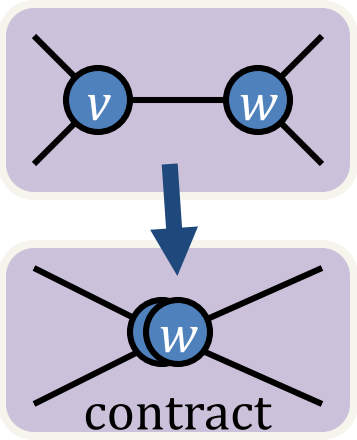
Bidimensionality (version 3)

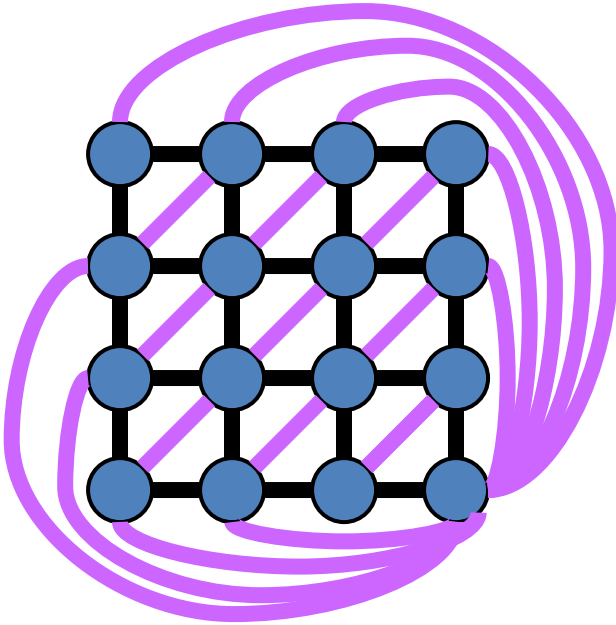
[Fomin, Golovach, Thilikos 2009]

- Parameter k is **contraction-bidimensional** if
  - **Closed under contractions:** k only decreases when contracting edges

and

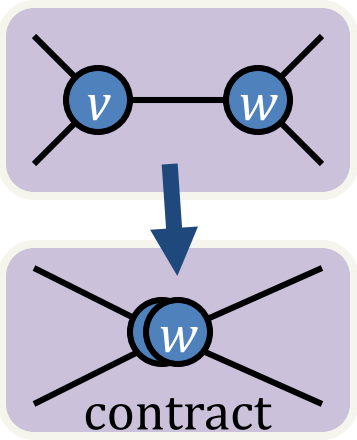
  - **Large on Γ graphs:** For naturally triangulated $r \times r$ grid graphs, $k = \Omega(r^2)$

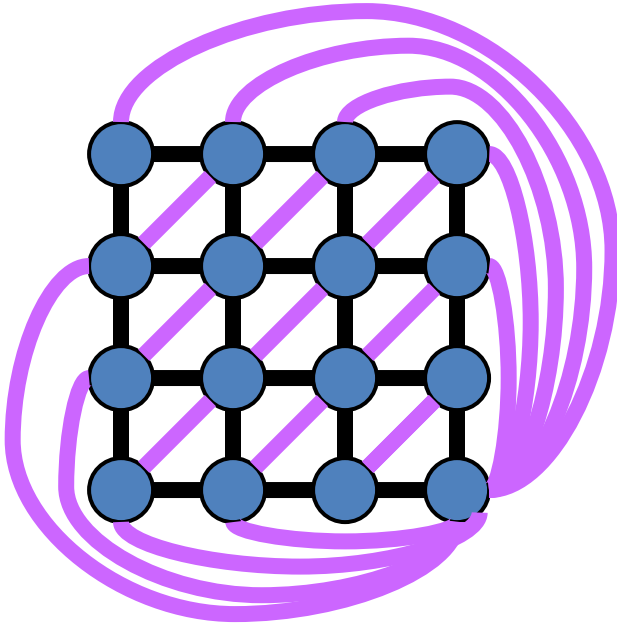




Contraction-Bidimensional Problems

- Minimum maximal matching
- Face cover (planar graphs)
- Dominating set
- Edge dominating set
- r-dominating set
- Connected … dominating set
- Unweighted TSP tour
- Chordal completion (fill-in)



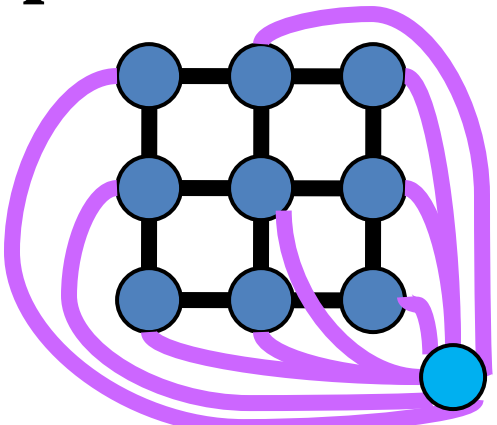


Contraction-Bidimensional ⇒ Good

- <u>Theorem 6:</u> As before, obtain
  - parameter-treewidth bound
  - subexponential FPT
  - efficient PTASs
  - linear kernel

  for **contraction-bidimensional** parameters in any graph family excluding an **apex minor**

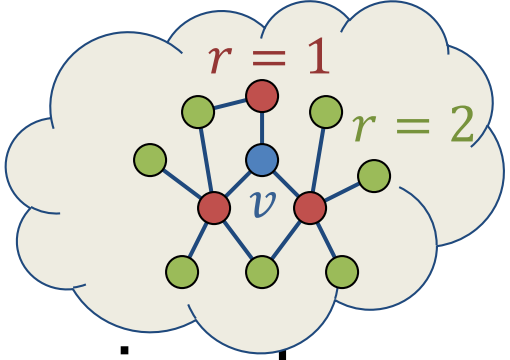
[Demaine, Fomin, Hajiaghayi, Thilikos 2004;
Demaine & Hajiaghayi 2005 & 2005;
Fomin, Golovach, Thilikos 2009;
Fomin, Lokshtanov, Saurabh, Thilikos 2009]



Separators via Bidimensionality

- **Number of vertices** is minor-bidimensional
- Parameter-treewidth bound
  $\Rightarrow$ treewidth $= O\left(\sqrt{k}\right) = O(\sqrt{n})$
- <u>Corollary:</u> For any fixed graph H, every H-minor-free graph has treewidth $O(\sqrt{n})$ and hence separator of size $O(\sqrt{n})$
  [Alon, Seymour, Thomas 1990; Grohe 2003]

Local Treewidth via Bidimensionality



- **Diameter** is a contraction-bidimensional parameter except $\Omega(r)$, not $\Omega(r^2)$, in $r \times r$ grid
- $\Rightarrow$ Treewidth $= O(\text{diameter})$ in apex-minor-free graphs
- $\Rightarrow$ Treewidth of radius-r neighborhood $= O(r)$ [Demaine & Hajiaghayi 2004]

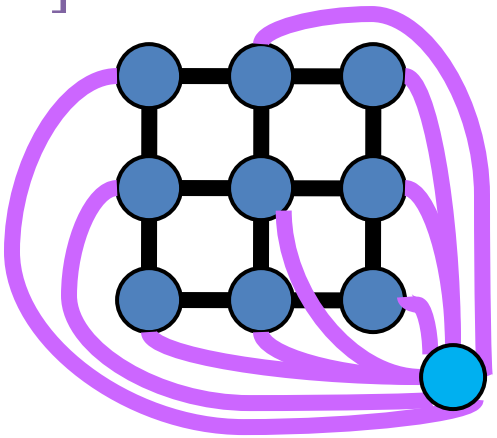
General Graphs via Bidimensionality

- <u>Theorem:</u> Any minor-closed graph parameter
  - nonzero on some planar graph X,
  - at least the sum over connected components in a disconnected graph, and
  - fixed-parameter tractable with respect to treewidth

  can be solved in $f(k)\,n$ by a known algorithm [Demaine & Hajiaghayi 2007]
- <u>Proof sketch:</u>
  - Treewidth $> c_1(x\,k)^{c_2} \Rightarrow (x\,k) \times (x\,k)$ grid minor $\Rightarrow k^2$ of $x \times x$ grid minors, with X minor $\Rightarrow \mathrm{OPT} \geq k^2 \Rightarrow$ answer NO

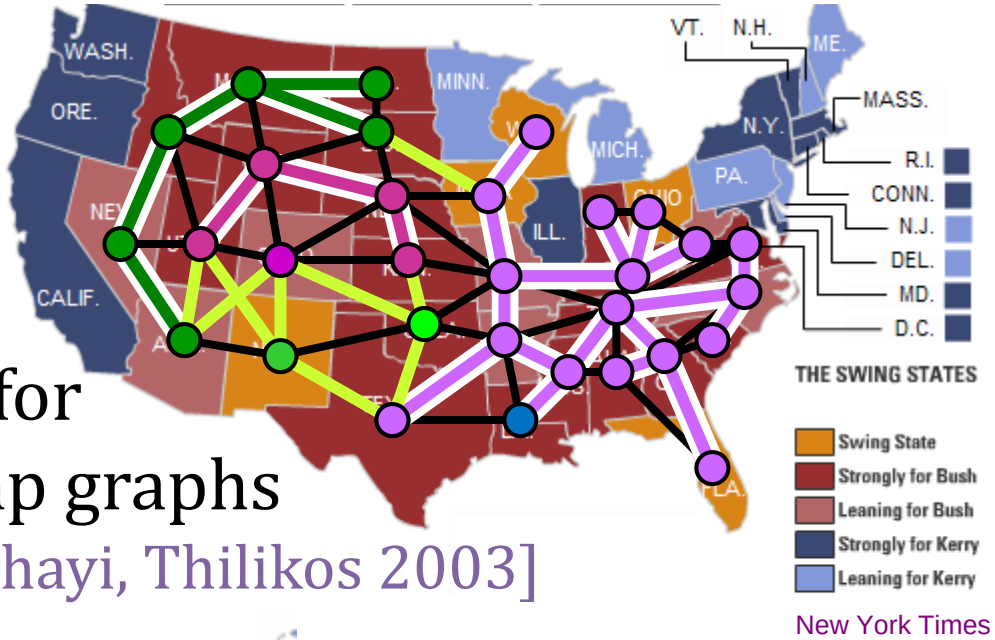
Beyond Apex-Minor-Free

- <u>Conjecture:</u> Algorithms for contraction-bidimensional problems generalize to H-minor-free graphs, even for a **non-apex graph** H
  - True for e.g. dominating set subexponential FPT (but parameter-treewidth bound *false*) [Demaine, Fomin, Hajiaghayi, Thilikos 2004]
  - Stronger form of contraction-bidimensionality works [Fomin, Golovach, Thilikos 2009]



Beyond H-Minor-Free Graphs

- <u>Conjecture:</u> Most of bidimensionality theory generalizes to fixed powers of H-minor-free graphs, e.g., map graphs
  - Treewidth $\Omega(r^7)$ $\Rightarrow r \times r$ grid minor [Demaine, Hajiaghayi, Kawarabayashi 2009]
  - Subexponential FPT for dominating set in map graphs [Demaine, Fomin, Hajiaghayi, Thilikos 2003]
  - Subexponential FPT & EPTASs for map graphs by "cleaning" cliques [Fomin, Lokshtanov, 2012]



New York Times

Beyond *H*-Minor-Free Graphs

- **Excluded topological minors**
  - Linear kernel for (connected) dominating set [Fomin, Lokshtanov, Saurabh, Thilikos 2013]
  - General framework for linear kernels [Langer, Reidl, Rossmanith, Sikdar 2012]

- **Directed graphs**
  - Subexponential fixed-parameter algorithms for e.g. Directed Hamiltonian Path via bidimensionality [Dorn, Fomin, Lokshtanov, Raman, Saurabh 2010]
  - PTASs?

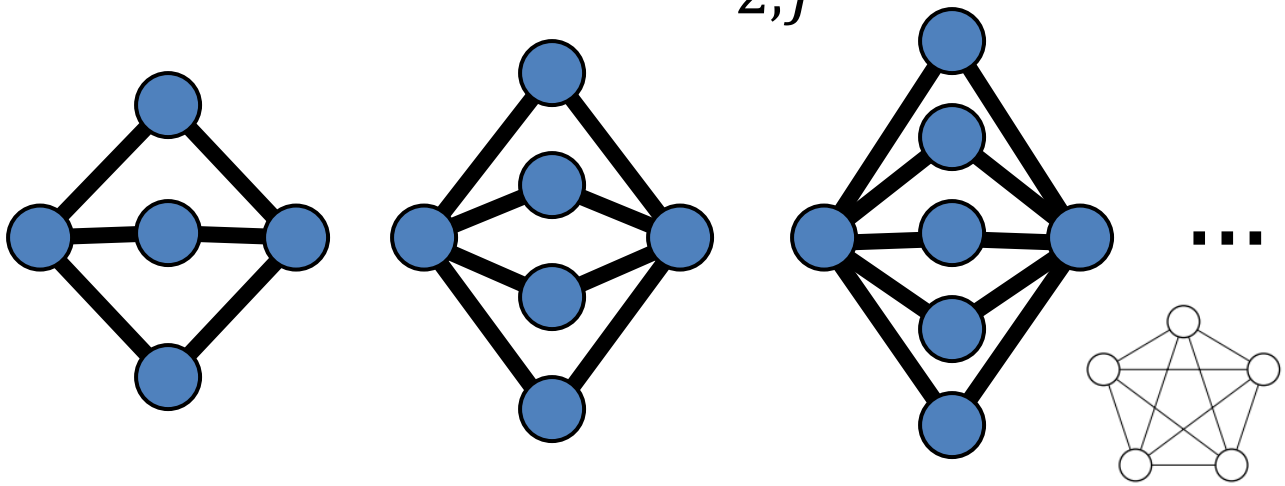
Beyond: Subset Problems

- What if only some of the nodes are "critical"?
- Steiner tree, subset TSP, etc. have PTASs up to bounded-genus graphs
  [Borradaile, Mathieu, Klein 2007; Borradaile, Demaine, Tazari 2009]
- Steiner forest has PTAS in planar graphs
  [Bateni, Hajiaghayi, Marx 2010]
- <u>Wanted:</u> General framework

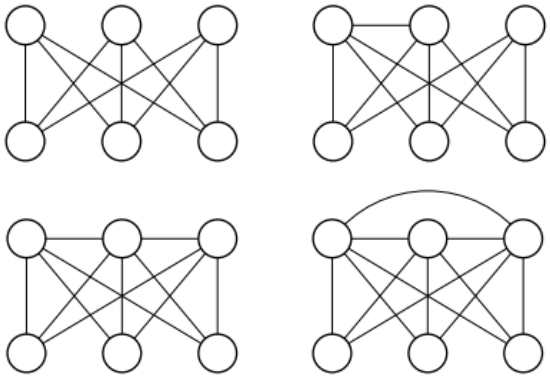
Contractions not Well-Quasi-Ordered

[Demaine, Hajiaghayi, Kawarabayashi 2009]

- No $K_{2,k}$ can be contracted into $K_{2,j}$



- Well-quasi-ordered for
  - **triangulated** planar graphs
  - 2-connected **outerplanar** graphs
  - **trees**

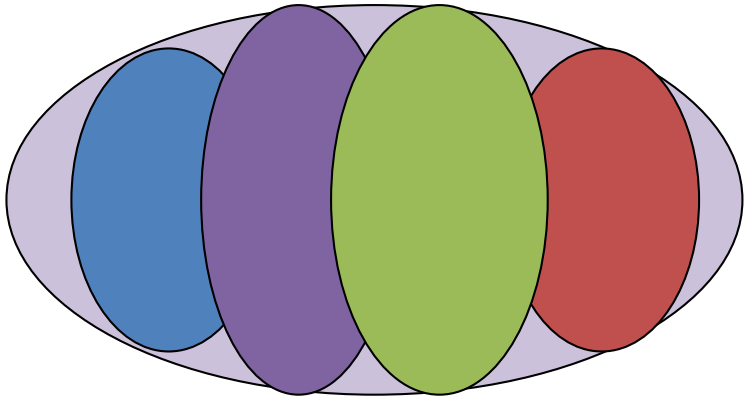


- Minor-closed ⇒ finite excluded contractions

Simplifying Graph Decomposition

[Demaine, Hajiaghayi, Kawarabayashi 2005; DeVos et al. 2004]

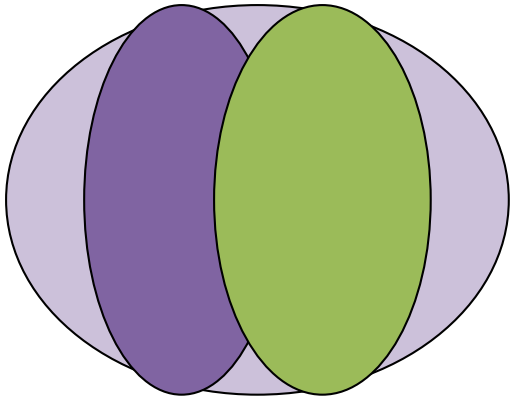
- H-minor-free graphs can have vertices or edges partitioned into ***k* pieces** such that **deleting any one piece** results in **bounded treewidth**
  - For PTAS, set $k \approx 1/\varepsilon$
  - $n^{f(H)}$ algorithm
  - $f(H)\, n^{O(1)}$ algorithm [Tazari 2012]
- Previously known for
  - planar graphs [Baker 1994]
  - apex-minor-free [Eppstein 2000]



Simplifying Graph Decomposition

[Demaine, Hajiaghayi, Kawarabayashi 2010]

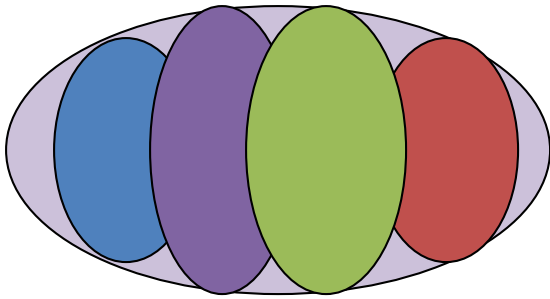
- **Odd-**H-minor-free graphs can have their vertices or edges partitioned into **2 pieces** such that deleting any one piece results in bounded treewidth
  - $n^{f(H)}$ algorithm
  - $f(H)\, n^{O(1)}$ algorithm [Tazari 2012]



Contraction Decomposition

[Demaine, Hajiaghayi, Kawarabayashi 2011]

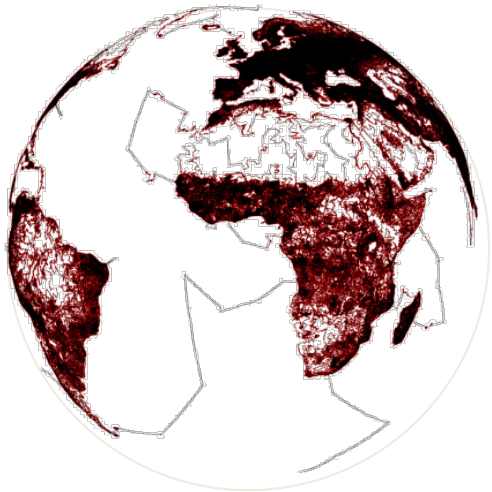
- <u>Theorem:</u> H-minor-free graphs can have their edges partitioned into ***k* pieces** such that **contracting any one piece** results in **bounded treewidth**



  - Polynomial-time algorithm
  - Previously known for
    - planar [Klein 2005, 2006]
    - bounded-genus [Demaine, Hajiaghayi, Mohar 2007]
    - apex-minor-free [Demaine, Hajiaghayi, Kawarabayashi 2009]

Applications of Contraction Decomposition

- **PTAS for Traveling Salesman Problem** in weighted H-minor-free graphs
  - Decade-old problem by [Grohe]
- PTAS for minimum-size c-edge-connected submultigraph
- Fixed-parameter algorithm for ***k*-cut** & **bisection**



[Demaine, Hajiaghayi, Kawarabayashi 2011]

Fixed-Parameter Algorithms via Contraction Decomposition

- **Bisection:** Cut graph into equal halves with $\leq k$ edges between them
- FPT in H-minor-free graphs:
  - Closed under contraction (but not minors)
  - Contraction decomposition with $k + 1$ layers avoids OPT in some contraction
  - Vertices become weighted
  - Can still solve weighted bounded-treewidth
  - $\Rightarrow$ Solve in $2^{\tilde{O}(k)}\, n + n^{O(1)}$ time

Fixed-Parameter Algorithms via Contraction Decomposition

- ***k*-cut:** Remove fewest edges to make at least k connected components
- FPT in H-minor-free graphs:
  - Average degree $c_H = O\left(H\sqrt{\lg H}\right)$
  - $\Rightarrow$ OPT $\leq c_H\, k$
  - $\Rightarrow$ Contraction decomposition with $c_H\, k + 1$ layers avoids OPT in some contraction
  - $\Rightarrow$ Solve in $2^{\tilde{O}(k)}\, n + n^{O(1)}$ time
- Generalization to arbitrary graphs [Kawarabayashi & Thorup 2011]

Radial Coloring for Bounded Genus

[Demaine, Hajiaghayi, Kawarabayashi 2011]

- Start with **tight** precolored edges:
  - Treewidth of radius-r neighborhood is $f(r)$
  - $c = O(1)$ induced connected components
- Color edges at radial distance r with color $r \bmod k$

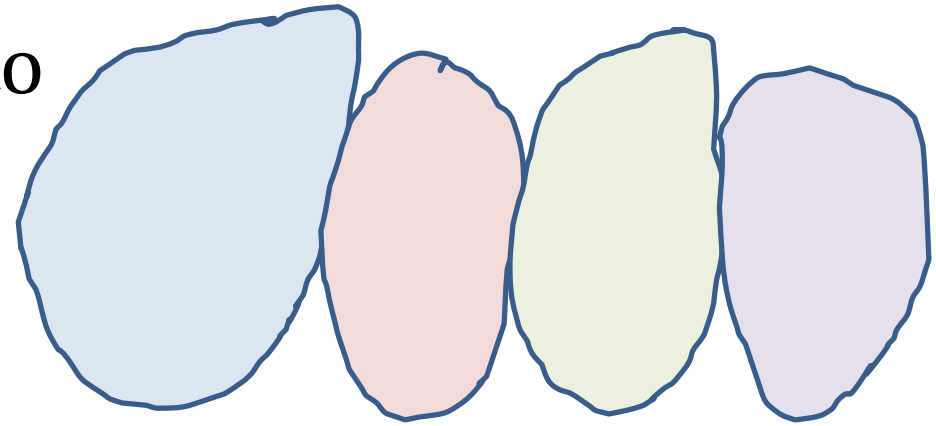
Radial Coloring for Bounded Genus

[Demaine, Hajiaghayi, Kawarabayashi 2011]

- Contract color i
  - Each connected component → articulation point
  - Split these apart → **blobs**, connected in DAG
- Outdegree > 1 ⇒ **split**
- Indegree > 1 ⇒ **rejoin**
  - ⇒ Indegree $\leq c + g$ (initial component or handle)

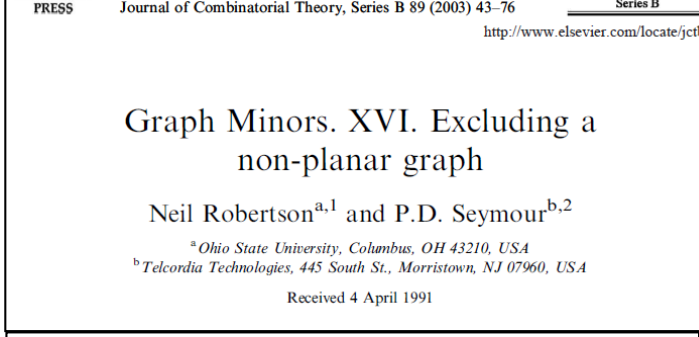
Radial Coloring for Bounded Genus

[Demaine, Hajiaghayi, Kawarabayashi 2011]



- Split nonroot blob into $\leq c + g$ **chunks** by closest BFS seed
- Each chunk has radial diameter $O(k)$
- Nonroot blob has radial diameter, so treewidth $O\big(k\,(c+g)\big)$ [Eppstein 2000]
- Blob DAG = tree + $\leq g$ extra edges
  - Tree of 1-sums doesn't affect treewidth
  - Add extra edges to all bags $\Rightarrow +O(g)$

Contraction Decomp. for *H*-Minor-Free


PRESS Journal of Combinatorial Theory, Series B 89 (2003) 43–76 Series B

http://www.elsevier.com/locate/jctb

Graph Minors. XVI. Excluding a non-planar graph

Neil Robertson[a,1] and P.D. Seymour[b,2]

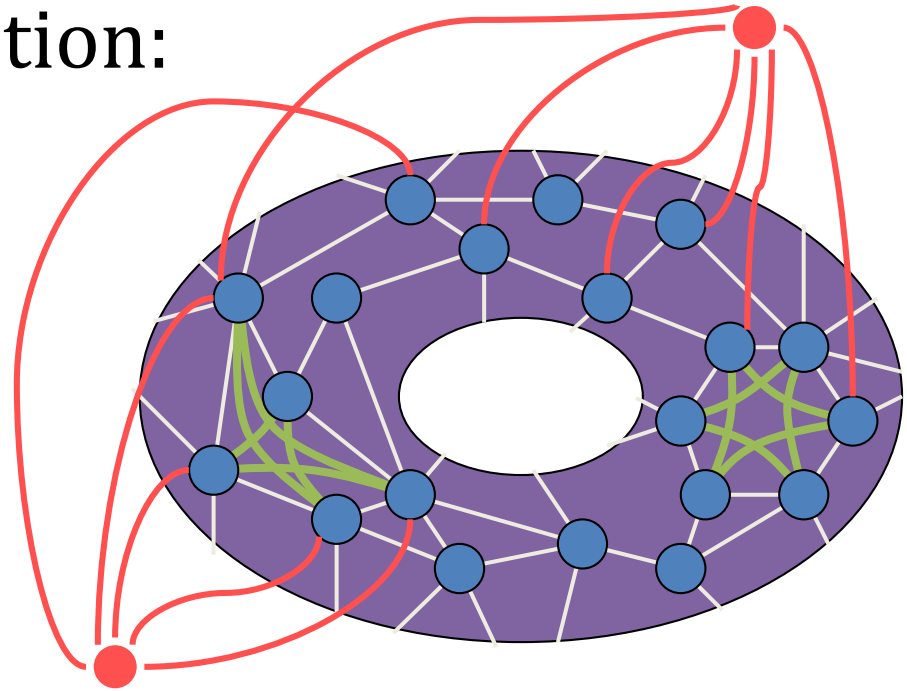
[a] Ohio State University, Columbus, OH 43210, USA
[b] Telcordia Technologies, 445 South St., Morristown, NJ 07960, USA

Received 4 April 1991

This paper contains the corner stone theorem of the series.

- H-minor-free graph = "tree" of "almost-embeddable graphs" [GM16]
- Each almost-embeddable graph has contraction decomposition:
  - Bounded genus done
  - Apices easy: increase treewidth of anything by $O(1)$
  - Vortices similar

[Demaine, Hajiaghayi, Mohar 2007]

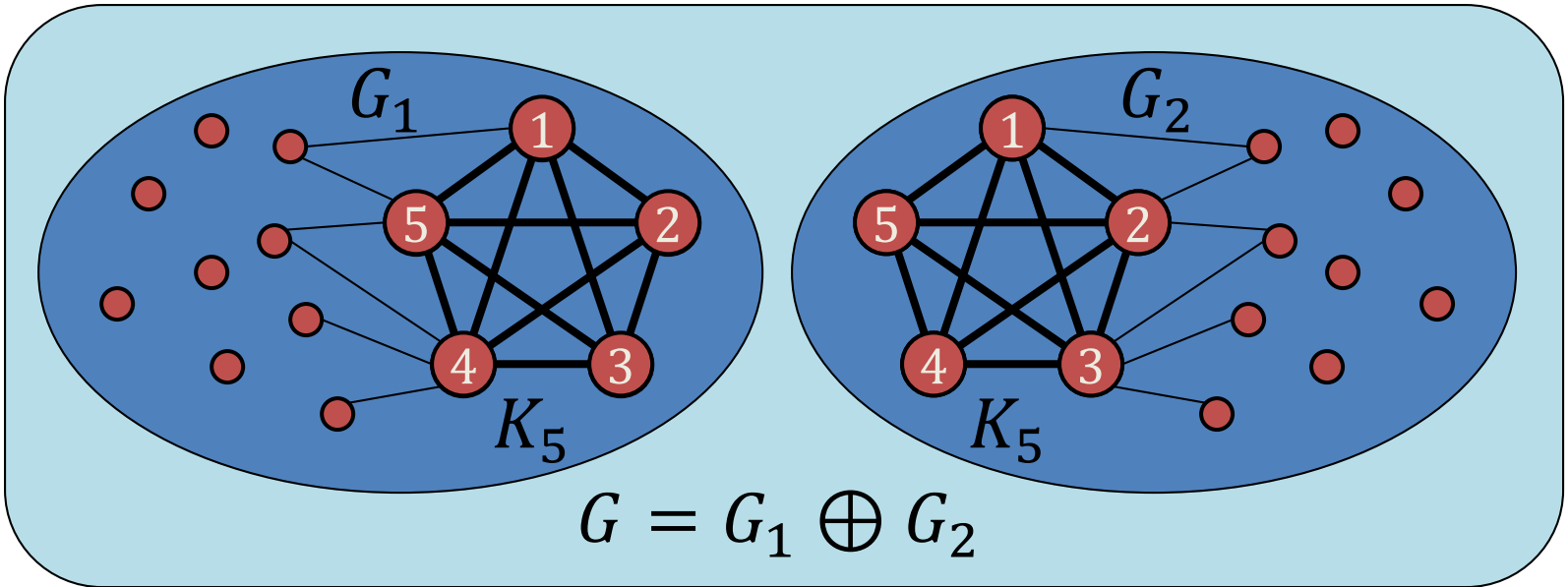


Hard Part of the Problem

[Demaine, Hajiaghayi, Mohar 2007]

- Tree "joins" between two graphs are **clique sums**:
  - Attach at clique of same size
  - Delete any desired edges

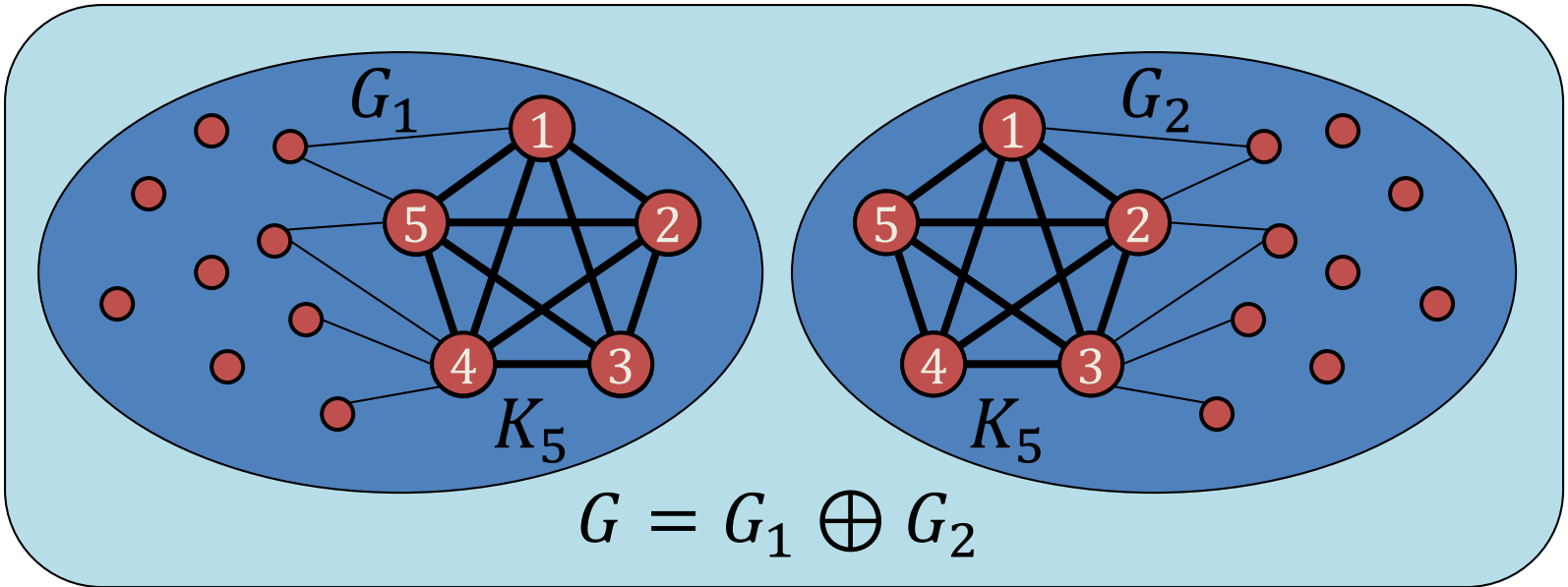
destroys contraction property



Solution Idea

[Demaine, Hajiaghayi, Kawarabayashi 2011]

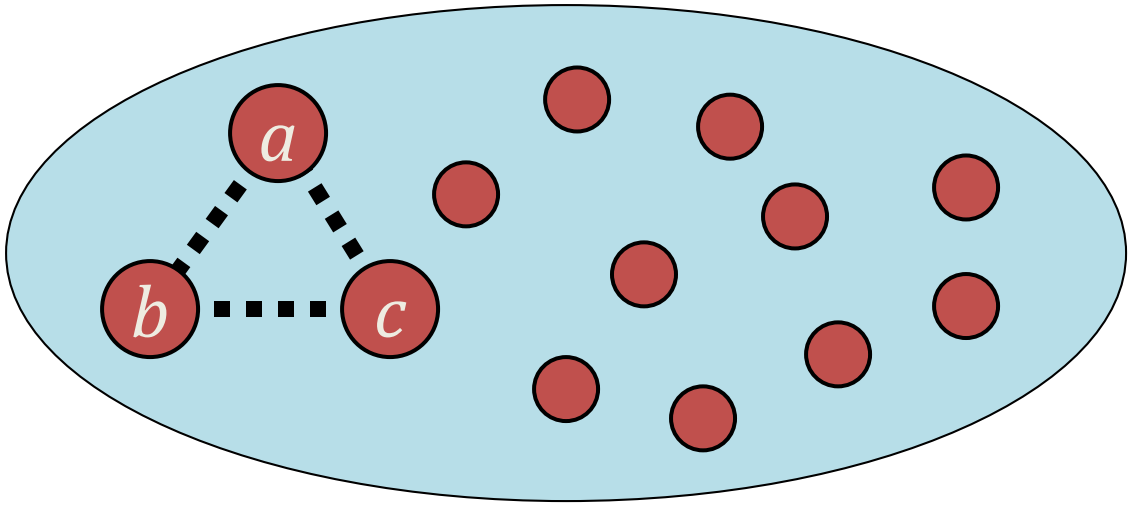
- Clique sums either: [Graph Minors]
  - Vortices ↔ apices
    ⇒ *no contraction necessary*
  - K_3 in bounded-genus part
    (hard part)



Solution Idea

[Demaine, Hajiaghayi, Kawarabayashi 2011]

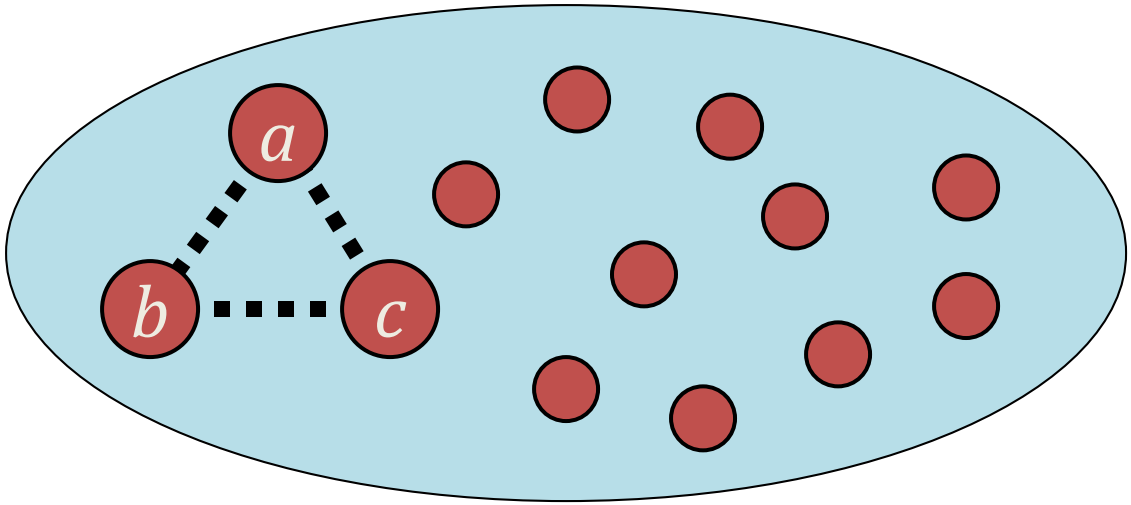
- Find a path in child G_2 to simulate the effect of each deleted edge in K_3
  - Can make pieces G_i 3-edge-connected
- Need these paths to all contract together, by putting them all in piece 1



Solution Idea

[Demaine, Hajiaghayi, Kawarabayashi 2011]

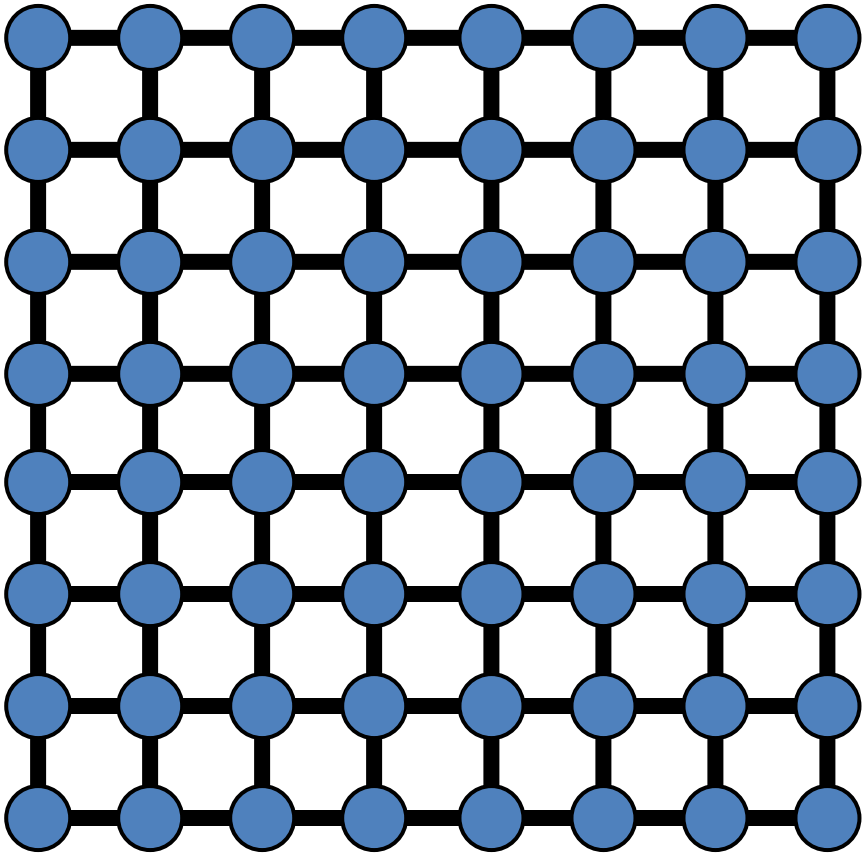
- <u>Claim:</u> Precoloring $O(1)$ **shortest paths** doesn't hurt contraction decomposition
- Shortcut paths to shortest paths (in radial graph ≈ primal + dual)
  - Two paths $a \to b$; one path $c \to a$-b path



$O(1)$ Shortest Paths are Tight

[Demaine, Hajiaghayi, Kawarabayashi 2011]

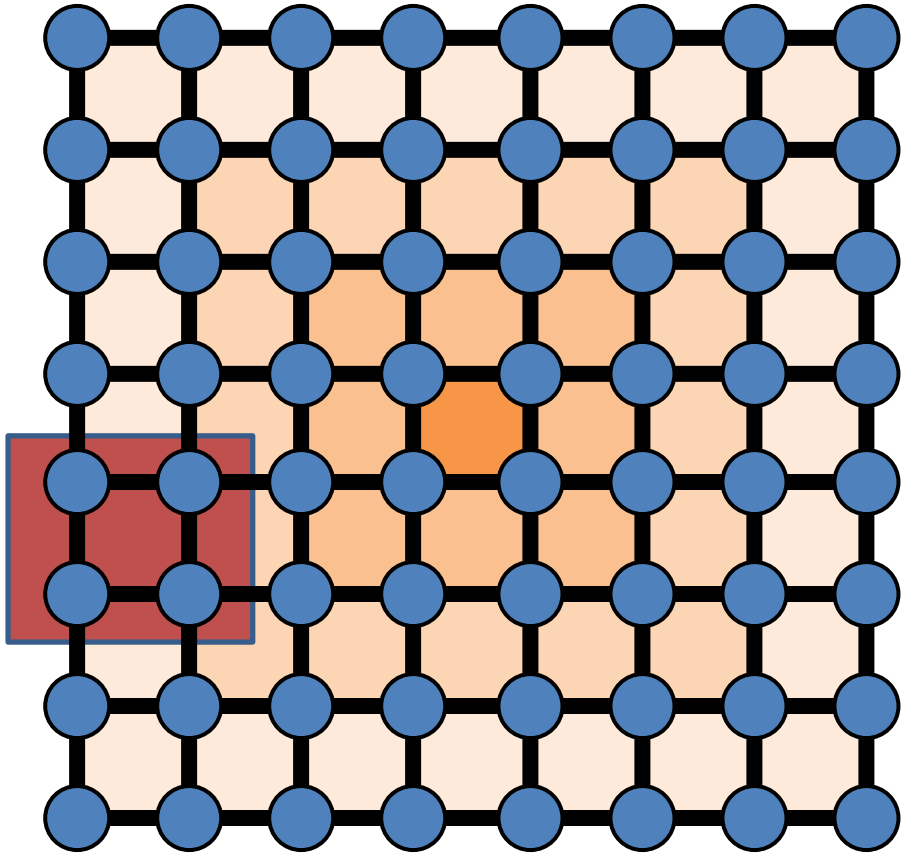
- <u>Theorem:</u> Treewidth of radius-r neighborhood of $O(1)$ shortest paths is $f(r)$
- If not, large grid within the neighborhood
- <u>Intuition:</u> Then there should be a shortcut: $2r \ll f(r)$
  - Contradiction



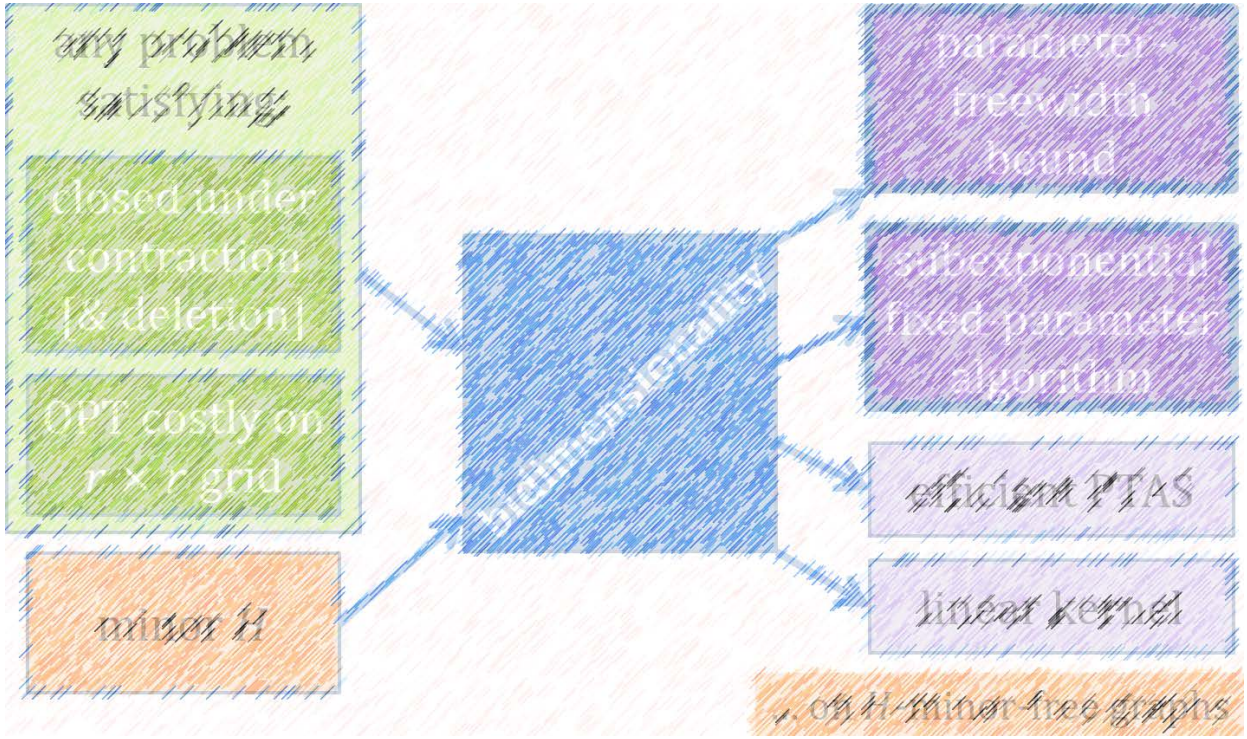
$O(1)$ Shortest Paths are Tight

[Demaine, Hajiaghayi, Kawarabayashi 2011]

- <u>Theorem:</u> Treewidth of radius-r neighborhood of $O(1)$ shortest paths is $f(r)$
  - Path is r-dive into $\Theta(r) \times \Theta(r)$ wall
  - $\Rightarrow r$-rainbow
  - By Menger's Theorem, get $\Theta(r)$ "cross paths"
  - Build $\Omega(r) \times \Omega(r)$ wall $\geq 2\,r$ away from shortest path
  - Repeat for each path



Minors & Algorithms



Erik Demaine

M.I.T.