# The moment-LP and moment-SOS approaches

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ICERM, Providence, February 2014

- Why polynomial optimization?
- LP- and SDP- CERTIFICATES of POSITIVITY
- The moment-LP and moment-SOS approaches
- An alternative characterization of nonnegativity

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# Why Polynomial Optimization?

**After all ...**

the polynomial optimization problem:

$$f^* = \min\{f(\mathbf{x}) : \quad g_j(\mathbf{x}) \geq 0,\ j = 1, \ldots, m\}$$

is just a particular case of Non Linear Programming (NLP)!

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## When searching for a local minimum ...

Optimality conditions and descent algorithms use basic tools from REAL and CONVEX analysis and linear algebra

The focus is on how to improve $f$ by looking at a NEIGHBORHOOD of a nominal point $\mathbf{x} \in \mathbf{K}$, i.e., LOCALLY AROUND $\mathbf{x} \in \mathbf{K}$, and in general, no GLOBAL property of $\mathbf{x} \in \mathbf{K}$ can be inferred.

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## BUT for GLOBAL Optimization

... the picture is different!

Remember that for the GLOBAL minimum $f^*$:

$$f^* = \sup \{ \lambda : f(\mathbf{x}) - \lambda \geq 0 \quad \forall \mathbf{x} \in \mathbf{K} \}.$$

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**REAL ALGEBRAIC GEOMETRY helps!!!!**

Indeed, POWERFUL CERTIFICATES OF POSITIVITY EXIST!

**Moreover .... and importantly,**

Such certificates are amenable to PRACTICAL COMPUTATION!

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# SOS-based certificate

$$\mathbf{K} = \{\mathbf{x} : \, g_j(\mathbf{x}) \, \geq \, 0, \quad j = 1, \ldots, m\}$$

**Theorem (Putinar's Positivstellensatz)**

*If* $\mathbf{K}$ *is compact (+ a technical Archimedean assumption) and* $f > 0$ *on* $\mathbf{K}$ *then:*

$$\dagger \quad f(\mathbf{x}) \, = \, \sigma_0(\mathbf{x}) + \sum_{j=1}^{m} \sigma_j(\mathbf{x}) \, g_j(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathbb{R}^n,$$

*for some SOS polynomials* $(\sigma_j) \subset \mathbb{R}[\mathbf{x}]$.

Testing whether † holds for some

SOS $(\sigma_j) \subset \mathbb{R}[\mathbf{x}]$ with a degree bound, is SOLVING an SDP!

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$$\mathbf{K} = \{\mathbf{x} : \; g_j(\mathbf{x}) \geq 0; \; (1 - g_j(\mathbf{x})) \geq 0, \quad j = 1, \ldots, m\}$$

**Theorem (Krivine-Vasilescu-Handelman's Positivstellensatz)**

*Let $\mathbf{K}$ be compact and the family $\{g_j, (1 - g_j)\}$ generate $\mathbb{R}[\mathbf{x}]$. If $f > 0$ on $\mathbf{K}$ then:*

$$\star \quad f(\mathbf{x}) = \sum_{\alpha,\beta} c_{\alpha\beta} \prod_{j=1}^{m} g_j(\mathbf{x})^{\alpha_j} \, (1 - g_j(\mathbf{x}))^{\beta_j}, , \qquad \forall \mathbf{x} \in \mathbb{R}^n,$$

*for some NONNEGATIVE scalars $(c_{\alpha\beta})$.*

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allow to infer GLOBAL Properties of
FEASIBILITY and OPTIMALITY,

## ... the analogue of (well-known) previous ones

valid in the CONVEX CASE ONLY!

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**. . . modeled as**

particular instances of the so called Generalized Moment Problem, among which: Probability, Optimal and Robust Control, Game theory, Signal processing, multivariate integration, etc.

$$(GMP): \quad \inf_{\mu_i \in M(\mathbf{K}_i)} \{ \sum_{i=1}^{s} \int_{\mathbf{K}_i} f_i \, d\mu_i : \sum_{i=1}^{s} \int_{\mathbf{K}_i} h_{ij} \, d\mu_i \overset{\leq}{=} b_j, \quad j \in J \}$$

with $M(\mathbf{K}_i)$ space of Borel measures on $\mathbf{K}_i \subset \mathbb{R}^{n_i}$, $i = 1, \ldots, s$.

$$\text{Global OPTIM} \quad \rightarrow \inf_{\mu \in M(\mathbf{K})} \{ \int_{\mathbf{K}} f \, d\mu : \int_{\mathbf{K}} 1 \, d\mu = 1 \}.$$

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For instance, one may also want:

• To approximate sets defined with QUANTIFIERS, like .e.g.,

$$R_f := \{x \in \mathbf{B} \quad : \quad f(x, y) \leq 0 \text{ for all } y \text{ such that } (x, y) \in \mathbf{K}\}$$

$$D_f := \{x \in \mathbf{B} \quad : \quad f(x, y) \leq 0 \text{ for some } y \text{ such that } (x, y) \in \mathbf{K}\}$$

where $f \in \mathbb{R}[x, y]$, **B** is a simple set (box, ellipsoid).

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## The moment-LP and moment-SOS approaches

consist of using a certain type of positivity certificate (Krivine-Vasilescu-Handelman's or Putinar's certificate) in potentially any application where such a characterization is needed. (Global optimization is only one example.)

## In many situations this amounts to

solving a HIERARCHY of :

- LINEAR PROGRAMS, or
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... of increasing size!.

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Replace $f^* = \sup_{\lambda,\sigma_j} \{ \lambda : f(\mathbf{x}) - \lambda \geq 0 \quad \forall \mathbf{x} \in \mathbf{K}\}$ with:

**The SDP-hierarchy indexed by $d \in \mathbb{N}$:**

$$f_d^* = \sup \{ \lambda : f - \lambda = \underbrace{\sigma_0}_{SOS} + \sum_{j=1}^{m} \underbrace{\sigma_j}_{SOS} g_j ; \quad \deg(\sigma_j \, g_j) \leq 2d \}$$

**or, the LP-hierarchy indexed by $d \in \mathbb{N}$:**

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# LP- and SDP-hierarchies for optimization

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## Theorem

*Both sequence ($f_d^*$), and ($\theta_d$), $d \in \mathbb{N}$, are MONOTONE NON DECREASING and when $\mathbf{K}$ is compact (and satisfies a technical Archimedean assumption) then:*

$$f^* = \lim_{d\to\infty} f_d^* = \lim_{d\to\infty} \theta_d.$$

• What makes this approach exciting is that it is at the crossroads of several disciplines/applications:

- Commutative, Non-commutative, and Non-linear ALGEBRA
- Real algebraic geometry, and Functional Analysis
- Optimization, Convex Analysis
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• Has already been proved useful and successful in applications with modest problem size, notably in optimization, control, robust control, optimal control, estimation, computer vision, etc. (If sparsity then problems of larger size can be addressed)

• HAS initiated and stimulated new research issues:

- in Convex Algebraic Geometry (e.g. semidefinite representation of convex sets, algebraic degree of semidefinite programming and polynomial optimization)
- in Computational algebra (e.g., for solving polynomial equations via SDP and Border bases)
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# A remarkable property of the SOS hierarchy: I

When solving the optimization problem

$$\mathbf{P}: \quad f^* = \min \{ f(\mathbf{x}) : \; g_j(\mathbf{x}) \geq 0, \; j = 1, \ldots, m \}$$

one does NOT distinguish between CONVEX, CONTINUOUS NON CONVEX, and 0/1 (and DISCRETE) problems! A boolean variable $x_i$ is modelled via the equality constraint "$x_i^2 - x_i = 0$".

In Non Linear Programming (NLP),

modeling a 0/1 variable with the polynomial equality constraint

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and applying a standard descent algorithm would be considered "stupid"!

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Each class of problems has its own *ad hoc* tailored algorithms.

Even though the moment-SOS approach DOES NOT SPECIALIZE to each class of problems:

- It recognizes the class of (easy) SOS-convex problems as FINITE CONVERGENCE occurs at the FIRST relaxation in the hierarchy.
- Finite convergence also occurs for general convex problems and generically for non convex problems
- $\rightarrow$ (NOT true for the LP-hierarchy.)
- The SOS-hierarchy dominates other lift-and-project hierarchies (i.e. provides the best lower bounds) for hard 0/1 combinatorial optimization problems!

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# A remarkable property: II

FINITE CONVERGENCE of the SOS-hierarchy is GENERIC!

... and provides a GLOBAL OPTIMALITY CERTIFICATE,

the analogue for the NON CONVEX CASE of the
KKT-OPTIMALITY conditions in the CONVEX CASE!

## Theorem (Marshall, Nie)

*Let* $\mathbf{x}^* \in \mathbf{K}$ *be a global minimizer of*

$$\mathbf{P}: \quad f^* = \min\{f(\mathbf{x}) : g_j(\mathbf{x}) \geq 0, \; j = 1, \ldots, m\}.$$

*and assume that:*

(i) *The gradients* $\{\nabla g_j(\mathbf{x}^*)\}$ *are linearly independent,*

(ii) *Strict complementarity holds (*$\lambda_j^* \, g_j(\mathbf{x}^*) = 0$ *for all j.)*

(iii) *Second-order sufficiency conditions hold at* $(\mathbf{x}^*, \lambda^*) \in \mathbf{K} \times \mathbb{R}_+^m$.

*Then* $f(\mathbf{x}) - f^* = \sigma_0^*(\mathbf{x}) + \sum_{j=1}^m \sigma_j^*(\mathbf{x}) g_j(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n$, *for some SOS polynomials* $\{\sigma_j^*\}$.

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Certificates of positivity already exist in convex optimization

$$f^* = f(\mathbf{x}^*) = \min \{ f(\mathbf{x}) : g_j(\mathbf{x}) \geq 0, \quad j = 1, \ldots, m \}$$

when $f$ and $-g_j$ are CONVEX. Indeed if Slater's condition holds there exist nonnegative KKT-multipliers $\lambda_j^* \in \mathbb{R}_+^m$ such that:

$$\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j{}^* g_j(\mathbf{x}^*) = 0; \quad \lambda_j{}^* g_j(\mathbf{x}^*) = 0, \, j = 1, \ldots, m.$$

... and so ... the Lagrangian

$$L_{\lambda^*}(\mathbf{x}) := f(\mathbf{x}) - f^* - \sum_{j=1} \lambda_j{}^* g_j(\mathbf{x}),$$

**satisfies**

$L_{\lambda^*}(\mathbf{x}^*) = 0$ and $L_{\lambda^*}(\mathbf{x}) \geq 0$ for all $\mathbf{x}$. Therefore:

$$L_{\lambda^*}(\mathbf{x}) \geq 0 \Rightarrow f(\mathbf{x}) \geq f^* \quad \forall \mathbf{x} \in \mathbf{K}!$$

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when $f$ and $-g_j$ are CONVEX. Indeed if Slater's condition holds there exist nonnegative KKT-multipliers $\lambda_j^* \in \mathbb{R}_+^m$ such that:

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... and so ... the Lagrangian

$$L_{\lambda^*}(\mathbf{x}) := f(\mathbf{x}) - f^* - \sum_{j=1} \lambda_j{}^* g_j(\mathbf{x}),$$

**satisfies**

$L_{\lambda^*}(\mathbf{x}^*) = 0$ and $L_{\lambda^*}(\mathbf{x}) \geq 0$ for all $\mathbf{x}$. Therefore:

$$L_{\lambda^*}(\mathbf{x}) \geq 0 \Rightarrow f(\mathbf{x}) \geq f^* \quad \forall \mathbf{x} \in \mathbf{K}!$$

In summary:

| KKT-OPTIMALITY when $f$ and $-g_j$ are CONVEX | PUTINAR's CERTIFICATE in the non CONVEX CASE |
|---|---|
| $\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \lambda_j^* \nabla g_j(\mathbf{x}^*) = 0$ | $\nabla f(\mathbf{x}^*) - \sum_{j=1}^m \sigma_j(\mathbf{x}^*) \nabla g_j(\mathbf{x}^*) = 0$ |
| $f(\mathbf{x}) - f^* - \sum_{j=1}^m \lambda_j^* g_j(\mathbf{x})$ | $f(\mathbf{x}) - f^* - \sum_{j=1}^m \sigma_j^*(\mathbf{x}) g_j(\mathbf{x})$ |
| $\geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$ | $(= \sigma_0^*(\mathbf{x})) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. |
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So even though both LP- and SDP-relaxations were not designed for solving specific hard problems ...

The SDP-relaxations behave reasonably well ("efficiently"?) in very different contexts in contrast to LP-relaxations.

However they also have limits to their efficiency and may be algorithms tailored to specific hard problems with ad-hoc tools are needed.

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# A Lagrangian interpretation of LP-relaxations

Consider the optimization problem

$$\mathbf{P}: \quad f^* = \min \{ f(\mathbf{x}) : \mathbf{x} \in \mathbf{K} \},$$

where $\mathbf{K}$ is the compact basic semi-algebraic set:

$$\mathbf{K} := \{ \mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \geq 0; \, j = 1, \ldots, m \}.$$

**Assume that:**

• For every $j = 1, \ldots, m$ (and possibly after scaling), $g_j(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in \mathbf{K}$.

• The family $\{g_j, 1 - g_j\}$ generate $\mathbb{R}[\mathbf{x}]$.

# Lagrangian relaxation

The dual method of multipliers, or Lagrangian relaxation consists of solving: $\rho := \max_{\mathbf{u}}\{\, G(\mathbf{u}) \,:\, \mathbf{u} \geq 0\,\}$,

$$\text{with} \quad \mathbf{u} \mapsto \quad G(\mathbf{u}) \,:=\, \min_{\mathbf{x}} \left\{ f(x) - \sum_{j=1}^{m} u_j\, g_j(x) \right\}.$$

Equivalently:

$$\rho = \max_{\mathbf{u},\lambda} \{\lambda :\ f(x) - \sum_{j=1}^{m} u_j\, g_j(x) \geq \lambda, \quad \forall x.\}$$

**In general, there is a DUALITY GAP, i.e., $\rho \,<\, f^*$,**

except in the CONVEX case where $f$ and $-g_j$ are all convex (and under some conditions).

With $d \in \mathbb{N}$ fixed, consider the new optimization problem $\mathbf{P}_d$:

$$f_d^* = \min_x \left\{ f(x) : \quad \prod_{j=1}^m g_j(x)^{\alpha_j}(1 - g_j(x))^{\beta_j} \geq 0 \right.$$
$$\left. \forall \alpha, \beta : |\alpha + \beta| = \sum_j \alpha_j + \beta_j \leq 2d \right\}$$

**Of course**

$\mathbf{P}$ and $\mathbf{P}_d$ are equivalent and so $f_d^* = f^*$.

... because $\mathbf{P}_d$ is just $\mathbf{P}$ with additional redundant constraints!

The Lagrangian relaxation of $\mathbf{P}_d$ consists of solving:

$$\rho_d = \max_{\mathbf{u}\geq 0,\lambda} \{\lambda : \quad f(x) - \sum_{\alpha,\beta} u_{\alpha\beta} \prod_{j=1}^{m} g_j(x)^{\alpha_j}(1-g_j(x))^{\beta_j} \geq \lambda, \quad \forall x.$$

$$|\alpha+\beta| \leq 2d\}$$

**Theorem**

*$\rho_d \leq f^*$ for all $d \in \mathbb{N}$, and if $\mathbf{K}$ is compact and the family of polynomials $\{g_j, 1-g_j\}$ generates $R[\mathbf{x}]$, then:*

$$\lim_{d\to\infty} \rho_d = f^*.$$

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## The LP-hierarchy may be viewed as

the BRUTE FORCE SIMPLIFICATION of

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$$\theta_d = \max_{\mathbf{u}\geq 0,\lambda} \{\lambda : \quad f(x) - \sum_{\alpha,\beta} u_{\alpha\beta} \prod_{j=1}^{m} g_j(x)^{\alpha_j}(1-g_j(x))^{\beta_j} - \lambda = 0, \quad \forall x$$

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and indeed, ... with $|\alpha + \beta| \leq 2d$,

the set of $(\mathbf{u}, \lambda)$ such that $\mathbf{u} \geq 0$ and

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is a CONVEX POLYTOPE!

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**However as already mentioned**

- For most easy convex problems (except LP) finite convergence is impossible!
- Other obstructions to exactness occur

Typically, if $\mathbf{K}$ is the polytope $\{\mathbf{x} : g_j(\mathbf{x}) \geq 0,\ j = 1, \ldots, m\}$ and $f^* = f(\mathbf{x}^*)$ with $g_j(\mathbf{x})^* = 0,\ j \in J(\mathbf{x}^*)$, then finite convergence is impossible as soon as the exists $\mathbf{x} \neq \mathbf{x}^*$ with $J(\mathbf{x}) = J(\mathbf{x}^*)$ ($\mathbf{x}$ not necessarily in $\mathbf{K}$)

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# A less brutal simplification

With $k \geq 1$ FIXED, consider the LESS BRUTAL SIMPLIFICATION of

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# Why such a simplification?

- With $k$ fixed, $\rho_d^k = f^*$ as $d \to \infty$.
- Computing $\rho_d^k$ is now solving an SDP (and not an LP any more!)
- However, the size of the LMI constraint of this SDP is $\binom{n+k}{n}$ (fixed) and does not depend on $d$!
- For convex problems where $f$ and $-g_j$ are SOS-CONVEX polynomials, the first relaxation in the hierarchy is exact, that is, $\rho_1^k = f^*$ (never the case for the LP-hierarchy)

• A polynomial $f$ is SOS-CONVEX if its Hessian $\nabla^2 f(x)$ factors as $L(x)\, L(x)^T$ for some polynomial matrix $L(x)$. For instance, separable polynomials $f(x) = \sum_{i=1}^n f_i(x_i)$, with convex $f_i$'s are SOS-CONVEX.

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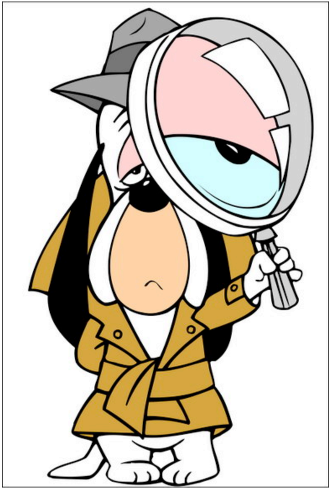
# An alternative moment-approach



## So far we have considered

LP- and SDP-moment approaches based on
CERTIFICATES of POSITIVITY on $\mathbf{K}$

## That is:

One approximates FROM INSIDE the (convex cone) $C_d(\mathbf{K})$ of polynomials nonnegative on $\mathbf{K}$: For instance if $\mathbf{K} = \{\mathbf{x} : \, g_j(\mathbf{x}) \geq 0, \, j = 1, \ldots, m\}$, by the convex cones:

$$C_d^k(\mathbf{K}) \;=\; \{\underbrace{\sigma_0}_{SOS} + \sum_{j=1}^m \underbrace{\sigma_j}_{SOS} \, g_j \; : \; \deg(\sigma_j g_j) \leq 2k \,\} \cap \mathbb{R}[\mathbf{x}]_d$$

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## An alternative is to try

## to approximate $C_d(\mathbf{K})$ FROM OUTSIDE!

Given a sequence $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$:

• Let $L_y : \mathbb{R}[\mathbf{x}] \to \mathbb{R}$ be the Riesz linear functional:

$$g\,(= \sum_\beta g_\beta\, \mathbf{x}^\beta) \;\mapsto\; L_y(g) \;:=\; \sum_\beta g_\beta\, y_\beta$$

• Define the localizing matrix $\mathbf{M}_k(g\,y)$ with respect to $y$ and $g \in \mathbb{R}[\mathbf{x}]$ is the real symmetric matrix with rows and columns indexed by $\alpha \in \mathbb{N}^n$ and with entries

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## Theorem

*Let* $\mathbf{K} \subset \mathbb{R}^n$ *be compact and let* $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, *be the moments of a Borel measure whose support is* $\mathbf{K}$. *Then a polynomial* $g_j$ *is nonnegative on* $\mathbf{K}$ *if and only if:*

$$\mathbf{M}_k(g_j\, y) \succeq 0, \quad k = 0, 1, \ldots$$

- So if $y$ is known checking whether $\mathbf{M}_k(g_j\, y) \succeq 0$ is just computing the smallest eigenvalue of the matrix $\mathbf{M}_k(g_j\, y)$!
- The set $\Delta_k \subset \mathbb{R}[x]_d$ defined by:

$$\Delta_k := \{g \in \mathbb{R}[\mathbf{x}]_d : \mathbf{M}_k(g_j\, y) \succeq 0\}, \qquad k = 0, 1, \ldots$$

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- Of course $C_d(\mathbf{K}) \subset \Delta_k \subset \Delta_{k-1}$, for all $k = 0, 1, \ldots$, and so
- $C_d(\mathbf{K}) = \cap_{k=0}^{\infty} \Delta_k$, i.e.,

The convex cones $\Delta_k$ form a nested sequence of INNER APPROXIMATIONS of $C_d(\mathbf{K})$.

Examples of sets for which the moments of a measure $\mu$ can be computed easily include:

- In the compact case: hyper-Rectangle $[a, b]^n$, Ellipsoid $\{\mathbf{x} : (\mathbf{x} - m)^T Q(\mathbf{x} - m) \leq 1\}$, simplex $\{\mathbf{x} \geq 0 : \sum_i a_i x_i \leq b\}$, hypercube $\{-1, 1\}^n$ with $\mu$ being uniformly distributed, and
- in the non-compact case: $\mathbb{R}^n$ with $d\mu = \exp(-\|\mathbf{x}\|^2) d\mathbf{x}$, and $\mathbb{R}^n_+$ with $d\mu = \exp(-\sum_i |x_i|) d\mathbf{x}$.

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The convex cones $\Delta_k$ form a nested sequence of INNER APPROXIMATIONS of $C_d(\mathbf{K})$.

Examples of sets for which the moments of a measure $\mu$ can be computed easily include:

- In the compact case: hyper-Rectangle $[a, b]^n$, Ellipsoid $\{\mathbf{x} : (\mathbf{x} - m)^T Q(\mathbf{x} - m) \leq 1\}$, simplex $\{\mathbf{x} \geq 0 : \sum_i a_i x_i \leq b\}$, hypercube $\{-1, 1\}^n$ with $\mu$ being uniformly distributed, and
- in the non-compact case: $\mathbb{R}^n$ with $d\mu = \exp(-\|\mathbf{x}\|^2)d\mathbf{x}$, and $\mathbb{R}^n_+$ with $d\mu = \exp(-\sum_i |x_i|)d\mathbf{x}$.

# Application to optimization

Let $f^* = \min_{\mathbf{x}}\{f(\mathbf{x}) : \mathbf{x} \in \mathbf{K}\}$ and let $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, be the moments of a measure $\mu$ whose support is $\mathbf{K}$.

**For each $d \in \mathbb{N}$ consider the optimization problem:**

$$\rho_d = \max_{\lambda}\{\lambda : \mathbf{M}_d(f\,y) \succeq \lambda\,\mathbf{M}_d(y)\}.$$

with the single unknown $\lambda$.

- Computing $\rho_d$ is solving a generalized eigenvalue problem associated with $\mathbf{M}_d(f\,y)$ and $\mathbf{M}_d(y)$.
- $\rho_d \geq f^*$ for all $d$ and $\rho_d \to f^*$ as $d \to \infty$

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In other words: the sequence $(\rho_d)$, $d \in \mathbb{N}$, provides a converging sequence of upper bounds on $f^*$!

Example: MAX-CUT problem: $f(x) = \mathbf{x}^T Q \mathbf{x}$ and $\mathbf{K} = \{-1, 1\}^n$. Take for $\mu$ the measure uniformly distributed on $\mathbf{K}$ with weights $1/2$, and so with moments:

$$y_\alpha = \int_{\{-1,1\}^n} \mathbf{x}^\alpha \, d\mathbf{x} = \begin{cases} 0 \text{ if } \alpha_i \text{ is odd for some } i \\ 1 \text{ otherwise} \end{cases}$$

Then build up the localizing matrix $\mathbf{M}_d(f\, y)$ and solve

$$\rho_d = \max_\lambda \{\lambda : \mathbf{M}_d(f\, y) \succeq \lambda\, \mathbf{M}_d(y)\}.$$

**In fact, same as computing**

the smallest eigenvalue of $\widehat{\mathbf{M}}_d(f\, y)$ (keeping only the rows and columns of $\mathbf{M}_d(f\, y)$ indexed by square-free monomials $(\mathbf{x}^\alpha)$.

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THANK YOU!!